Homework 3

Due Date: December 6, 2011.

1. Let $G$ be a $d$-regular $n$-vertex graph and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of its adjacency matrix.

   (a) What can you say about $G$ if $\lambda_n = -d$?

   (b) Show that the size any independent set in $G$ is at most:

   $$\frac{-\lambda_n}{d - \lambda_n} \cdot n.$$

   This is called the Hoffman bound.

   (c) How do parts (a) and (b) relate to each other?

2. Let $G$ be a connected $d$-regular $n$-vertex graph. Let $\lambda_2$ be the second eigenvalue of the adjacency matrix of $G$. Let $f : V_G \to \mathbb{R}$ be a function with $\sum_v f(v) = 0$ and $\sum_v f(v)^2 = 1$.

   (a) Show that $\max_{u,v \in V_G}(f(u) - f(v)) \geq \frac{C}{\sqrt{n}}$, for some universal constant $C$.

   (b) Let $u$ and $v$ be vertices witnessing the above inequality. Suppose $\ell$ is the distance between $u$ and $v$. Show that

   $$\langle f, Lf \rangle \geq \Omega\left(\frac{1}{n\ell}\right),$$

   where $L$ is the Laplacian of $G$.

   (c) Thus show that in any graph $\lambda_2 \leq d - \frac{1}{n^2}$.

   (d) Define a lazy random walk on a graph to be the following process. $v_0$ is a fixed vertex; for each $k \geq 0$, $v_{k+1}$ is chosen as follows: with probability $1/2$, $v_{k+1} = v_k$, and with probability $1/2$, $v_{k+1}$ is a uniformly random neighbor of $v_k$.

   Use the bound on $\lambda_2$ to show that for any vertex $w$, the lazy random walk visits $w$ within $O(n^2 \log n)$ steps with probability $1 - o(1)$.

3. Read the proof of Cheeger’s inequality from the notes.

4. Let $G$ be an $n$ vertex graph. Let $\alpha(G)$ denote the size of the largest independent set in $G$. Let $\chi(G)$ denote the chromatic number of $G$. Recall that for every graph $G$, we have $\chi(G) \geq \frac{n}{\alpha(G)}$.

   Show that if $G$ is a Cayley graph $\text{Cay}(\Gamma, S)$ (for some $n$-element group $\Gamma$ and some $S \subseteq \Gamma$), then:

   $$\chi(G) \leq O\left(\frac{n}{\alpha(G)} \cdot \log n\right).$$
5. Recall the result of Ajtai-Komlos-Szemeredi stating that if $G$ is an $n$-vertex triangle-free graph with maximum degree at most $d$, then $G$ has an independent set of size $\Omega(n \cdot \frac{\log d}{d})$.

Let $\epsilon > 0$ be a constant. Now suppose $G$ is a $d$-regular graph which has $nd^{2-\epsilon}$ triangles. Use the above result to show that $G$ has an independent set of size $\Omega(\epsilon \cdot \frac{n \cdot \log d}{d})$. 