Suggested problems on inequalities

1. Prove that for any real number $x \geq -1$, and any positive integer $n$, $(1 + x)^n \geq 1 + nx$.

2. Prove that $n! \geq (n/e)^n$ and that $n! \leq ((n + 1)/e)^{n+1}$.

3. Suppose that $a_1, a_2, \ldots$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that for any $p > 1/2$, $\sum_{n \geq 1} \sqrt[n]{a_n}/n^p$ also converges.

4. Let $a_1, a_2, \ldots, a_n$ be positive real numbers and let $s$ denote their sum. Show that $(1 + a_1)(1 + a_2)\ldots(1 + a_n) \leq \sum_{i=0}^{n} s^i/i!$.

5. Let $p_1, \ldots, p_n$ be distinct points in the closed unit disc in the plane. Let $d_k$ be the distance from $p_k$ to the nearest other point. Show that $\sum_{k=1}^{n} (d_k)^2 \leq 16$.

6. For $n$ positive real numbers with minimum $m$ and maximum $M$, let $A$ and $G$ denote their arithmetic and geometric means. Prove that $A - G \leq (\sqrt{M} - \sqrt{m})^2/n$.

7. Let $x_1, \ldots, x_n$ be positive real numbers and $k$ a positive integer. Prove $\frac{1}{n} \sum x_i^k \leq \frac{\sum x_i^{k+1}}{\sum x_i}$.

8. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be nonnegative real numbers. Show $(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \leq [(a_1 + b_1) \cdots (a_n + b_n)]^{1/n}$.

9. Let $x_1, \ldots, x_n$ be real numbers in $[0, \pi]$ Let $x$ be their average. Prove that: $\Pi_{i=1}^{n} \sin(x_i)/x_i \leq (\sin(x)/x)^n$. 

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