Characterization of Poisson Process, Notes for Rutgers Math 338, Spring 2005, E,D. Sontag

We consider a process $\{N(t), t \geq 0\}$ that counts the number of "hits" happening in the interval $[0,t]$.[1] We write $P_n(t) = P(N(t) = n)$ for any time t and any nonnegative integer n and assume the following properties:

1. $P_0(0) = 1$ (i.e., zero probability of an hit occurring in an interval of length 0).
2. Hits occurring in disjoint time intervals are independent.
3. $P(N(t) - N(s) = n) = P_n(t-s)$ (i.e., the probability of k hits happening on an interval $[t,s]$ depends only on $t-s$), for all pairs $t > s$.
4. $P(N(t) > 1) = o(t)$. (Recall that a function $f(t)$ is said to be "$o(t)$" if $\lim_{t\to 0} \frac{f(t)}{t} = 0$.)

We also assume that $P(N(t) = n)$ is differentiable in t, for any nonnegative integer n.

Claim: $N(t)$ must be a Poisson process. In other words:

$$P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{1}$$

for all n. The proof is as follows.

Note that $P_n(0) = 0$ for every $n > 0$, because $P_n(0)$ is the probability that $N(0) = n$, and the first property says that $P(N(0) = 0) = 1$.

Expand $P_1(t) = P_1(0) + P_1'(0)t + o(t) = P_1'(0)t + o(t)$ into a first-order Taylor series. Now define $\lambda = P_1'(0)$, so that

$$P_1(t) = \lambda t + o(t)\,.$$

Also,

$$P_0(t) = P(N(t) = 0) = 1 - P(N(t) = 1) - P(N(t) > 1) = 1 - \lambda t + o(t)\,,$$

and therefore $P_0'(0) = -\lambda$. So we have that:

$$\begin{aligned} P_0(t+h) &= P(N(t+h) = 0) \;=\; P[(N(t) = 0) \text{ and } (N(t+h) - N(t) = 0)] \\ &= P(N(t) = 0)P(N(t+h) - N(t) = 0) \;=\; P(N(t) = 0)P(N(h) = 0) \;=\; P_0(t)P_0(h) \end{aligned}$$

so taking derivatives with respect to h at $h = 0$ we obtain: $P_0'(t) = -\lambda P_0'(t)$, and hence, solving the differential equation:

$$P_0(t) = P_0(0)e^{-\lambda t} = e^{-\lambda t}. \tag{2}$$

We will show, by induction on n, that (1) is true. For $n = 0$, this is just (2). For any $n \geq 1$:

$$\begin{aligned} P_n(t+h) &= P(N(t+h) = n) \;=\; \sum_{k=0}^{n} P[(N(t) = n-k) \text{ and } (N(t+h) - N(t) = k)] \\ &= \sum_{k=0}^{n} P(N(t) = n-k)P(N(t+h) - N(t) = k) \\ &= \sum_{k=0}^{n} P_{n-k}(t)P_k(h) \;=\; P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \;=\; P_n(t)e^{-\lambda h} + P_{n-1}(t)(\lambda h + o(h)) + o(h) \end{aligned}$$

so taking derivatives with respect to h at $h = 0$ we obtain:

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)\,.$$

We view this as a differential equation on $P_n(t)$ and use an integrating factor:

$$\frac{d}{dt}[e^{\lambda t}P_n(t)] = e^{\lambda t}[P_n'(t) + \lambda P_n(t)] = \lambda e^{\lambda t}P_{n-1}(t) = \lambda e^{\lambda t}e^{-\lambda t}\frac{(\lambda t)^{(n-1)}}{(n-1)!}$$

(induction hypothesis used at the end). So

$$\frac{d}{dt}[e^{\lambda t}P_n(t)] = \frac{(\lambda t)^{(n-1)}}{(n-1)!}$$

and hence integrating:

$$e^{\lambda t}P_n(t) = \frac{(\lambda t)^n}{n!} + P_n(0) = \frac{(\lambda t)^n}{n!}$$

(remember that $P_n(0) = 0$ for all $n > 0$), so multiplying both sides by $e^{-\lambda t}$ we obtain the desired formula.

[1] We use the terminology "hits" instead of "events" in order not to confuse with the probabilistic use of the word "event".