

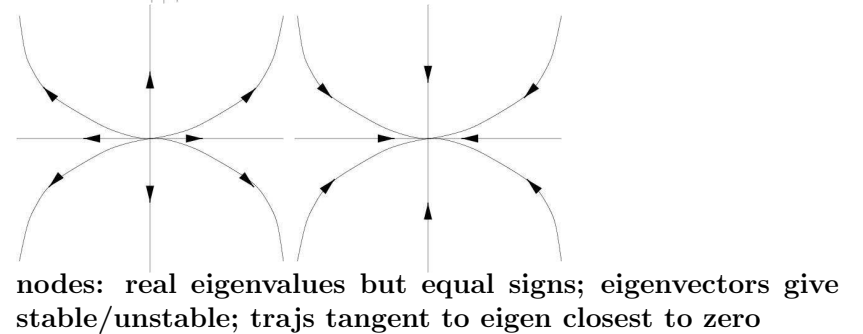
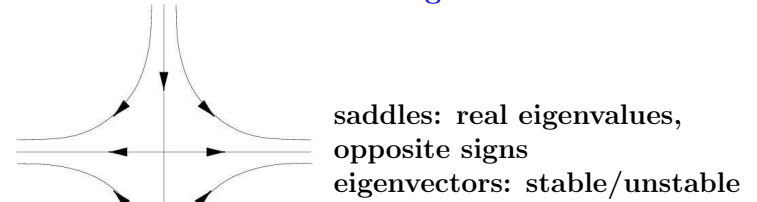
Rutgers 642:613 - Fall 2003

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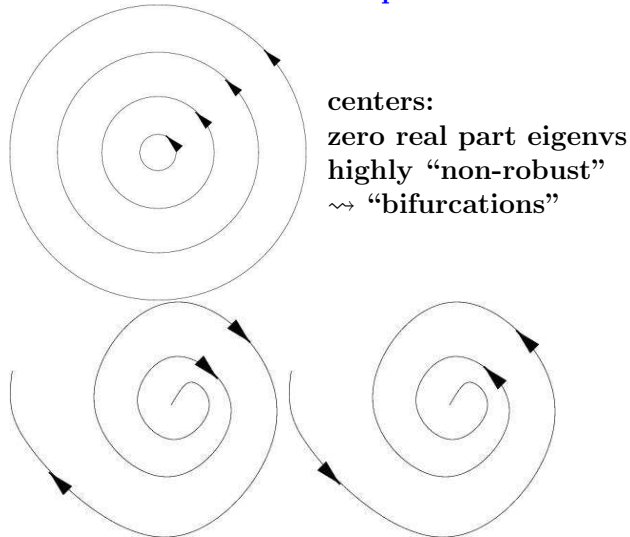
Review: Trace/Determinant Plane, Linear Phase-Planes
 Review: Period Orbits, Limit Cycles

<http://www.math.rutgers.edu/~sontag/613.html>

Linear Phase Planes: Real Eigens



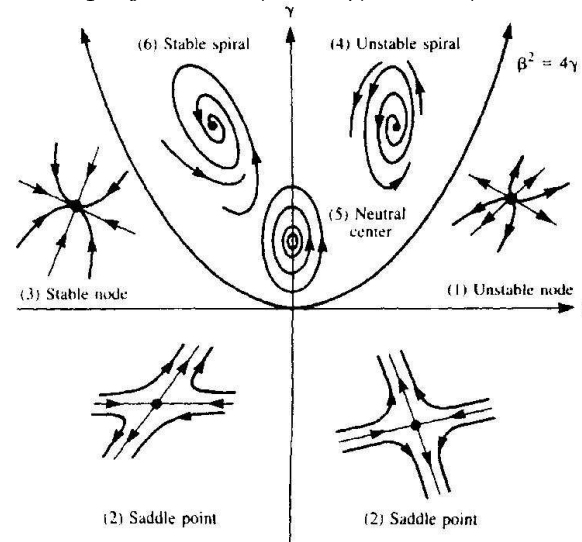
Linear Phase Planes: Complex λ 's



stable or unstable spirals
 stability depends on real part > 0 or < 0
 orientation: easiest is to plot vector at e.g. $(1, 0)$

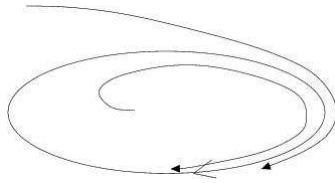
Trace/Determinant Plane

char poly is $\lambda^2 - \beta\lambda + \gamma$, where $\beta = \text{trace}$, $\gamma = \text{det}$

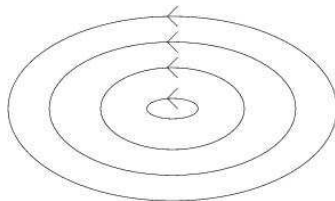


Periodic Orbits and Limit Cycles

(stable) *limit cycle* := a periodic trajectory which attracts other solutions to it:



a member of a family of “parallel” periodic solutions (as for linear centers) is *not* called a limit cycle



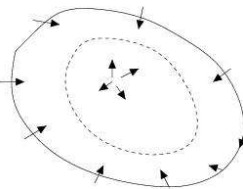
limit cycles **robust** in two ways (& linear periodic sols not):

Poincaré-Bendixon Theorem

for systems of two equations, \exists very powerful criterion

– we give a simple version sufficient for our purposes

suppose a bounded region D in the plane is so that no trajectories can exit D [on ∂ , v.f. points inside or tangentially] and either \nexists no steady states inside or \exists unique steady state that is repelling then there is a periodic orbit inside D



(also: if unique periodic orbit, then limit cycle)

idea: start near boundary: go towards inside, cannot cross back, must keep going, cannot approach source - must approach periodic (proof in grad diff eq course!)

Robustness of Limit Cycles

(1) if perturbation moves state to different initial state away from the cycle, system will return to cycle e.g. circadian rhythm: study late, but later get back to normal pattern (\rightsquigarrow jet lag!)

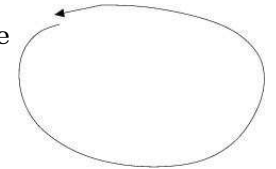
compare linear: will simply start oscillating along a different orbit, and never come back by itself; particular oscillation depends on initial conditions

(2) if dynamics changes a little, a limit cycle will still exist (can be proved as theorem)

compare linear: a small perturbation like

$$dx/dt = y, \quad dy/dt = -x + \varepsilon y$$

changes to spiral (stable or unstable)



Example of Limit Cycle

consider this system (not biological - just math!)

$$\begin{aligned} \dot{x} &= x + y - x(x^2 + y^2) \\ \dot{y} &= -x + y - y(x^2 + y^2) \end{aligned}$$

easier to understand in polar coordinates:

$$x = r \cos \varphi, \quad y = r \sin \varphi$$

one obtains: $\dot{r} = r(1 - r^2)$, $\dot{\varphi} = -1$,

so $r = 1$ (unit circle) is limit cycle

since all trajectories rotate clockwise at unit speed while point at distance r decreases towards 1 if $r > 1$ or increases towards 1 if $r < 1$

Poincaré-Bendixon: only equil: $(0, 0)$ unstable spiral on circle $x^2 + y^2 = 2$: normal is (x, y) , dot product:

$$\begin{aligned} [x + y - x(x^2 + y^2)]x + [-x + y - y(x^2 + y^2)]y \\ = (1 - (x^2 + y^2))(x^2 + y^2) < 0 \end{aligned}$$

so v.f. points inside $\Rightarrow \exists$ periodic orbit;

using a more subtle argument can prove limit cycle:

use annular regions $1 - \varepsilon < x^2 + y^2 < 1 + \varepsilon$ so unique

Bendixon's Criterion

given region D simply-connected (no holes)

if *the divergence of the vector field*

is always positive or is always negative inside D ,

then there cannot be a periodic orbit inside D

Proof: suppose \exists ; describes simple closed curve C ,

recall divergence of $F(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$ is: $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$

Gauss divergence theo (or Green's theorem) \Rightarrow

$$\int \int_D \operatorname{div} F(x, y) \, dx dy = \int_C \vec{n} \cdot F$$

(line integral of dot prod of outward normal with F)

along periodic *orbit*: F is tangent \Rightarrow dot prod = 0

so integral of div is zero, \therefore must change sign

$\dot{x} = x, \dot{y} = y$: div = 2 \Rightarrow \nexists periodic orbits

$\dot{x} = x, \dot{y} = -y$: div = 0 (but \nexists)

$\dot{x} = y, \dot{y} = -x$: div = 0 (but \exists)