Rutgers 642:613 - Fall 2003

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Very Quick Intro to Bursting

http://www.math.rutgers.edu/~sontag/613.html

we start by modifying the FitzHugh-Nagumo model

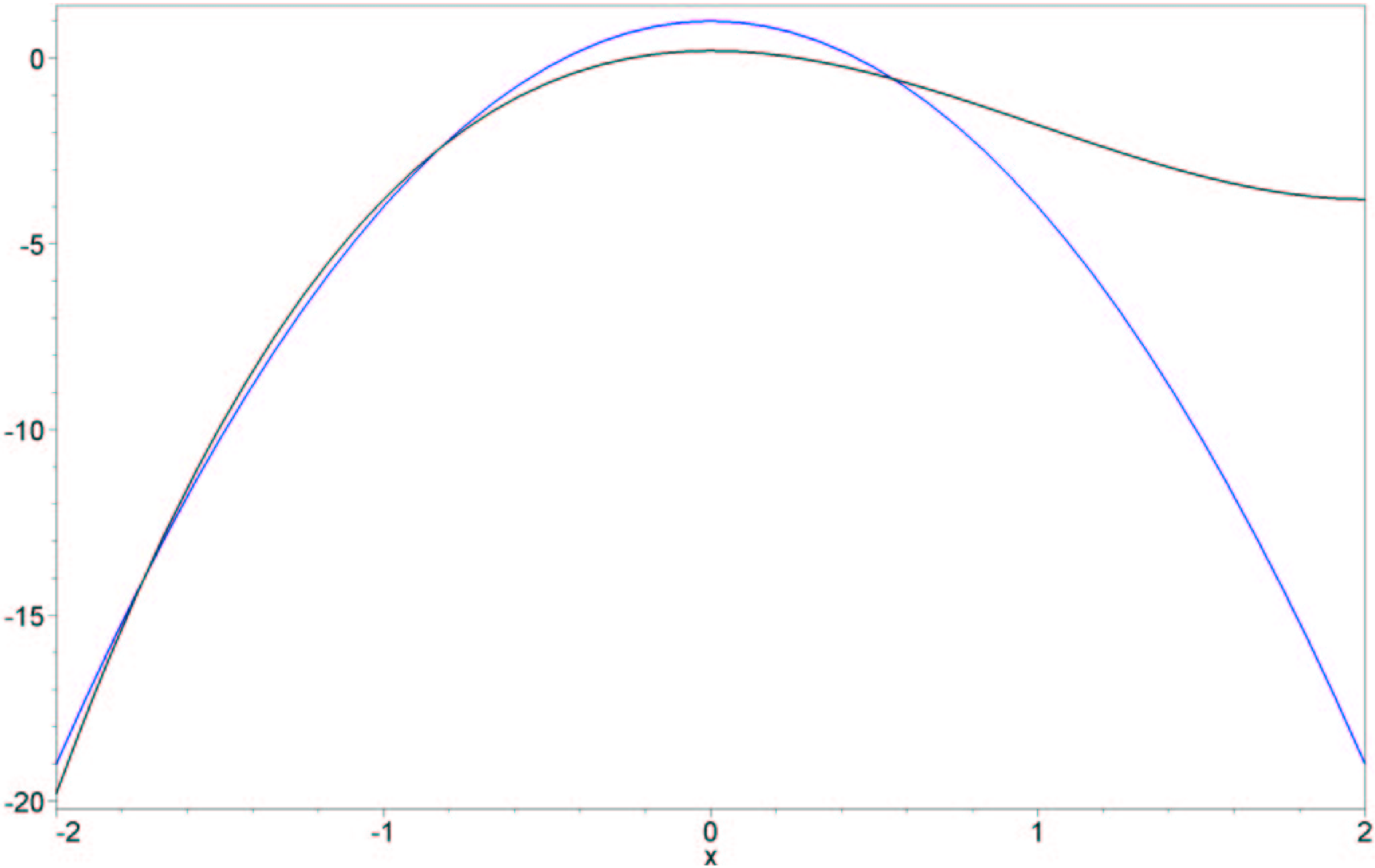
we deform the y nullcline (used to be a line) so that there are three steady states instead of one

this allows for “bistability” between a stable state and an unstable one (with a saddle in between) where the unstable one is sorrounded by a limit cycle

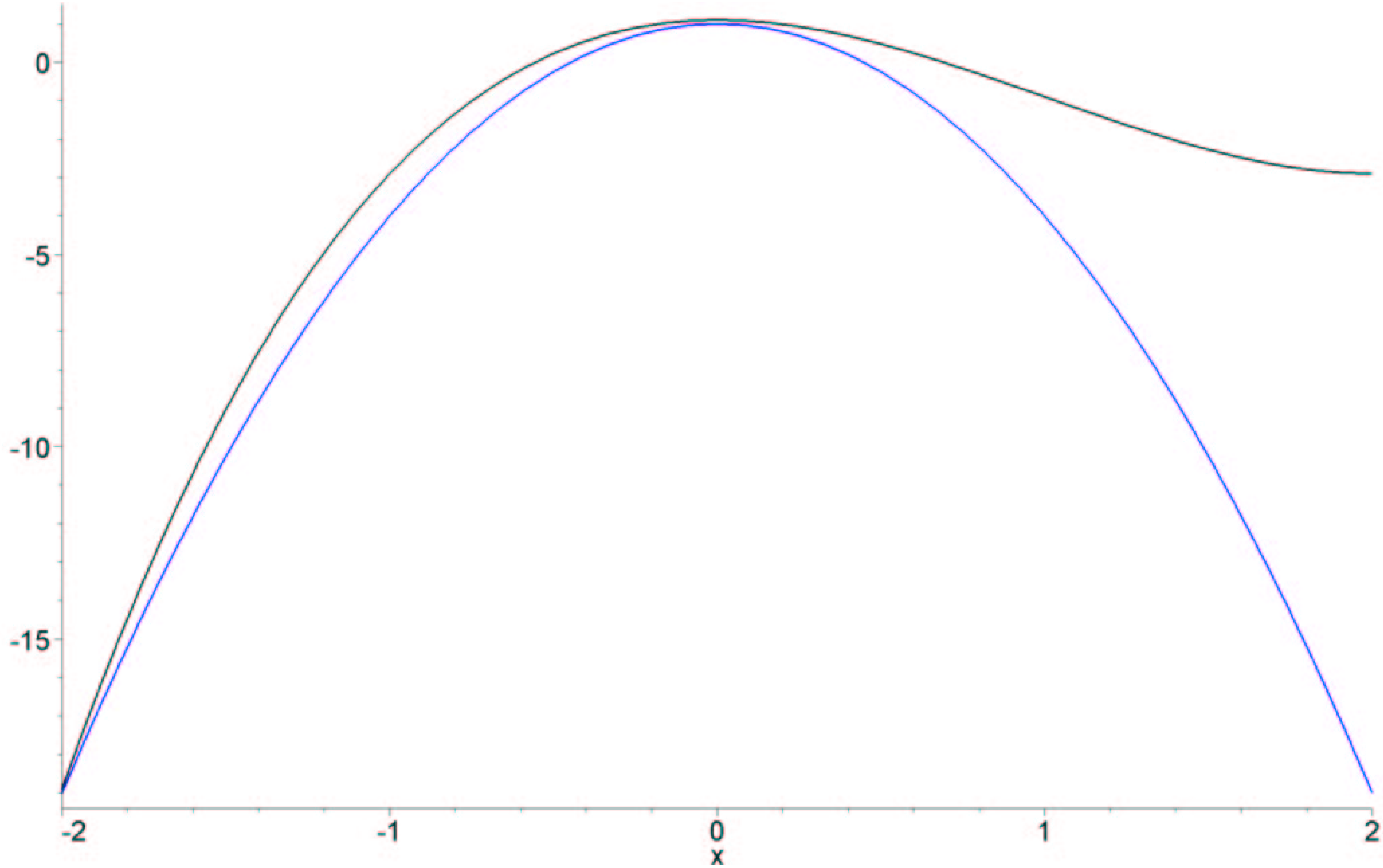
thus, depending on where we start, we have either stability or periodic behavior

$$\dot{x} = y - x^3 + 3x^2 + I$$
$$\dot{y} = 1 - 5x^2 - y$$

with $I = 1.1$:



but when $I = 0.2$:



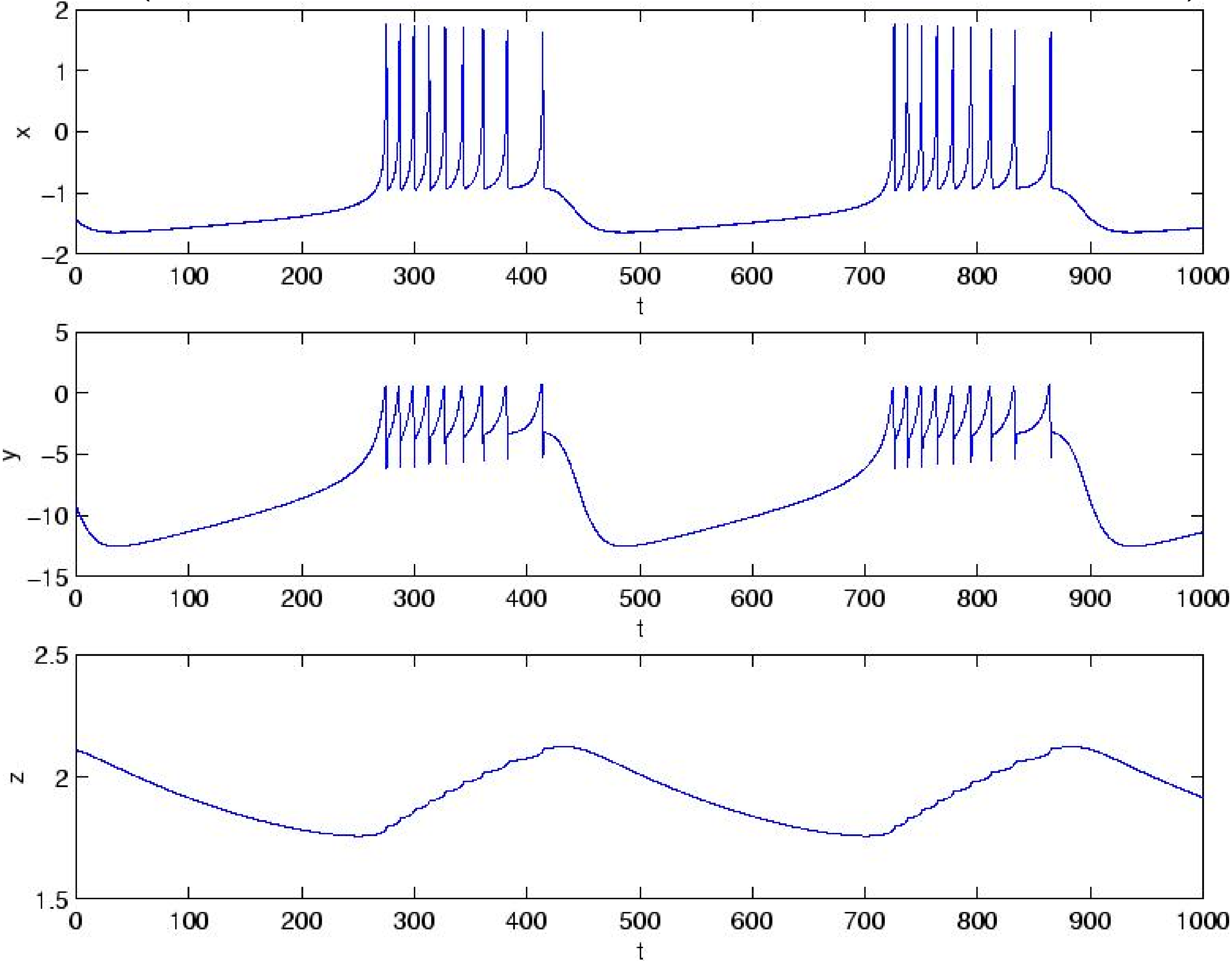
the saddle and unstable equilibrium have dissapeared, and only a stable equilibrium remains; thus, if I is temporarily brought down, trajectories will tend to stabilize - until I goes back up and oscillations might restart (depending on the initial condition; separatrix is almost horizontal)

we next modify system to allow I to change slowly, depending on the values of x, y:

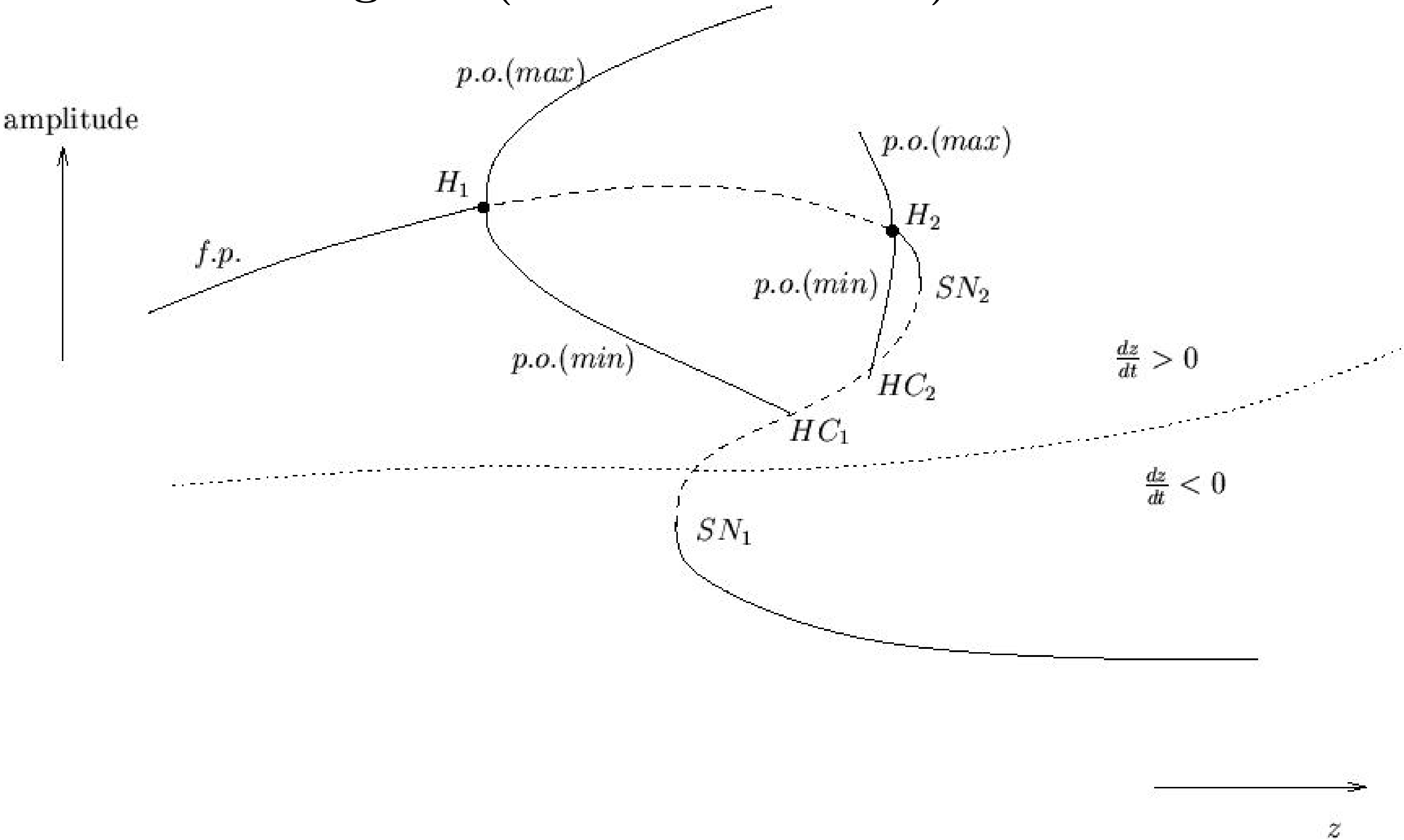
$$\begin{aligned} \dot{x} &= y - x^3 + 3x^2 + I - z \\ \dot{y} &= 1 - 5x^2 - y \\ \dot{z} &= r(s(x - x_1) - z) \end{aligned}$$

where $x_1 = -\frac{1}{2}(1 + \sqrt{5})$, $I = 2$, $r = 0.001$, $s = 4$
"type I bursting" behavior, analogous to β-cells in pancreas (but actual equations are just phenomenological — look at book for more realistic models)

plots (from Jeff Moehlis's Princeton APC591, 2001)



bifurcation diagram (from same source):

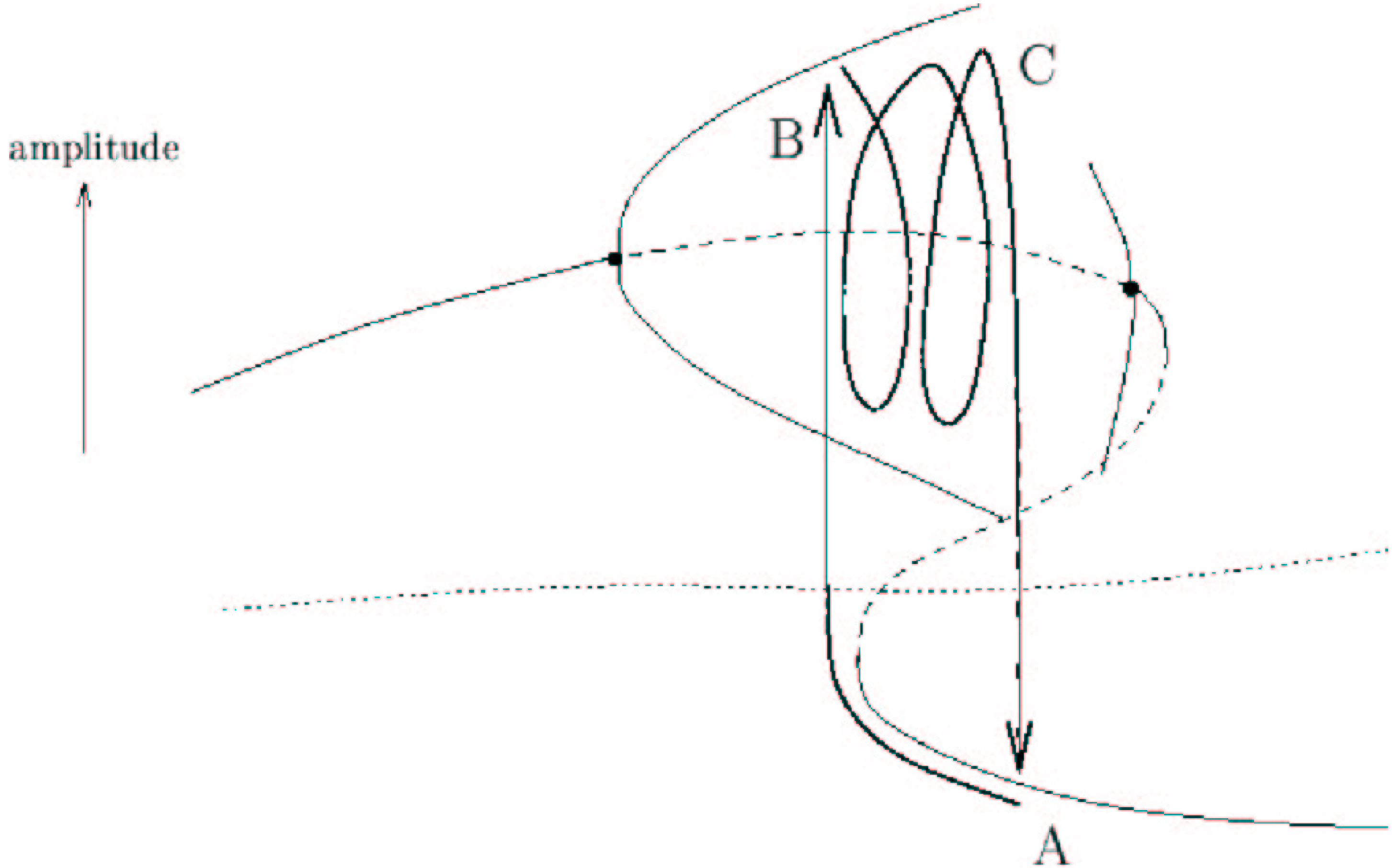


bifurcation diagram for equations with $I = 2$ and z treated as a bifurcation parameter. Solid lines indicate stable solutions, while dashed lines indicate unstable solutions.
Fixed points are labelled f.p., and periodic orbits (amplitudes) are labelled p.o. The dotted line shows the z-nullcline.

the bifurcations are as follows:

- H_1: **Hopf bifurcation at** $z \approx -9.59$
- H_2: **Hopf bifurcation at** $z \approx 2.926$
- SN_1: **saddlenode bifurcation of fixed points at**
- SN_2: **saddlenode bifurcation of fixed points at** $z = 3$
- HC_1: **homoclinic bifurcation at** $z \approx 2.086$
- HC_2: **homoclinic bifurcation at** $z \approx 2.926$

the behavior of the complete system can be understood as follows (again borrowing figure from above):



Suppose starting at A (stable point, rest state of neuron). Then z will slowly decrease; follow this branch of fixed points until reaching the saddlenode bifurcation. Here the system makes a jump to point B, the stable periodic orbit (active state). Next, z will slowly increase, with the neuron actively firing until reaching homoclinic bifurcation at C. Here, periodic orbit ceases to exist, and system makes a jump back to stable point A. Behavior repeats, giving a sequence of bursts. Note that as the homoclinic bifurcation is approached, the period of the periodic orbits increases

review of homoclinic bifurcation (again from above ref):

homoclinic orbit: trajectory which approaches a fixed point both as $t \to +\infty$ and as $t \to -\infty$. The formation of a homoclinic orbit as a parameter is varied, called a homoclinic bifurcation, can lead to the creation or destruction of a periodic orbit. For the homoclinic bifurcation shown below, a homoclinic orbit forms at $\lambda = 0$, and a periodic orbit exists for $\lambda > 0$ but not for $\lambda < 0$. As $\lambda \to 0^+$, the period of the periodic orbit diverges to infinity