5. Sketch the nullclines in the xy phase plane, identify steady states, and draw directions of arrows on the nullclines for the following systems of first-order equations:

(a) $\frac{dx}{dt} = y^2 - x^2,$

$\frac{dy}{dt} = x - 1.$

(b) $\frac{dx}{dt} = x(y^2 - y),$

$\frac{dy}{dt} = x - y.$

(c) $\frac{dx}{dt} = x^2 + y,$

$\frac{dy}{dt} = -y.$

(d) $\frac{dx}{dt} = -xy,$

$\frac{dy}{dt} = (1 + x)(1 - y).$

(e) $\frac{dx}{dt} = x^2 - y,$

$\frac{dy}{dt} = y^2 - x.$

(f) $\frac{dx}{dt} = \frac{-xy}{1 + x} + x,$

$\frac{dy}{dt} = \frac{xy}{1 + x} - y.$

(g) $\frac{dx}{dt} = xy(1 - x) + C,$

$\frac{dy}{dt} = y\left(1 - \frac{y}{x}\right).$

6. For problem 5(a–g) find the Jacobian of each system of equations and determine stability properties of each steady state.

7. Sketch the phase-plane behavior of the following systems of linear equations and classify the stability characteristic of the steady state at (0, 0):

(a) $\frac{dx}{dt} = -2y,$

$\frac{dy}{dt} = x.$

(b) $\frac{dx}{dt} = 3x + 2y,$

$\frac{dy}{dt} = 4x + y.$

(c) $\frac{dx}{dt} = 2x + y,$

$\frac{dy}{dt} = x + 2y.$

(d) $\frac{dx}{dt} = 5x + 8y,$

$\frac{dy}{dt} = -3x - 5y.$

(e) $\frac{dx}{dt} = -4 - 2y,$

$\frac{dy}{dt} = 3x - y.$

(f) $\frac{dx}{dt} = x - 4y,$

$\frac{dy}{dt} = x + y.$

10. *Looking at the phase-plane,* describe in words what would happen if we set up the chemostat to contain the following:
(a) A small number of bacteria with excess nutrient in the growth chamber.
(b) A large number of bacteria with very little nutrient in the growth chamber.

11. In drawing the phase-plane diagram of the chemostat, we assumed that $\alpha_2 > 1/(\alpha_1 - 1)$.
(a) Show that $(\bar{N}_2, \bar{C}_2)$ is a saddle point whenever this inequality is satisfied.
(b) Now suppose this inequality is not satisfied. Sketch the resulting phase-plane diagram and interpret the biological meaning.

21. In this problem we examine a continuous plant-herbivore model. We shall define q as the chemical state of the plant. Low values of q mean that the plant is toxic; higher values mean that the herbivores derive some nutritious value from it. Consider a situation in which plant quality is enhanced when the vegetation is subjected to a low to moderate level of herbivory, and *declines* when herbivory is extensive. Assume that herbivores whose density is I are small insects (such as scale bugs) that attach themselves to one plant for long periods of time. Further assume that their growth rate depends on the quality of the vegetation they consume. Typical equations that have been suggested for such a system are

$$\frac{dq}{dt} = K_1 - K_2 qI(I - I_0),$$

$$\frac{dI}{dt} = K_3 I\left(1 - \frac{K_4 I}{q}\right).$$

(a) Explain the equations, and suggest possible meanings for K_1, K_2, I_0, K_3, and K_4.
(b) Show that the equations can be written in the following dimensionless form:

$$\frac{dq}{dt} = 1 - KqI(I - 1),$$

$$\frac{dI}{dt} = \alpha I\left(1 - \frac{I}{q}\right).$$

Determine K and α in terms of original parameters.

(c) Find qualitative solutions using phase-plane methods. Is there a steady state? What are its stability properties?
(d) Interpret your solutions in part (c).

23. The following equations were given by J. S. Griffith (1971, pp. 118–122), as a model for the interactions of messenger RNA M and protein E:

$$\dot{M} = \frac{aKE^m}{1 + KE^m} - bM, \qquad \dot{E} = cM - dE.$$

(a) Show that by changing units one can rewrite these in terms of dimensionless variables, as follows

$$\dot{M} = \frac{E^m}{1 + E^m} - \alpha M, \qquad \dot{E} = M - \beta E.$$

Find α and β in terms of the original parameters.
(b) Show that one steady state is $E = M = 0$ and that others satisfy $E^{m-1} = \alpha\beta(1 + E^m)$. For $m = 1$ show that this steady state exists only if $\alpha\beta \leq 1$.
(c) *Case 1.* Show that for $m = 1$ and $\alpha\beta > 1$, the only steady state $E = M = 0$ is stable. Draw a phase-plane diagram of the system.
(d) *Case 2.* Show that for $m = 2$, at steady state

$$E = \frac{1 \pm (1 + 4\alpha^2\beta^2)^{1/2}}{2\alpha\beta}.$$

Conclude that there are two solutions if $2\alpha\beta < 1$, one if $2\alpha\beta = 1$, and none if $2\alpha\beta > 1$.
(e) *Case 2 continued.* For $m = 2$ and $2\alpha\beta < 1$, show that there are two stable steady states (one of which is at $E = M = 0$) and one saddle point. Draw a phase-plane diagram of this system.