Math 338, Problem Solution, Week 14; Spring 2008

Consider a hidden Markov model whose hidden chain evolves in the state space $S = \{0, 1, 2, 3\}$ with state transition matrix

$$A \triangleq \begin{bmatrix} 0 & .2 & .3 & .5 \\ 0 & .3 & .3 & .4 \\ 0 & .4 & .4 & .2 \\ 0 & .1 & .6 & .3 \end{bmatrix}$$

Assume that the states output one of three letters $\{A, B, C\}$ with probabilities,

$$\begin{bmatrix} e_1(A) & e_1(B) & e_1(C) \\ e_2(A) & e_2(B) & e_2(C) \\ e_3(A) & e_3(B) & e_3(C) \end{bmatrix} = \begin{bmatrix} .25 & .35 & .4 \\ .1 & .25 & .65 \\ .5 & .45 & .05 \end{bmatrix}.$$

Find the probability that the first three letters produced by this model are ABC.

For this, we need to run the forward algorithm on the hidden chain, as defined in Chapter 8, to compute $f_k(t)$ for $t \leq 3$. The probability of the output sequence ABC is then obtained as the sum $\sum_{k=1}^{3} f_k(3)$.

Using equation (8.10) of Theorem 1 in Chapater 8 and the intial conditions $f_0(0) = 1$ $f_k(0) = 0$, $k \geq 1$, we obtain, as in equation (8.12),

$$\begin{aligned} f_1(1) &= a_{01}e_1(A) = (.2)(.25) = 0.05 \\ f_2(1) &= a_{02}e_2(A) = (.3)(.1) = 0.03 \\ f_3(1) &= a_{03}e_3(A) = (.5)(.5) = 0.25 \end{aligned}$$

As for $t = 2$, application of (8.10) again yields

$$\begin{aligned} f_1(2) &= (f_1(1)a_{11} + f_2(1)a_{21} + f_3(1)a_{31})\, e_1(B) = (((.05)(.3) + (.03)(.4) + (.25)(.1))(.35) = 0.0182 \\ f_2(2) &= (f_1(1)a_{12} + f_2(1)a_{22} + f_3(1)a_{32})\, e_2(B) = ((.05)(.3) + (.03)(.4) + (.25)(.6))(.25) = 0.0354 \\ f_3(2) &= (f_1(1)a_{13} + f_2(1)a_{23} + f_3(1)a_{33})\, e_3(B) = ((.05)(.4) + (.03)(.2) + (.25)(.3))(.45) = 0.04545 \end{aligned}$$

Similarly

$$\begin{aligned} f_1(3) &= (f_1(2)a_{11} + f_2(2)a_{21} + f_3(2)a_{31})\, e_1(C) = 0.009666 \\ f_2(3) &= (f_1(2)a_{12} + f_2(2)a_{22} + f_3(2)a_{32})\, e_2(C) \\ &= ((.0182)(.3) + (.0354)(.4) + (.04545)(.6))(.65) = 0.0304785 \\ f_3(2) &= (f_1(2)a_{13} + f_2(2)a_{23} + f_3(2)a_{33})\, e_3(C) \\ &= ((.0182)(.4) + (.0354)(.2) + (.04545)(.3))(.05) = .00139975 \end{aligned}$$

The probability of seeing the path ABC is, after rounding,

$$f_1(3) + f_2(3) + f_3(3) = 0.0415.$$