Rutgers 642:613 - Fall 2003

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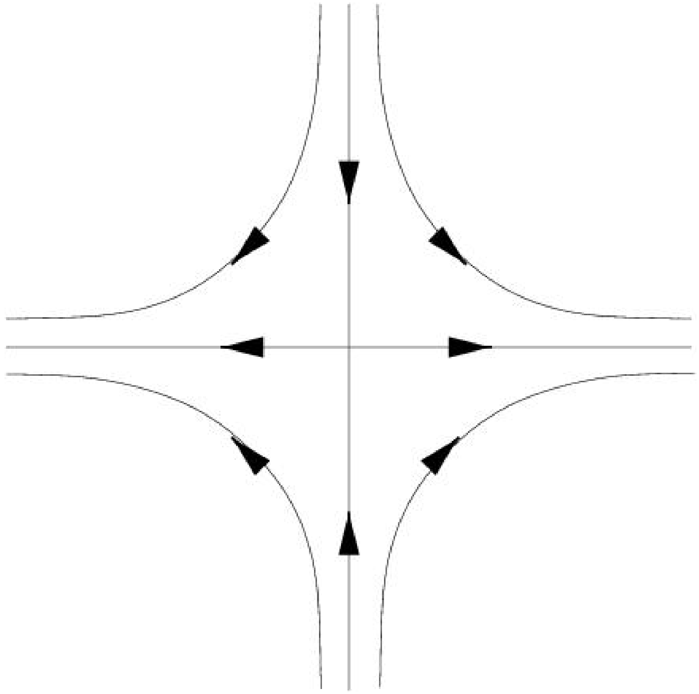
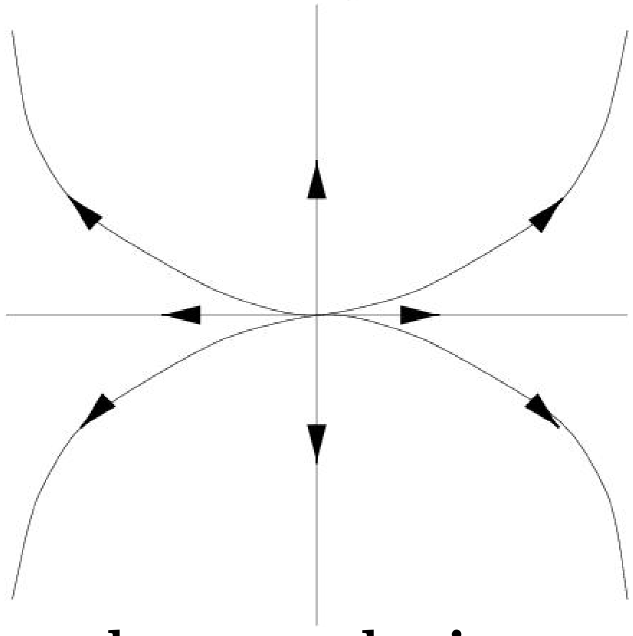
Review: Trace/Determinant Plane, Linear Phase-Planes
Review: Period Orbits, Limit Cycles

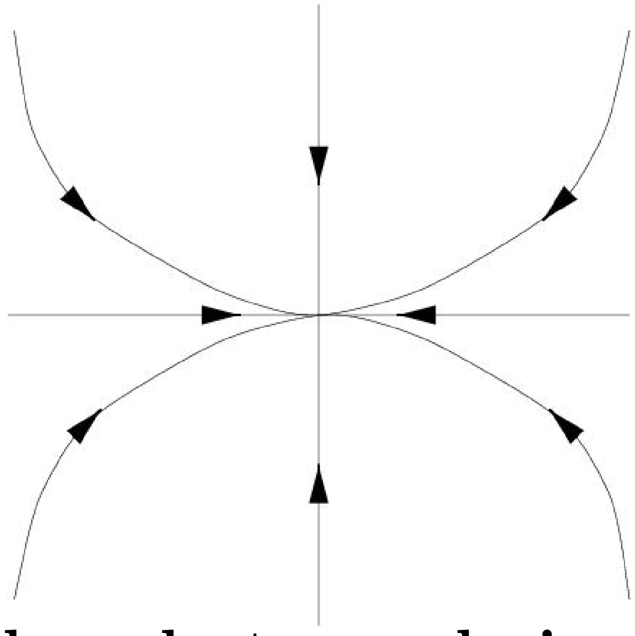
http://www.math.rutgers.edu/~sontag/613.html

Linear Phase Planes: Real Eigens



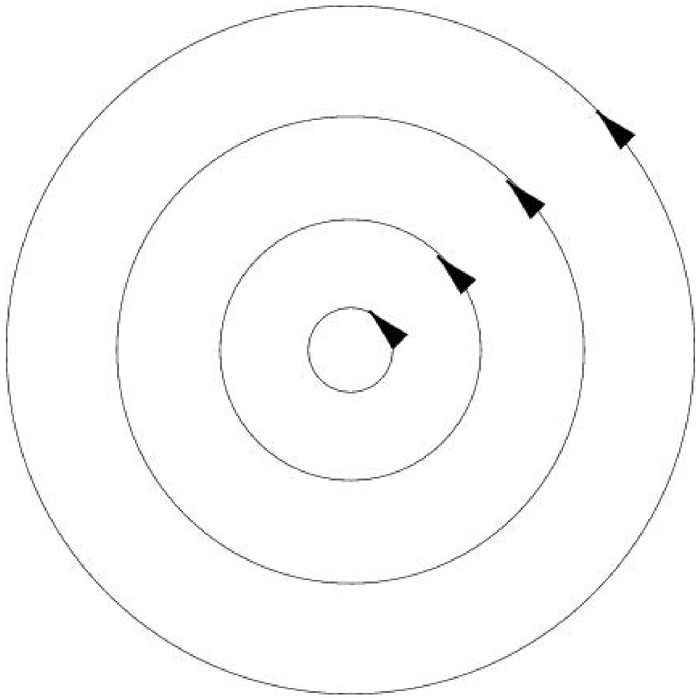
saddles: real eigenvalues, opposite signs eigenvectors: stable/unstable



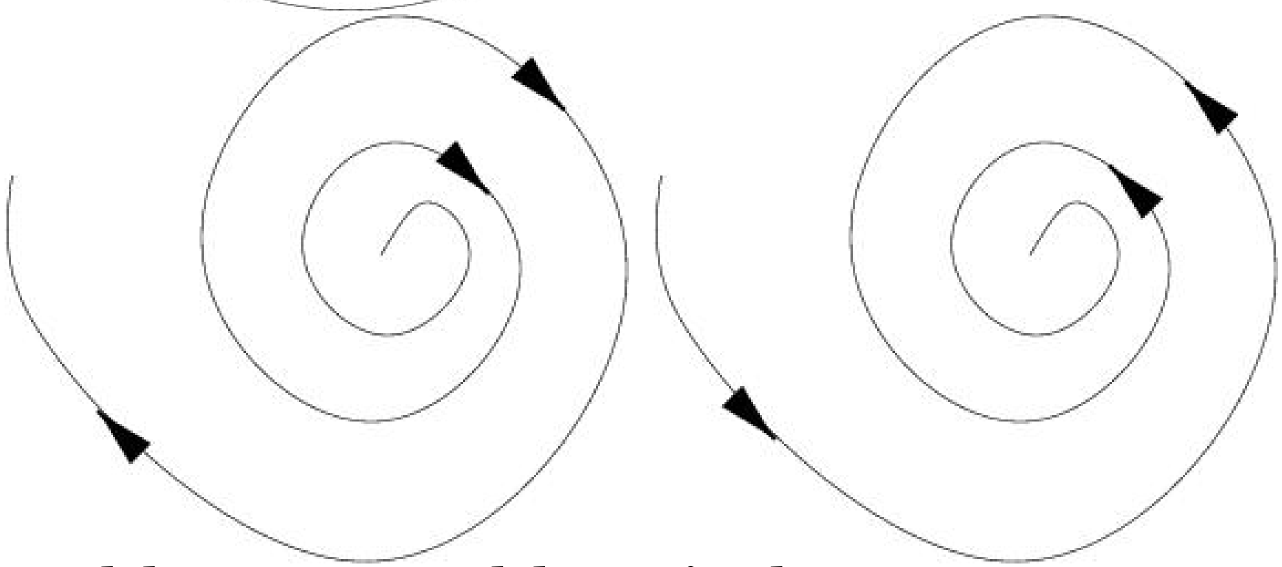


nodes: real eigenvalues but equal signs; eigenvectors give stable/unstable; trajs tangent to eigen closest to zero

Linear Phase Planes: Complex λ's



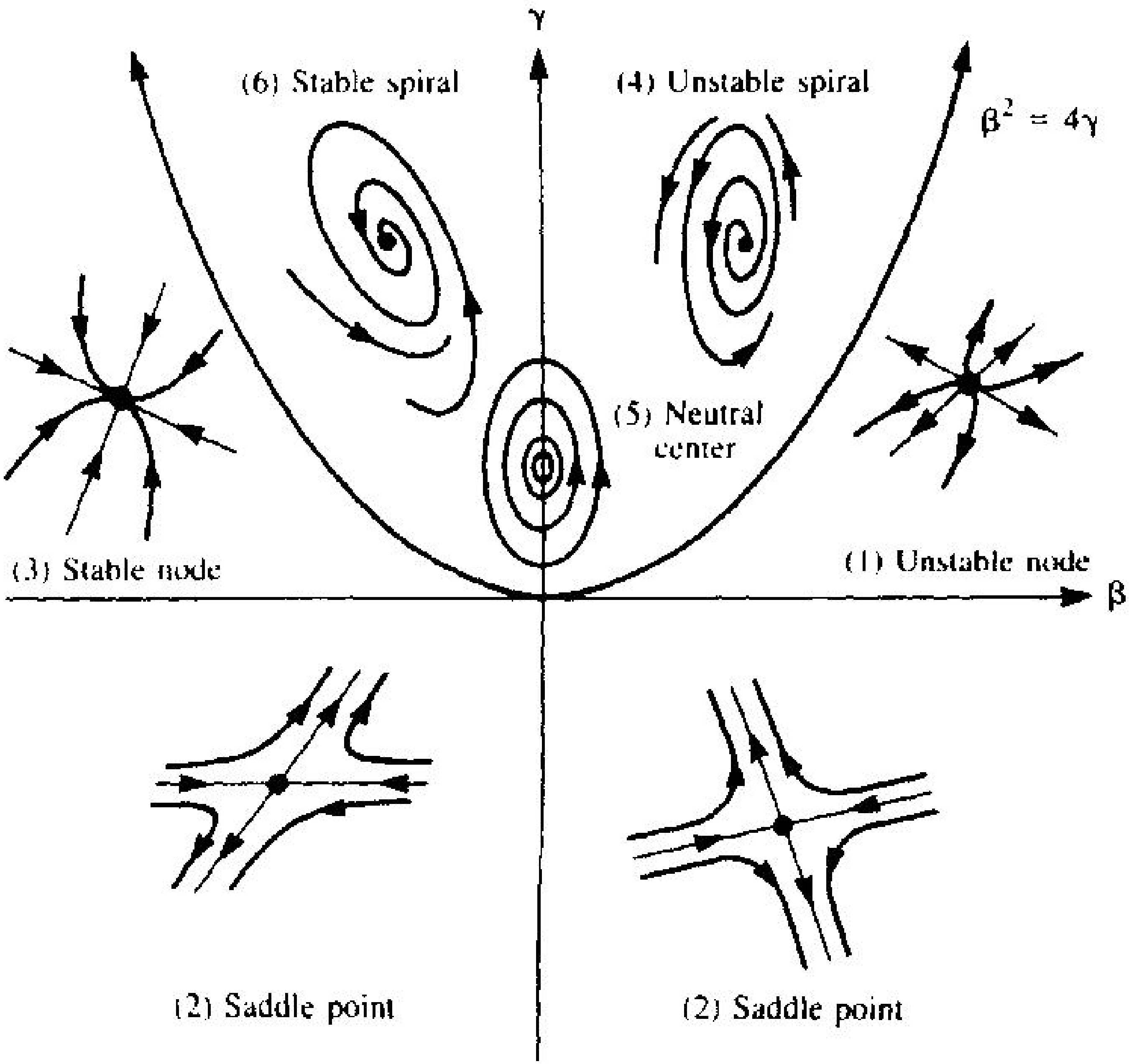
centers:
zero real part eigenvs
highly "non-robust"
$\rightsquigarrow$ "bifurcations"



stable or unstable spirals
stability depends on real part > 0 or < 0
orientation: easiest is to plot vector at e.g. $(1, 0)$

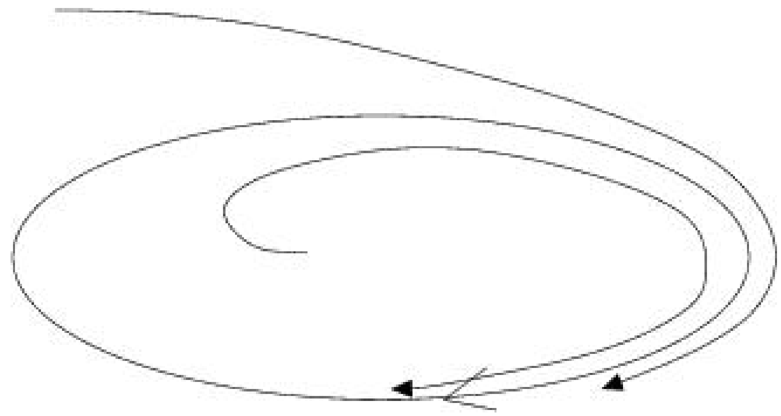
Trace/Determinant Plane

char poly is $\lambda^2 - \beta\lambda + \gamma$, where β = trace, γ = det

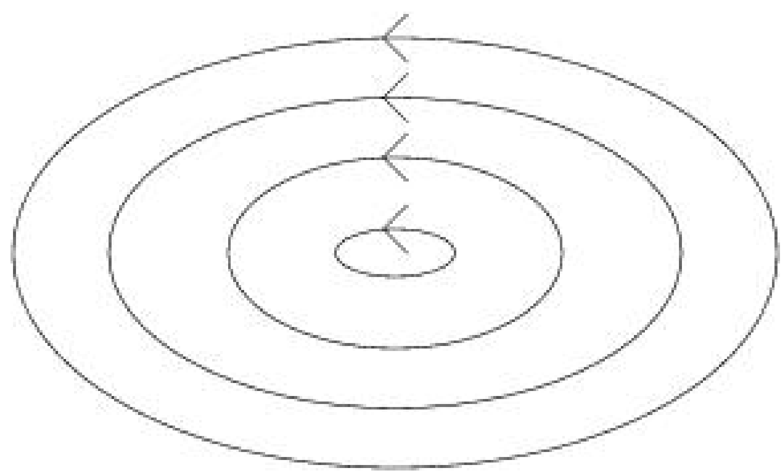


Periodic Orbits and Limit Cycles

(stable) limit cycle **:= a periodic trajectory which attracts other solutions to it:**



a member of a faily of "parallel" periodic solutions (as for linear centers) is *not* called a limit cycle



limit cyles robust in two ways (& linear periodic sols not):

Robustness of Limit Cycles

(1) if perturbation moves state to different initial state away from the cycle, system will return to cycle
e.g. circadian rythm: study late,
but later get back to normal pattern (⇝ jet lag!)

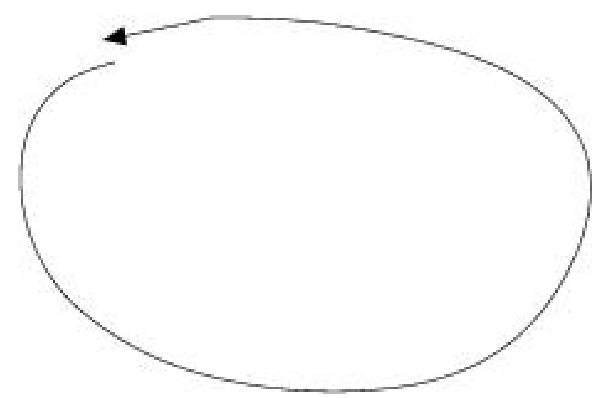
compare linear: will simply start oscillating along a different orbit, and never come back by itself; particular oscillation depends on initial conditions

(2) if dynamics changes a little, a limit cycle will still exist (can be proved as theorem)

compare linear: a small perturbation like

$$dx/dt = y, \quad dy/dt = -x + \varepsilon y$$

changes to spiral (stable or unstable)

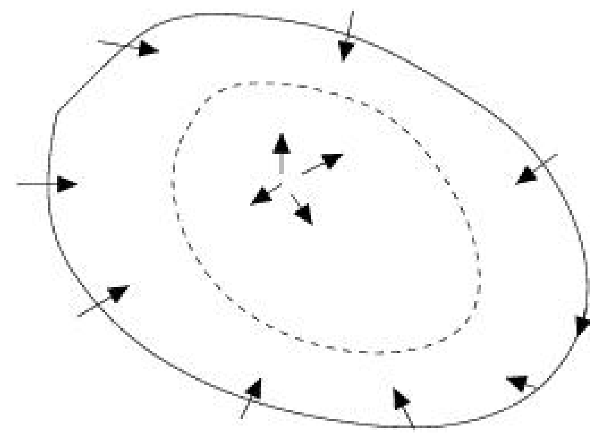


Poincaré-Bendixon Theorem

for systems of two equations, ∃ very powerful criterion

– we give a simple version sufficient for our purposes

suppose *a bounded region D in the plane is so that* no trajectories can exit *D* *[on ∂, v.f. points inside or tangentially]* *and either* ∄ no steady states *inside* *or* ∃ *unique steady state that is repelling* then *there is a periodic orbit inside D*



(also: if unique periodic orbit, then limit cycle)

idea: start near boundary: go towards inside,
cannot cross back, must keep going,
cannot approach source - must approach periodic
(proof in grad diff eq course!)

Example of Limit Cycle

consider this system (not biological - just math!)

$$\dot{x} = x + y - x(x^2 + y^2)$$
$$\dot{y} = -x + y - y(x^2 + y^2)$$

easier to understand in polar coordinates:

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

one obtains: $\dot{r} = r(1 - r^2)\,, \quad \dot{\varphi} = -1\,.$,
so $r = 1$ (unit circle) is limit cycle
since all trajectories rotate clockwise at unit speed
while point at distance r decreases towards 1 if > 1
or increases towards 1 if < 1
Poincaré-Bendixon: only equil: $(0, 0)$ unstable spiral
on circle $x^2 + y^2 = 2$: normal is (x, y), dot product:

$$[x + y - x(x^2 + y^2)]\,x + [-x + y - y(x^2 + y^2)]\,y$$
$$= (1 - (x^2 + y^2))(x^2 + y^2) < 0$$

so v.f. points inside $\Rightarrow \exists$ periodic orbit;
using a more subtle argument can prove limit cycle:
use annular regions $1 - \varepsilon < x^2 + y^2 < 1 + \varepsilon$ so unique

Bendixon's Criterion

given region D simply-connected (no holes)
if *the divergence of the vector field*
is always positive or is always negative inside D,
then *there cannot be a periodic orbit inside* D
Proof: suppose $\exists$; describes simple closed curve C,
recall divergence of $F(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ is: $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$
Gauss divergence theo (or Green's theorem) $\Rightarrow$

$$\int\int_D \operatorname{div} F(x,y)\,dxdy \;=\; \int_C \vec{n} \cdot F$$

(line integral of dot prod of outward normal with F)
along periodic *orbit*: F is tangent $\Rightarrow$ dot prod $= 0$
so integral of div is zero, $\therefore$ must change sign
$\dot{x} = x$, $\dot{y} = y$: div $= 2 \Rightarrow \nexists$ periodic orbits
$\dot{x} = x$, $\dot{y} = -y$: div $= 0$ (but $\nexists$)
$\dot{x} = y$, $\dot{y} = -x$: div $= 0$ (but $\exists$)