Some Remarks on Phase Planes

A technique which is often very useful in order to analyze the phase plane behavior of a two-dimensional autonomous system

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

is as follows. We would like to understand the graphs of solutions $(x(t), y(t))$ look like. One possibility is to try to see these graphs as the level sets of some function $h(x,y)$.

For example, take

$$\begin{aligned} \frac{dx}{dt} &= x \\ \frac{dy}{dt} &= -y \end{aligned}$$

(that is, $f(x,y) = x$ and $g(x,y) = y$). If we could solve for t as a function of x, by inverting the function $x(t)$, and substitute the expression that we obtain into $y(t)$, we would end up with an expression $y(x)$ for the y-coordinate in terms of the x coordinate, eliminating t. This cannot be done in general, but it suggests that we may want to look at dy/dx. Formally (or, more precisely, using the chain rule), we have that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y}{x}$$

which is a differential equation for y as a variable dependent on x. This equation is separable:

$$\int \frac{dy}{y} = \int -\frac{dx}{x}$$

so we obtain, taking antiderivatives,

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

where c is an undetermined constant. *In conclusion, the solutions* $(x(t), y(t))$ *all lie in the circles* $x^2 + y^2 = c$ *of different radii and centered at zero.* Observe that we have **not** solved the differential equation, since we did determine the forms of x and y as functions of t (which, as a matter of fact, are trigonometric functions). What we have done is just to find curves (the above-mentioned circles) which contain all solutions. Even though this is less interesting (perhaps) than the actual solutions, it is still very interesting. We know what the general phase plane picture looks like.

Another example is this:

$$\begin{aligned} \frac{dx}{dt} &= y^5 e^x \\ \frac{dy}{dt} &= x^5 e^x . \end{aligned}$$

Here, $dy/dx = x^5/y^5$ so we get again a separable equation, and we see that the solutions all stay in the curves

$$x^6 + y^6 = c\,.$$

More interesting is the general case of predator-prey equations:

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned}$$

where a, b, c, d are all positive constants. Then

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)}$$

so

$$\int \left(\frac{a}{y} - b\right) dy = \int \left(-\frac{c}{x} + d\right) dx$$

and from here we conclude that the solutions all stay in the sets

$$a\ln(y) - by + c\ln(x) - dx = c$$

for various values of the constant c. It is not obvious what these sets look like, but if you graph the level sets of the function

$$h(x, y) = a\ln(y) - by + c\ln(x) - d$$

you'll see that the level sets look like the orbits of the predator-prey system shown, for the special values $a = 2$, $b = 1.2$, $c = 1$, and $d = 0.9$ in page 144 of the book. (Of course, the scales will be different, for different values of the constants, but the picture will look the same, in general terms.) This argument is used to prove that predator-prey systems always lead to periodic orbits, no matter what the coefficients of the equation are.

Homework: In each of the following problems, a system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

is given. Solve the equation

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$$

and use the information to sketch what the orbits of the original equation should look like.

1. $\frac{dx}{dt} = y(1 + x^2 + y^2)$, $\frac{dy}{dt} = x(1 + x^2 + y^2)$.
2. $\frac{dx}{dt} = 4y(1 + x^2 + y^2)$, $\frac{dy}{dt} = -x(1 + x^2 + y^2)$.
3. $\frac{dx}{dt} = y^3 e^{x+y}$, $\frac{dy}{dt} = -x^3 e^{x+y}$.
4. $\frac{dx}{dt} = y^2$, $\frac{dy}{dt} = (2x + 1)y^2$.
5. $\frac{dx}{dt} = e^{xy}\cos(x)$, $\frac{dy}{dt} = e^{xy}$.