Rutgers 642:613 - Fall 2003

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Sketch of Approximation Arguments as $\varepsilon \approx 0$

http://www.math.rutgers.edu/~sontag/613.html

I was asked in the last class for a sketch of how to *prove* that trajectories track the slow dynamics

I'll give simple arguments rather than refer to general singular perturbation theory

let's start with this simplified version:

$$\varepsilon\dot{x} = g(y) - x$$
$$\dot{y} = f(x, y)$$

and assume that the reduced equation

$$\dot{y} = f(g(y), y)$$

has a solution where $y(0) = y_0$ and $y(T) = y_T$

we wish to prove that, for $0 < \varepsilon \ll 1$, the solution of the original system such that

$$x(0) = g(y_0),\ y(0) = y_0$$

is such that $y(T) \approx y_T$ (and also $x(T) \approx g(y_T)$)

let us write $z(t) = x(t) - g(y(t))$, so $x = z + g(y)$ and the system becomes (since $\dot{x} = (1/\varepsilon)(g(y) - x)$:

$$\dot{z} = -\frac{1}{\varepsilon}z - F(z,y)$$
$$\dot{y} = f(z + g(y), y)$$

where $F(z,y) := g'(y)f(z+g(y),y)$, $z(0) = x(0) - g(y_0) = 0$

we may assume without loss of generality that

$$|F(z,y)| \leq c$$

for some constant $c > 0$ (not necessarily "small")

This is because we can modify the equations outside a compact neighborhood of the compact set of points of the form $x = g(y)$, $y = y(t)$ where $y(\cdot)$ is the solution that we started with (so all functions have compact suport and are continuous). Since we'll prove that we can stay arbitrarily close to this set, and using uniqueness of solutions, a solution of the modified system is also a solution of the original system; (see e.g. my control theory textbook for examples of this argument in similar contexts, e.g. in the proof of Theorem 1 there).

claim: for any $\delta > 0$, we can find $\varepsilon_0 > 0$ such that

$$|z(t)| = |x(t) - g(y(t))| \leq \delta \,,\; 0 \leq t \leq T$$

provided that $0 < \varepsilon \leq \varepsilon_0$

we write $\dot{z} = -\frac{1}{\varepsilon}z - F(z, y)$ as

$$\dot{z} + \frac{1}{\varepsilon}z = -F(z, y)$$

so $(d/dt)\left(e^{t/\varepsilon}z\right) = -e^{t/\varepsilon}F(z, y)$ and so (recall $z(0) = 0$)

$$e^{t/\varepsilon}z(t) = -\int_0^t e^{s/\varepsilon}F(z(s), y(s))\, ds$$

and hence:

$$|z(t)| \leq ce^{-t/\varepsilon}\int_0^t e^{s/\varepsilon}\, ds = \varepsilon\left(1 - e^{-t/\varepsilon}\right)$$

so that this is $< \delta$ is ε is sufficiently small

now we look at the y equation: $\dot{y} = f(z + g(y), y)$
since $z(t)$ is uniformly small, uniform continuity of solutions on inputs (time-varying parameters; see e.g. my control textbook) completes the result

but in FitzHugh's, not $\varepsilon\dot{x} = g(y) - x$, so let's generalize:

$$\begin{aligned} \varepsilon\dot{x} &= g(x) - y \\ \dot{y} &= f(x, y) \end{aligned}$$

difference is now we do not have $x - g(x)$ in explicit form,
but suppose we solve $g(x) = y$ for $y = \gamma(x)$
i.e. we have $g(\gamma(y)) = y$ for y in a certain range
we also assume $\gamma'(x) \leq -c < 0$ in the domain of interest

(e.g. solve one of the two decreasing branches of the cubic, away from the two "elbows", getting W_+ or W_-)
also, we assume that all functions are bounded (same trick as earlier)

would like to prove that $x(t) \approx \gamma(y(t))$,
so we can approximate the second equation by:

$$\dot{y} = g(\gamma(y), y)$$

and argue as before

we take $z = x - \gamma(y)$ and again start on the nullcline: $z(0)=0$

$$\dot{z} = \frac{1}{\varepsilon}(g(x) - y) - \gamma'(y) f(x, y)$$

and using $\gamma(y) = x - z \Rightarrow y = g(\gamma(y)) = g(x - z)$:

$$\dot{z} = \frac{1}{\varepsilon}\left(g'(x)z + g''(\xi)z^2\right) - \gamma'(y) f(x, y)$$

where $\xi = \xi(t)$ is between x and $x - z$

(may pick as continuous function of t if desired)

so

$$\dot{z} = \frac{1}{\varepsilon} g'(x)z + \frac{1}{\varepsilon}\alpha(t)z^2 + \beta(t)$$

where (all functions bounded) $|\alpha(t)| \leq a$ and $|\beta(t)| \leq b$ for all $0 \leq t \leq T$

let $w := z^2/2$, so:

$$\dot{w} = \frac{2}{\varepsilon} g'(x)w + \frac{2}{\varepsilon}\alpha(t)wz + \beta(t)z$$

and pick w_0, ε s.t.:

$$\sqrt{2w_0} < \frac{c}{2a}, \quad \varepsilon < \frac{c\sqrt{w_0}}{b\sqrt{2}}$$

we had $\dot{w} = \frac{2}{\varepsilon} g'(x) w + \frac{2}{\varepsilon} \alpha(t) w z + \beta(t) z$, so:

$$\dot{w} = \frac{2w}{\varepsilon} \left(\frac{1}{2} g'(x) - \alpha(t) z \right) + \left(\frac{1}{2\varepsilon} g'(x) z^2 + \beta(t) z \right) \; , \quad w(0) = 0$$

claim: there cannot be any t_0 such that $w(t_0) = w_0$; else:

$$\dot{w}(t_0) = \frac{2w_0}{\varepsilon} A + B$$

where

$$A = (1/2) g'(x(t_0)) - \alpha(t_0) z_0 \, , \; B = (1/2\varepsilon) g'(x(t_0)) z_0^2 + \beta(t_0) z_0$$

and:

$$A \leq -c/2 + a\sqrt{2w_0} < 0$$

(since $|\alpha(t_0) z_0| \leq a\,|z_0| = a\sqrt{2w_0}$ and $\sqrt{2w_0} < \frac{c}{2a}$), and

$$B \leq -(c/\varepsilon) w_0 + b\sqrt{2w_0} = \sqrt{w_0} \left(-c\sqrt{w_0}/\varepsilon + \sqrt{2} b \right) < 0$$

(since $|\beta(t_0) z_0| \leq b\,|z_0| = b\sqrt{2w_0}$ and $\varepsilon < \frac{c\sqrt{w_0}}{b\sqrt{2}}$)

so conclude:
if $w(t_0) = w_0$ *then* $\dot{w}(t_0) < 0$

this implies that $w(t) \leq w_0$ for all t (else look at first t_0 such that $w(t_0) = w_0$; since $\dot{w}(t_0) < 0$ it follows that $w(t) > w_0$ for $t \approx t_0^-$, contradicting minimality of t_0)

as w_0 could be picked arbitrarily small,
this shows that we can pick $\varepsilon > 0$ such that
for every $0 < \varepsilon \leq \varepsilon_0$ the trajectories have that
$|x - \gamma(y)| = |z| = \sqrt{2w}$ is arbitrarily small