

Problems for weeks 3 and 4

1. We consider the analysis of one locus with three alleles, A_1 , A_2 , and A_3 ; f_{ij} is used to denote $f_{A_i A_j}$ and f_i is used to denote f_{A_i} .

a) Suppose it is given that $f_{11}(0) = 0.1$, $f_{12}(0) = 0.2$, $f_{13}(0) = 0.25$, $f_{22}(0) = 0.15$, $f_{23}(0) = 0.25$, $f_{33}(0) = 0.05$. Calculate the allele frequencies $f_1(0)$, $f_2(0)$, and $f_3(0)$.

b) Suppose generation one is created from generation zero by random mating. Make the infinite population assumption. Compute all the genotype frequencies $f_{ij}(1)$ in the first generation. Explain the formula that you use to compute these frequencies and explain how it comes from the random mating assumption.

c) Calculate the allele frequencies $f_1(1)$, $f_2(1)$, $f_3(1)$ from the genotype frequencies computed in part (b), and verify that these allele frequencies are the same as those of generation zero.

d) Determine the genotype frequencies $f_{ij}(t)$ for all future generations.

2. In part (d) of problem 3.3.9, we showed that $\lim_{t \rightarrow \infty} f_A^m(t) = \lim_{t \rightarrow \infty} f_A^f(t) = (2/3)f_A^f(0) + (1/3)f_A^m(0)$. What does this answer represent? Consider a (finite) human population divided into two equal subpopulations of male and female. Consider a locus on the X chromosome with alleles A and a . The population allele pool is obtained by combining the allele pool of the males and that of the females. Let $f_A(0)$ be the frequency of A in the combined allele pool and let $f_A^f(0)$ and $f_A^m(0)$ be the frequencies in the female and male subpopulations, respectively. Show that $f_A(0) = (2/3)f_A^f(0) + (1/3)f_A^m(0)$. This is the limiting frequency of $f_A(t)$ in each subpopulation is just the overall frequency of A in the initial population. This is intuitive because the male and female allele pools get mixed in each random mating.

3. A population with age structure. Imagine a monocious population composed of two subpopulations; subpopulation I of sexually immature, nonbreeding individuals, and subpopulation II of mature, breeding individuals. We want to create a model in which it takes individuals 1 generation to enter the breeding subpopulation. Assume: (a) subpopulation I is a fraction $1/4$ of the entire population and subpopulation II constitutes the remaining $3/4$; (b) subpopulation I at $t+1$ is created by random mating of subpopulation II at t ; (c) subpopulation II at $t+1$ is created by removing a third of subpopulation II at t and replacing it by subpopulation I at t —this corresponds to subpopulation at t entering the breeding population of $t+1$. Let $f_{A,I}(t)$ and $f_{A,II}(t)$ be the frequencies of A in the subpopulations I and II, respectively, at time t . From the written description, derive difference equations that express $f_{A,I}(t+1)$ and $f_{A,II}(t+1)$ in terms of $f_{A,I}(t)$ and $f_{A,II}(t)$. Finally, solve these equations assuming $f_{A,I}(0) = 1/4$ and $f_{A,II}(0) = 1/2$.

4. Problem 3.3.17 in page 31. *Hint:* pick an individual I, and let T be the random variable that says what is the length of I's life. So, $T = k$ means that individual I did not die in the first $k - 1$ generations but does exactly at the k th one. Compute the expected value T .

Next two are Wright-Fisher problems

5. Problem 3.5.2 in page 47. Skip the second part ("Find an expression ..."), but do the last part (probability that 'a' disappears).

6. This problem asks you to formulate a variant of the Wright-Fisher model with mutation. In the Wright-Fisher model, generation $t + 1$ is created from generation t by N independent random matings. From the point of view of the allele pool, we know this is equivalent to creating the allele pool of generation $t + 1$ by a random sample of size $2N$ from the allele pool of generation t . Assume now however, that each A selected may mutate to a with probability u , and each a may mutate to A with probability v , and that whether mutation occurs or not is independent for the different alleles chosen. Write down the transition probabilities $\mathbb{P}(X(t+1)=j \mid X(t)=i)$ for this model.

Week 4 problems:

7. This is basically problem 3.4.4 on page 46, but rewritten. Consider the recursive equation

$$x(n + 1) = f(x(n)), \tag{1}$$

where $f(x) = 4x(1 - x)$. Graph this function on the unit interval $[0, 1]$ and notice that f maps the unit interval into itself. Convince yourself by cobwebbing that there is a non-constant solution to (1) with period 2; this means that $x(2) = x(0)$ but $x(1) \neq x(0)$. Next find the period-two solution analytically, by observing that for the periodic solution

$$x(2) = f(f(x(0))) = x(0), \quad f(x(0)) \neq x(0).$$

That is, the starting value $x(0) = x_0$ is a fixed point of the function $f(f(x))$, but $x(0)$ is not a fixed point of f . (Notice that the fixed points of f , which you can find are also fixed points of $f(f(x))$; this will help you factor $f(f(x)) - x$ to find a new fixed point of $f(f(x))$.)

8. Exercise 3.4.1, page 45. Analyze also $w_{aa} = 1.2$, $w_{Aa} = 1$, $w_{AA} = 0.9$; and $w_{Aa} = 1$, $w_{aa} = .8$, $w_{AA} = .9$.

9. Exercise 3.4.2, page 45. Do this problem but **change the data** to $w_{AA} = 0.3$ and $w_{aa} = 0.2$.

10. Exercise 3.4.3. (Note that the hint on this problem had a small error: the formula for $\phi'(p)$ used to calculate $\phi'(0)$ is on Chapter 3, on page 42.) The idea is to show that $\phi'(0) > 1$. Use this and the fact that you may take as given that $\phi(p)$ is increasing in p , to draw the general shape of the curve.