Problems on power series

1. Use the technique of generating functions to solve the recurrence relation $a_0 = 1, a_1 = 0, a_2 = -5$ and for $n > 3$,
   $$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}.$$ 

2. Prove that the value of the $n$th derivative of $x^3/(x^2 - 1)$ at $x = 0$ is 0 for $n$ even, and $-n!$ for $n$ odd.

3. Let $p, q$ with $p \in (0, 1/2]$ be real numbers satisfying $1/p - 1/q = 1$. Prove that $p + p^2/2 + p^3/3 + \cdots = q - q^2/2 + q^3/3 - \cdots$.

4. Show that $\sum_{n=0}^{\infty} \sin(n \theta)/n! = \sin(\sin \theta) \cos \theta$.

5. Let $T_n = \sum_{i=1}^{n} (-1)^{i+1} / (2i - 1)$, and $T = \lim_{n \to \infty} T_n$. Show that:
   $$\sum_{n=1}^{\infty} T_n - T = \frac{\pi}{8} - \frac{1}{4}.$$ 

6. Putnam problem: For $0 < x < 1$, express $\sum_{n=0}^{\infty} x^{2n} / 1 - x^{2n}$ as a rational function of $x$.

7. Putnam problem: Define $S_0 = 1$. For $n \geq 1$, let $S_n$ be the number of $n \times n$ symmetric matrices with nonnegative entries, and all row sums equal to 1. Prove:
   (a) $S_{n+1} = S_n + nS_{n-1}$
   (b) $\sum_{n=0}^{\infty} S_n x^n / n! = e^{x^2/2}$.

8. Putnam problem: If $n$ is a positive integer, let $(B(n))$ be the number of ones in the base 2 expression for $n$. For example, $B(6) = 2, B(15) = 4$. Determine whether or not $e^{\sum_{n=1}^{\infty} B(n)/n(n+1)}$ is a rational number.

9. Putnam problem: For each positive integer $n$, let $f_n(x)$ denote the function:
   $$f_n(x) = \sum_{0 \leq k \leq n/2} \binom{n}{k} x^k / \sum_{0 \leq k \leq n/2} \binom{n}{2k+1} x^k$$
   Express $f_{n+1}(x)$ rationally in terms of $f_n(x)$ and $x$. Determine $\lim_{n \to \infty} f_n(x)$ for all of the values of $x$ that you can.

10. Putnam problem: Let $f_0(x) = e^x$ and $f_{n+1}(x) = x f_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that
    $$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$