## Problems on polynomials

1. Suppose $P(X) \in \mathbb{C}[X]$ is such that for every $x \in \mathbb{R}, P(x) \in \mathbb{R}$. Show that all the coefficients of $P(X)$ are real numbers.
2. Let $P(X)=X^{r}+a_{1} X^{r-1}+\ldots+a_{r-1} X+a_{r}$ be a polynomial with complex coefficients such that $P(X)$ divides $X^{n}-1$. Show that $\left|a_{i}\right| \leq\binom{ r}{i}$.
3. Show that $X^{100}-X-5$ cannot be written as the product of two polynomials of degree at least one with integer coefficients.
4. Find all complex numbers $a, b$ such that the roots of $x^{2}+a x+b$ are $\{a, b\}$.
5. Suppose $P(X)$ is a polynomial such that $P(1)=0, d P / d X(1)=0, \ldots, d^{k} P / d X^{k}(1)=$ 0 . Then show that $(X-1)^{k}$ divides $P(X)$.
6. Is there a polynomial with integer coefficients which has $\sqrt{2}+\sqrt{3}$ as a root?
7. Find all polynomials $P(X)$ such that $P(P(X))=X$.
8. Find all polynomials $P(X)$ such that $P(P(P(X)))=X$.
9. Show that for any real number $a, b, c$,

$$
a^{2}+b^{2}+c^{2} \geq a b+b c+c a .
$$

10. Show that

$$
\sum_{i=0}^{n-1}(i+1)\binom{n}{i+1} 2^{i}=n 3^{n-1}
$$

11. Let $P(X)$ be a quadratic polynomial with real coefficients such that $P(0), P(1)$ and $P(2)$ are all integers. Then show that $P(n)$ is an integer for all integers $n$.
Also show that there are quadratic polynomials $P(X)$ such that $P(0), P(1)$ and $P(3)$ are integers, but there are integers $n$ for which $P(n)$ is not an integer.
12. $P(X)$ is a polynomial such that $P(1)=1$ and $P(-1)=2$. What is the remainder when you divide $P(X)$ by $X^{2}-1$ ?
13. Show that the polynomial

$$
P(X)=X^{n}+X^{n-1}+X^{n-2}+a_{3} X^{n-3}+a_{4} X^{n-4}+\ldots+a_{n}
$$

cannot have all its roots real.

