Additional remark on controllability Gramians

Let us consider the problem of minimizing the energy (L^2 norm) of an input controlling 0 to x. The time T is now arbitrary.

Note that, since an input on time T may be seen as an input on time $T' > T$ (just use $u \equiv 0$ on an initial interval), this optimal cost is decreasing (to be more precise, non-increasing) as T increases.

The optimum for inputs of any given length T is $\omega = L^{\#}x$ and has norm $\sqrt{x^* W_T^{-1} x}$, where

$$W_T = \int_0^T \Phi(T,s)B(s)B(s)^*\Phi(T,s)^*\,ds\,, = \int_0^T e^{(T-s)A}BB^*e^{(T-s)A^*}\,ds\,.$$

Equivalently, substituting $T - s \to s$:

$$W_T == \int_0^T e^{sA}BB^*e^{sA^*}\,ds\,.$$

By the remark above on decreasing costs, $x^*W_T^{-1}x \leq x^*W_S^{-1}x$ is $T > S$.

If A is stable (all its eigenvalues have negative real part), this converges as $t \to +\infty$ to

$$W_\infty == \int_0^\infty e^{sA}BB^*e^{sA^*}\,ds$$

(which is finite).

[Remark: for unstable A we don't expect the min to be positive: if there is an eigenvector with eigenvalue with real part positive, first move a bit in the direction of such an eigenvector, and then let $\omega \equiv 0$, so that instability drives the motion; in this way, arbitrarily small ω is enough to reach at least some states.]

One can see that W_∞ is invertible. This can be proved as follows. Take any $p \neq 0$. Note that p^*W_Tp is nondecreasing, because it is the integral of a nonnegative function. Moreover, for any finite T, e.g. $T = 1$, the (symmetric) matrix W_T is invertible (because for time-invariant continuous-time systems, reachability implies reachability in arbitrarily small positive time), and hence positive definite. Thus $p^*W_\infty p \geq p^*W_1p > 0$ and W_∞ must also be positive definite.

Since (by definition of $\int_0^\infty$) $W_T \to W_\infty$, it follows that $W_T^{-1} \to W_\infty^{-1}$ too, and therefore

$$x^*W_T^{-1}x \to x^*W_\infty^{-1}x$$

for all x.

Because $x^*W_T^{-1}x$ decreases, this limit is also the infimum of the values $x^*W_T^{-1}x$. In other words, the least possible cost (not necessarily achievable exactly).

Note also, for $W = W_\infty$:

$$AW+WA^* = \int_0^\infty Ae^{sA}BB^*e^{sA^*}+e^{sA}BB^*e^{sA^*}A^*\,ds = \int_0^\infty \frac{d}{ds}(e^{sA}BB^*e^{sA^*})\,ds = e^{sA}BB^*e^{sA^*}|_0^\infty = -BB^*$$

(using stability), so

$$AW + WA^* + BB^* = 0\,.$$

This is a set of *linear* equations! (One can prove, and we do this later when talking about Lyapunov functions, that the matrix equation $AX + XA^* = -BB^*$ has a unique solution, which must therefore be $X = W$.)