Additional remark on distance to singularity (appendix A)

Lemma A.2.4 and Corollary A.2.5 show that $\sigma_{\min}$ measures, for a matrix with full row or column rank, the distance to lower-rank matrices. These results can be refined to the following fact:

$$\min_{X|\operatorname{rank} X=k} \|A - X\| = \sigma_{k+1} \tag{1}$$

(norm means induced operator norm with respect to Euclidean norm, and X ranges over matrices of the same size as A), which we prove next. (The Lemma and Corollary deal with the cases when $k = r - 1$, where r is the rank of A, assumed equal to the number of rows or columns.) The key fact is:

Lemma 0.1 For each X of rank $\leq k = \operatorname{rank} A$, $\|A - X\| \geq \sigma_{k+1}$.

Proof. Let $\ell = \operatorname{rank} X \leq k$. The nullspace of X has dimension $m - \ell$, where m is the number of columns of A and X. Let $A = U\Sigma V^*$ be an SVD of A and let $v_1, \ldots$ be the columns of V. Since

$$(m - \ell) + (k + 1) = (m + 1) + (k - \ell) \geq m + 1,$$

the span (of dimension $k + 1$) of $\{v_1, \ldots, v_{k+1}\}$ must intersect the nullspace of X. In other words, there is some vector $x = Vv$ in this span so that $Xx = 0$. Without loss of generality, we may take x to be a unit vector. Note that v is also a unit vector, since

$$1 = \|x\|^2 = v^*V^*Vv = v^*v = \|v\|^2 .$$

Since $\|(A - X)x\| \leq \|A - X\| \, \|x\| = \|A - X\|$, we have that

$$\|A - X\|^2 \geq \|(A - X)x\|^2 = \|Ax\|^2 = \|U\Sigma V^*Vv\|^2 = \|\Sigma v\|^2 = \sum_{i=1}^{k+1} \sigma_i^2 v_i^2 \geq \sum_{i=1}^{k+1} \sigma_{k+1}^2 v_i^2 = \sigma_{k+1}^2 \sum_{i=1}^{k+1} v_i^2 = \sigma_{k+1}^2 ,$$

which is what we wanted to prove. ∎

To show (1), it only remains to provide one X of rank k for which $\|A - X\| = \sigma_{k+1}$. This can be achieved using

$$X = U\Sigma_k V^*$$

where Σ_k is the matrix obtained from Σ by setting diagonal entries after the kth one to zero.

Observe that the minimizing X is not unique. For example, for

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

any matrix of the form

$$\begin{pmatrix} 2 + \varepsilon & 0 \\ 0 & 0 \end{pmatrix}$$

with $|\varepsilon| \leq 1$ has rank 1 and satisfies that $\|A - X\| = 1$ (because the induced 2-norm for a diagonal matrix is the maximum of the absolute values of its entries).