Rutgers 642:613 - Fall 2003

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Chapter 9, Excitation Fronts & Waves

http://www.math.rutgers.edu/~sontag/613.html

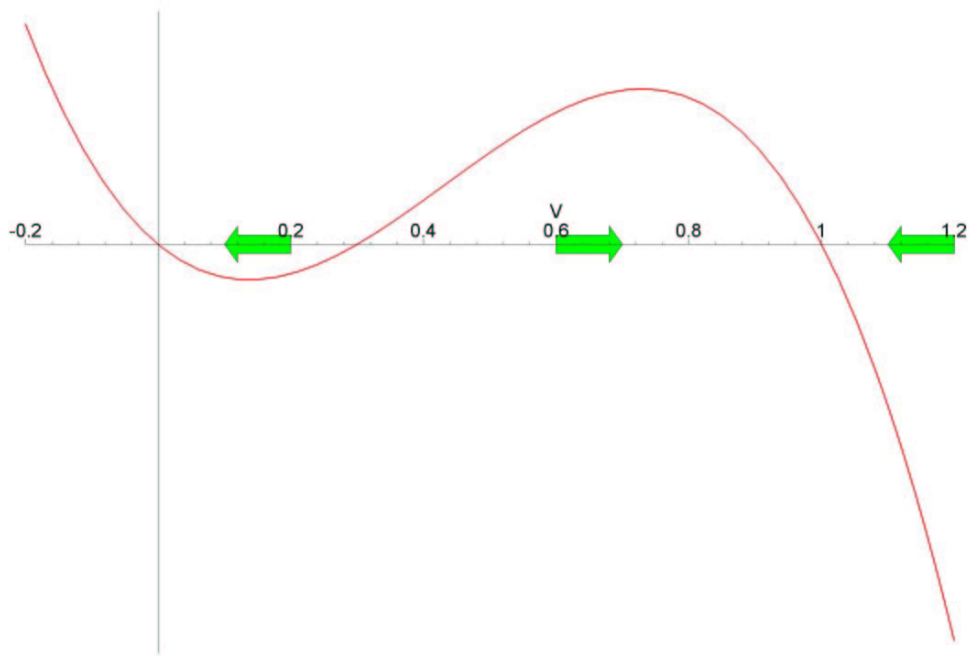
Travelling Waves for $cV_t = V_{xx} + f(V)$

we'll assume that f has zeroes at $0, \alpha, 1$, $0 < \alpha < 1$, and:

$$f'(0) < 0, \;\; f'(1) < 0, \;\; f'(\alpha) > 0$$

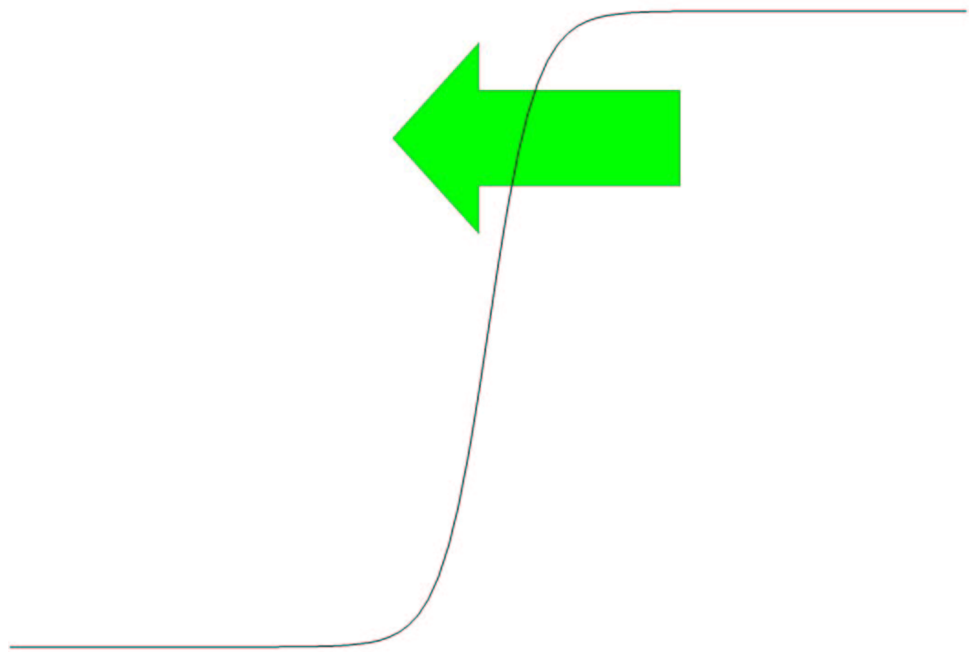
e.g.:

$$f(V) = V(V-1)(\alpha - V)$$



bistable system if no diffusion
leads to "excitable" system (threshold is α),
although with no "recovery" (since just one equation!)

we'd like to know if there's a solution like this:



i.e. front moving *left* (recall: no inactivation, stays up) this means that there is a solution $V(x,t)$ such that this picture holds for any fixed t, i.e. these conditions hold:

$$V(-\infty,t) = 0\,,\ V(+\infty,t) = 1$$

$$V_x(-\infty,t) = 0\,,\ V_x(+\infty,t) = 0$$

Reduction to ODE's

look for "waveform" $U(\xi)$ that "travels" at speed c:

$$V(x,t) = U(x+ct) = U(\xi)$$

so have this requirement (plug into $cV_t = V_{xx} + f(V)$):

$$cU' = U'' + f(U)$$

and $V(-\infty,t) = 0, V(+\infty,t) = 1, V_x(-\infty,t) = 0, V_x(+\infty,t) = 0$ means:

$$U(-\infty) = 0\,,\ U(+\infty) = 1\,,\ U'(-\infty) = 0\,,\ U'(+\infty) = 0$$

we'll only study in detail this very special case:
$f(V) = -A^2 V(V-\alpha)(V-1)$ (as in FHN model)
(could do other cubics —see book— but easier with this)

now, as $U' = 0$ when $U = 0, 1$, *guess* functional relation

$$U'(\xi) \ = \ -BU(\xi)\,(U(\xi)-1)$$

(minus sign just for convenience)

so, guessing $U' = -BU(U-1)$;
and eqn was $cU' = U'' - A^2(U-\alpha)(U-1) = 0$
substitute $U'' = B^2U(U-1)(2U-1)$ into eqn, and cancel $U(U-1) \Rightarrow$

$$B^2(2U-1) + cB - A^2(U-\alpha) = 0$$

this must hold $\forall\, U(\xi)$, so true for at least two values of $U(\xi)$ (unless U would be constant, which is not possible because $U(-\infty) = 0$ and $U(+\infty) = 1$)
so $2B^2 - A^2 = 0$, $-B^2 + cB + \alpha A^2 = 0$

$$\Rightarrow B = A/\sqrt{2}, \quad c = \frac{(1-2\alpha)A}{\sqrt{2}}$$

so back to:

$$U' = -BU(U-1) = -\frac{A}{\sqrt{2}}U(U-1)$$

solve this ODE with $U(-\infty) = 0$ by separation of variables and partial fractions, getting:

$$U(\xi) = \frac{1}{2}\left[1 + \tanh\left(\frac{A}{2\sqrt{2}}\xi\right)\right]$$

so $V(x,t) = U(x+ct)$ for this U and c

note that $c = \frac{(1-2\alpha)A}{\sqrt{2}} > 0$ if $\alpha \approx 0$,
i.e. "←" movement if α small
means activation increases:
intuitive because adjoining segment excites it

but if $\alpha \gg 0$ then not enough to excite, and wave moves backwards (everything becomes inactivated)

see book for stability: if $V(x,0)$ is more than α near $+\infty$ and less than α near $-\infty$, then solution asymptotic to above

General Case

look at (use "$'$" for $d/d\xi$):

$$U' = W$$
$$W' = -f(U) + cW$$

then Jacobian is

$$J = \begin{pmatrix} 0 & 1 \\ -f' & c \end{pmatrix}$$

and det$=f' < 0$, so saddles at $(0,0)$ and $(1,0)$

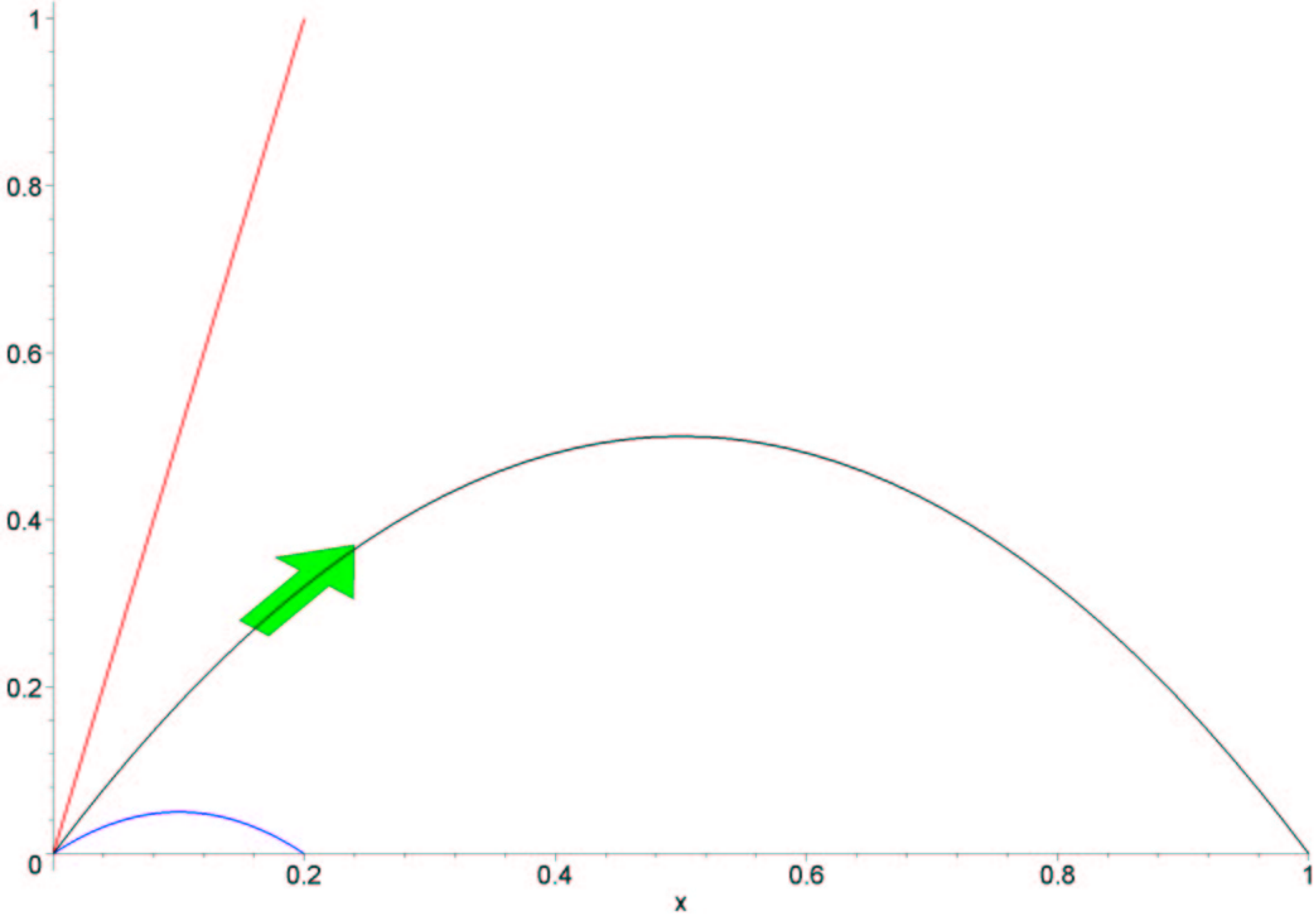
want: (U,W) heteroclinic connection between these:

$(U,W) \to (0,0)$ as $\xi \to -\infty$ and $(U,W) \to (1,0)$ as $\xi \to \infty$

Remark: even if did not ask that $W = U' \to 0$, this is automatically forced by the requirement that $U \to 0,1$. In general, $U \to$ limit does not imply $U' \to$ limit (e.g. $(1/t)(\sin e^t)$). But in this case yes. Proof: suppose that $|f(U(\xi))| < \varepsilon$ for all $\xi > X$. We claim that $|W(\xi)| < \varepsilon/c$ for all such ξ. Otherwise, say $cW(\xi) > \varepsilon$ for some such ξ. Then $W' = -f(U) + cW$ implies that $W'(\xi) > 0$, and hence $W(\xi') > \varepsilon/c$ for $\xi' > \xi$. This means that U diverges to $+\infty$. If instead $cW(\xi) < -\varepsilon$, we similarly get a contradiction.

Idea of Existence Proof (see book p.271 for details)

looking for heteroclinic orbit (black)



"shooting method": slope of black curve known from computing unstable and stable manifolds at both saddles; one shows (using only phase plane techniques and qualitative knowledge of f) that for $c \approx 0$ trajectories look like red curve, but for $c \gg 1$ they look like green (cannot stay in first quadrant); then a continuity argument implies that there is a c which gives exactly the desired connection

similar to Fisher equation: but there, it is a stable/saddle connection

Travelling Pulse

we saw cellular automaton (discrete) simulation of travelling pulses when inactivation is possible (also, spiral or scroll waves in 2-d)

let us sketch one approach to proving existence of travelling pulses for the continuous case (reaction-diffusion equation)

consider this model (now two equations):

$$\varepsilon \frac{\partial v}{\partial t} = \varepsilon^2 \frac{\partial^2 v}{\partial x^2} + f(v, w)$$
$$\frac{\partial w}{\partial t} = g(v, w)$$

comes from FHN-type equations

$$\varepsilon \dot{v} = f(v, w)$$
$$\dot{w} = g(v, w)$$

plus diffusion (term "ε^2" doesn't mean small diffusion – just rescale $x \longrightarrow \varepsilon x$, makes analysis easier) for voltage

one looks for solutions of the form $v(x - ct)$, $w(x - ct)$

equations for these are a 2-nd order ODE with a first order one, so equivalent to system of three equations

$$\varepsilon^2 v'' + cv' + f(v,w) = 0, \;\; cw' + g(v,w) = 0$$

very hard to analyze in general, but we sketch here one special case (see Sec 9.4) in which f is *piecewise linear*:

$$f(v,w) = -v - w \text{ if } v < \alpha \quad \text{ or } (1-v) - w \text{ if } v \geq \alpha$$

and $g(v,w) = v$

this f is motivated as follows:

first assume that $w \approx 0$ (inactivation channel not "on")
if we start at $v{\approx}0$ then $\dot{v}{<}0$, so back to $v = 0$
if we start at $v{>}\alpha$ then $\dot{v}{=}1 - v$, so $\longrightarrow$ excited state $v{=}1$
but then, with slower speed, w eventually becomes large
so $\dot{v}{=}{-w} - v < 0$ or $(1 - w) - v < 0$, and v goes back to 0

we look for a "pulse", meaning that
v is $\geq \alpha$ on some interval II, let us say $[0, \xi_1]$,
and $v < \alpha$ on $I = (-\infty, 0)$ and $III = (\xi_1, +\infty)$
(fig 9.5, p.283, in book)

equations on each of I, II, III are linear
(since both f, g are linear when away from $v = \alpha$)
and in fact the same equations except for a "+1" term in f
when in section II

eigenvalues (just look at matrix) are: one negative real λ_1
and two with real part positive λ_2, λ_3
so solution for v coordinate has general form

$$(1+) \sum_{i=1}^{3} a_i e^{\lambda_i \xi}$$

with different a_i's in each of I,II,III,
and with the +1 term in II but not in I,III

as $\lambda_2, \lambda_3 > 0$ and we want $v(+\infty) = 0$,
must have $a_2 = a_3 = 0$ in III (otherwise terms diverge)
similarly, $a_1 = 0$ on region I ($e^{\lambda_1 \xi} \to +\infty$ as $\xi \to -\infty$)
and anything is possible in the intermediate region

summary: have $1 + 3 + 2$ unknown parameters a_i,
plus c, ξ_1, total of 8 unknowns

and we have 8 equations:
the equations $v(0) = \alpha$, $v(\xi_1) = \alpha$
and 6 continuity constraints (for v, w, v' at $0, \xi_1$; note that we need continuity of v' since second order eqn for v)

see book for solving these (nonlinear) equations;
bottom line is for $\alpha < 1/2$ (the "easily excitable" case)
there are solutions
(for $\alpha \ll 1/2$ and $\varepsilon \ll 1$ there are *two* solutions, the slowest of which is unstable, cf. p.285)