

Homework problems, diffusion

Suppose that $c(x, t)$ denotes the density of a bacterial population undergoing random motions (diffusion). We assume in each case that we are working in dimension 1.

A. In the first problem, we think of the population as living in a thin tube along the x axis, with endpoints at $x = 0$ and $x = L$, and take the diffusion constant to be $D = 1$ (for simplicity).

For each of the following descriptions write down the appropriate diffusion equation, including boundary conditions, and then find one solution of the separated form $c(x, t) = X(x)T(t)$ which is bounded and nonzero.

1. $x = 0, L = 1$, both ends of the tube are open, with the outside density of bacteria being negligible.

Answer: $\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}, c(0, t) = c(1, t) = 0$

$e^{-k^2 \pi^2 t} \sin k\pi x, k$ any integer.

2. $x = 0, L = \pi/2$, end at $x = 0$ open and end at $x = L$ is closed.

Answer: $\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}, c(0, t) = \frac{\partial c}{\partial x}(\pi/2, t) = 0$

$e^{-k^2 t} \sin kx, k$ any odd integer.

3. $x = 0, L = \pi$, both ends are closed.

Answer: $\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}, \frac{\partial c}{\partial x}(0, t) = \frac{\partial c}{\partial x}(\pi/2, t) = 0$

$e^{-k^2 t} \cos kx, k$ any integer.

B. In the second problem, we suppose that the domain is infinite, $-\infty < x < \infty$, and that, besides diffusion, there is an air flow (in the positive x direction) with constant velocity v . We now let the diffusion coefficient be an arbitrary constant D .

1. Explain why we model this by the following equation:

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} - v \frac{\partial c}{\partial x}.$$

(This is called, by the way, an *advection-diffusion* equation, and may be interpreted probabilistically as describing a *random walk with drift*.)

2. You will now find the “fundamental solution” of this equation, as follows. Introduce the new variable $z = x - vt$ and the function

$$\alpha(z, t) = c(z + vt, t).$$

(You should recognize here the trick which was used in order to solve the transport equation, when there was no diffusion.) Show that α satisfies a diffusion equation, and show therefore how to obtain a solution for c by substituting back into the fundamental solution (Gaussian) for α .

Answer: Using subscripts for partial derivatives, in order to write less: with $\alpha(z, t) = c(z + vt, t)$

we have $\alpha_t = v c_x + c_t = D c_{xx}$ (chain rule first, and then using the advection-diffusion equation) and so $D \alpha_z = c_x, D \alpha_{zz} = D c_{xx} = \alpha_t$. So we have

$$\alpha(z, t) = \frac{C}{\sqrt{4\pi Dt}} e^{-\frac{z^2}{4Dt}}$$

and now using $c(x, t) = \alpha(x - vt, t)$:

$$c(x, t) = \frac{C}{\sqrt{4\pi Dt}} e^{-\frac{(x-vt)^2}{4Dt}}$$