Rutgers 642:613 - Fall 2003

Instructor: Eduardo D. Sontag

Review: Trace/Determinant Plane, Linear Phase-Planes
Review: Period Orbits, Limit Cycles

http://www.math.rutgers.edu/~sontag/613.html
Linear Phase Planes: Real Eigens

saddles: real eigenvalues, opposite signs
eigenvectors: stable/unstable

nodes: real eigenvalues but equal signs; eigenvectors give stable/unstable; trajs tangent to eigen closest to zero
Linear Phase Planes: Complex $\lambda$’s

centers:
zero real part eigenvs
highly “non-robust”
$\leadsto$ “bifurcations”

stable or unstable spirals
stability depends on real part $> 0$ or $< 0$
orientation: easiest is to plot vector at e.g. $(1, 0)$
Trace/Determinant Plane
char poly is $\lambda^2 - \beta \lambda + \gamma$, where $\beta = \text{trace}$, $\gamma = \text{det}$
Periodic Orbits and Limit Cycles

(stable) limit cycle := a periodic trajectory which attracts other solutions to it:

A member of a family of “parallel” periodic solutions (as for linear centers) is not called a limit cycle.

Limit cycles robust in two ways (& linear periodic sols not):
Robustness of Limit Cycles

(1) if perturbation moves state to different initial state away from the cycle, system will return to cycle
e.g. circadian rhythm: study late, but later get back to normal pattern (\(\sim\) jet lag!)

compare linear: will simply start oscillating along a different orbit, and never come back by itself;
particular oscillation depends on initial conditions

(2) if dynamics changes a little, a limit cycle will still exist (can be proved as theorem)

compare linear: a small perturbation like
\[
\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + \varepsilon y
\]
changes to spiral (stable or unstable)
Poincaré-Bendixson Theorem
for systems of two equations, ∃ very powerful criterion

– we give a simple version sufficient for our purposes

suppose a bounded region $D$ in the plane is so that no trajectories can exit $D$
[on $\partial$, v.f. points inside or tangentially]
and either $\exists$ no steady states inside
or $\exists$ unique steady state that is repelling
then there is a periodic orbit inside $D$

(also: if unique periodic orbit, then limit cycle)

idea: start near boundary: go towards inside,
cannot cross back, must keep going,
cannot approach source - must approach periodic
(proof in grad diff eq course!)
Example of Limit Cycle
consider this system (not biological - just math!)
\[\begin{align*}
\dot{x} &= x + y - x(x^2 + y^2) \\
\dot{y} &= -x + y - y(x^2 + y^2)
\end{align*}\]
easier to understand in polar coordinates:
\[x = r \cos \varphi, \quad y = r \sin \varphi\]
one obtains: \(\dot{r} = r(1 - r^2)\), \(\dot{\varphi} = -1\),
so \(r = 1\) (unit circle) is limit cycle
since all trajectories rotate clockwise at unit speed
while point at distance \(r\) decreases towards 1 if \(> 1\)
or increases towards 1 if \(< 1\)

Poincaré-Bendixon: only equil: \((0, 0)\) unstable spiral
on circle \(x^2 + y^2 = 2\): normal is \((x, y)\), dot product:
\[\begin{align*}
[x + y - x(x^2 + y^2)] x + [-x + y - y(x^2 + y^2)] y &= (1 - (x^2 + y^2))(x^2 + y^2) < 0
\end{align*}\]
so v.f. points inside \(\Rightarrow\) \(\exists\) periodic orbit;
using a more subtle argument can prove limit cycle:
use annular regions \(1 - \varepsilon < x^2 + y^2 < 1 + \varepsilon\) so unique
Bendixon’s Criterion

given region $D$ simply-connected (no holes)
if the divergence of the vector field is always positive or is always negative inside $D$, then there cannot be a periodic orbit inside $D$
Proof: suppose $\exists$; describes simple closed curve $C$,
recall divergence of $F(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$ is: $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$
Gauss divergence theo (or Green’s theorem) $\Rightarrow$

$$\int \int_D \text{div } F(x, y) \, dx \, dy = \int_C \vec{n} \cdot F$$

(line integral of dot prod of outward normal with $F$) along periodic orbit: $F$ is tangent $\Rightarrow$ dot prod $= 0$
so integral of div is zero, $\therefore$ must change sign
$\dot{x} = x, \dot{y} = y$: div $= 2$ $\Rightarrow$ $\nexists$ periodic orbits
$\dot{x} = x, \dot{y} = -y$: div $= 0$ (but $\nexists$)
$\dot{x} = y, \dot{y} = -x$: div $= 0$ (but $\exists$)