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Ensuring Stability of State-dependent Riccati Equation Controllers Via Satisficing

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Abstract

Controls based on solutions to the state-dependent Riccati equation (SDRE) have been shown to offer high performance, but they suffer from unproven stability properties. This paper combines SDRE with satisficing, a novel clf-based approach which analytically guarantees stability. Essentially, the SDRE controller is projected point-wise onto the satisficing set. It is shown that this projection onto a stabilizing set in the control space can be solved analytically, and an example demonstrates the performance of the resulting SDRE-satisficing controllers.

1 Introduction

Linear quadratic regulator theory has been successfully applied to a variety of applications in the past decades, but it is practically restricted to linear or linearized systems, and this limits its usefulness. The State-dependent Riccati equation (SDRE) approach, first implemented by Cloutier in [2] utilizes a Riccati equation similar to the one solved in LQR, except that it is solved continuously and uses matrices that are derived from a factorization of a nonlinear system. Unfortunately, this state-dependent Riccati equation approach doesn't guarantee closed-loop stability (see [3] for a thorough treatment of controllability and stability issues).

Satisficing, on the other hand, is a recently introduced [5, 6] parameterization of smooth Lyapunov-based control laws which are guaranteed to asymptotically stabilize the closed-loop system. The satisficing technique is based on control Lyapunov functions [7, 8] (clfs), and can be understood as a generalization of Sontag's Formula [9] and Freeman and Kokotovic's min-norm approach [10].

Satisficing is based on a point-wise cost/benefit comparison [12] where benefits are defined in terms of the clf, and costs reflect a penalty on the control effort and the state. It was shown in [6] that clf-based satisficing can be modified to parameterize a large class of inverse-optimal [10, 14] controllers which always possess desirable gain margins.

The power of satisficing is that it provides the designer maximal flexibility in choosing a *particular* satisficing controller, while guaranteeing that any valid choice is stabilizing (or inverse-optimal). In this paper, the demonstrated performance of the SDRE approach is combined with the analytical properties of satisficing to produce a method of generating high-performance, inverse-optimal Lyapunov-based control laws.

2 State-dependent Riccati Equation Control

Consider the following affine nonlinear system with multiple inputs and a disturbance, i.e.,

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f(0) = 0$. We will assume throughout the paper that f and g are locally Lipschitz functions, and the goal will be to regulate the state x to the origin.

Motivated by the efficacy of the linear quadratic regulation results from linear optimal control theory, a factorization of system (1) was introduced ([2]) such that it appears linear at any fixed state:

$$f(x) \triangleq A(x)x \tag{2}$$

$$g(x) \triangleq B(x). \tag{3}$$

Viewed in this manner, control gains at any state x can be computed using standard linear optimal control theory, by solving the algebraic Riccati equation:

$$A^T P + PA + Q - PBR^{-1}B^T P = 0, \tag{4}$$

point-wise at every state, where $R^T(x) = R(x) > 0$ penalizes control effort, $Q(x) = Q^T(x) > 0$ penalizes the state, and $P(x)$ is the positive-definite symmetric solution of (4). If (A, Q) is observable and (A, B) controllable at every state, then this Riccati equation has a unique positive definite solution $P(x)$ and the control gains become $u = -K(x)x = -B^T(x)P(x)x$.

In general, the SDRE technique requires that (4) be solved at every state, whereas in the linear case it must only be solved once. Another difference is that in the linear case, a solution of (4) guarantees a stabilizing (and optimal) control law, whereas in the nonlinear case the asymptotic stability of the closed-loop system has not been proven. It should be noted, though, that extensive simulations (see [16] for a hardware experiment) support the idea that SDRE controls will indeed stabilize a large class of nonlinear systems.

3 Satisficing

Clf-based satisficing is a complete parameterization of asymptotically stabilizing control laws (with certain regularity requirements) given a valid clf for the closed loop system. It was shown in [6] that the satisficing parameterization is easily modified to generate inverse-optimal controllers.

In particular, a C^1 function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be a control Lyapunov function (clf) for system (1) if $V(x)$ is positive definite, radially unbounded, and if

$$\inf_u V_x^T(f + gu) < 0,$$

for all $x \neq 0$.

Clfs can be constructed by the technique of integrator backstepping, if the system dynamics have a cascade structure, but finding clfs for more general nonlinear systems is an open problem.

Definition 3.1 *The* Satisficing Set*, denoted $S(x)$ is a state-dependent control value set containing all the points in $\mathbb{R}^m$ which satisfy:*

$$S(x) \triangleq \left\{ u \in \mathbb{R}^m : \; -V_x^T(f + gu) > \frac{1}{b}(l + u^T R u) \right\}, \tag{5}$$

for some positive value of b and the state-dependent functions $l(x) > 0$ and $R(x) = R^T(x) > 0$ are design parameters.

Definition 3.2 *A* satisficing control, *$k : \mathbb{R}^n \to \mathbb{R}^m$, is a function with $k(0) = 0$ that is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and such that $k(x) \in S(x)$ for all non-zero x.*

Theorem 3.3 *([6]) If*

1. *V is a clf,*
2. *ν: $\mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies $\|\nu(x)\| < 1$,*
3. *b: $\mathbb{R}^n \to \mathbb{R}^+$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and satisfies $\underline{b}(x) < b(x)$,*

then

$$k(x) = \begin{cases} 0, & \text{if } x = 0 \\ -\sigma_1(x, b(x)) + \sigma_2(x, b(x))\nu(x), & \text{otherwise} \end{cases} \tag{6}$$

is a satisficing control, where

$$\sigma_1(x, b) \triangleq \frac{1}{2} b R^{-1} g^T V_x$$

$$\sigma_2(x, b) \triangleq R^{-1/2} \sqrt{\frac{1}{4} b^2 V_x^T g R^{-1} g^T V_x - l - b V_x^T f},$$

and where $\underline{b}$ is defined as

$$\underline{b}(x) \triangleq \begin{cases} \frac{l}{-V_x^T f}, & \text{if } V_x^T g = 0 \\ \frac{2V_x^T f + 2\sqrt{(V_x^T f)^2 + l V_x^T g R^{-1} g^T V_x}}{V_x^T g R^{-1} g^T V_x}, & \text{otherwise} \end{cases}.$$

Furthermore, $k(x)$ asymptotically stabilizes the closed loop system $\dot{x} = f(x) + g(x)k(x)$.

The significance of Theorem 3.3 is that asymptotic stabilization has been reduced to choosing two smooth selection functions $b(x) > \underline{b}(x)$ and $\nu(x) : \|\nu(x)\| \leq 1$.

4 Robust Satisficing

Robustness to disturbances at the input can be measured in many ways, but a straightforward approach is to quantify the amount of amplification (and diminution) a control signal can experience before resulting in instability. Toward this end, we define stability margins as follows.

Definition 4.1 *An asymptotically stabilizing control law, $u = q(x)$, has stability margins (m_1, m_2) where $-1 \leq m_1 < m_2 \leq \infty$, if for every $\alpha \in (m_1, m_2)$, $u = (1+\alpha)q(x)$, also asymptotically stabilizes the system.*

Such margins are important in any practical application, and in [17, 20] it is shown that optimal control laws have stability margins of $(-\frac{1}{2}, \infty)$. In fact, one of the primary motivations for considering inverse optimal control laws [10, 21] is that they have these desirable stability margins. It is shown in [21, p.108] that all clf-based control laws $u = -k(x, V_x)$ satisfying:

- u is of the form $-k(x, V_x) = -\frac{1}{2}R^{-1}(x)g^T V_x$, with $R(x) = R^T(x) > 0$,
- u has gain margins of $(-\frac{1}{2}, \infty)$,

are optimal with respect to the meaningful cost function:

$$J(x) = \int_0^\infty l(x) + k^T R(x) k,$$
$$\text{with } l(x) \triangleq -V_x^T f + \frac{1}{2} g^T V_x k.$$

This result was used in [6] to delineate a subset of S called the robust satisficing set.

Definition 4.2 *The* robust satisficing set, *denoted $S_R(x)$, is a state-dependent control value set defined as*

$$S_R(x) = \{k(x, b, \nu) \in S(x) : \nu^T g^T V_x = 0\}$$

Definition 4.3 *A* robustly satisficing control, *k_R, is a function with $k(0) = 0$ that is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and such that $k(x) \in S_R(x)$ for all non-zero x.*

Theorem 4.4 *If $k_R(x)$ is a robustly satisficing control then*

- *$k_R(x)$ has gain margins of $[-\frac{1}{2}, \infty)$*
- *$k_R(x)$ is inverse optimal.*

5 Projection onto the Satisficing Set

5.1 The Boundaries of S and S_R

In order to project a control value onto S at some fixed x it is necessary to know the boundary of S (denoted ∂S).

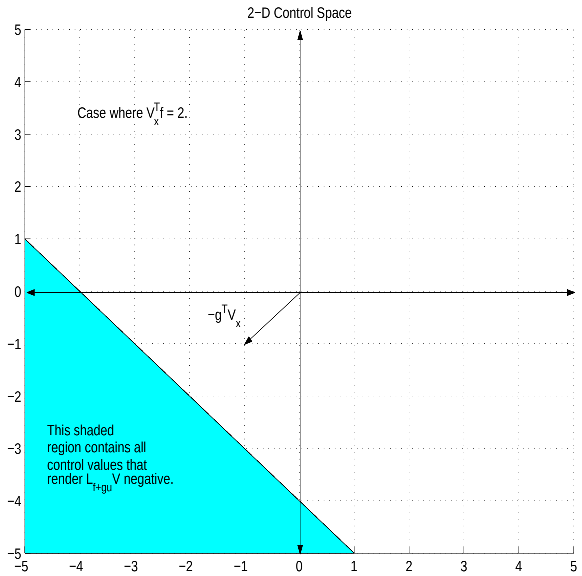

Figure 1: The Satisficing Set in a 2-D Control Space

The key to finding ∂S is to recognize that as the satisficing parameter b approaches infinity, the condition for membership in S becomes:

$$V_x^T f + V_x^T g u < 0.$$

The following theorem shows that ∂S is solely dependent on the vector $g^T V_x$.

Theorem 5.1 *An arbitrary control value $u = -\beta g^T V_x + \xi$ (with $\xi(x)$ orthogonal to $g^T V_x$) is in S if and only if*

$$\beta > \frac{V_x^T f}{V_x^T g g^T V_x}.$$

Proof:

$$\begin{aligned} & V_x^T f + V_x^T g u < 0 \\ \iff & V_x^T g u < -V_x^T f \\ \iff & -\beta V_x^T g g^T V_x < -V_x^T f \\ \iff & \beta > \frac{V_x^T f}{V_x^T g g^T V_x} \end{aligned}$$

■

The constraint on β (the part of u that is parallel to $-g^T V_x$) can be visualized geometrically as follows. The satisficing set is a region of the control space at every fixed x that contains all control values which render $\dot{V} = V_x^T f + V_x^T g u$ negative, and it is an open half space, bounded by a hyper-plane which lies perpendicular to the vector $-g^T V_x$.

The boundary of S_R (∂S_R) can be found through a similar analysis, only with the additionally constraint that an arbitrary control value (u_R) in S_R must satisfy: $\alpha u_R \in S$, where $\alpha \in [\frac{1}{2}, \infty)$.

Theorem 5.2 *An arbitrary control value* $u = -\beta g^T V_x + \xi$ *(with* $\xi(x)$ *orthogonal to* $g^T V_x$*) is in* S_R *if*

$$\beta > \max\left(0, \frac{2V_x^T f}{V_x^T g g^T V_x}\right).$$

Proof: First note that if $-u_R^T g^T V_x > 0$ (or $u_R = -\beta g^T V_x + \xi$ with $\beta > 0$) then it automatically has an infinite gain increase margin since $-\alpha u_R^T g^T V_x > 0$ for any $\alpha \geq 1$. Thus the requirement for an infinite gain increase margin for an arbitrary robust satisficing control reduces to $\beta \geq 0$. The requirement on u_R for a fifty percent gain reduction margin (assuming that $\beta \geq 0$ is already satisfied) can be found as follows:

$$\begin{aligned} & V_x^T f + \frac{1}{2} V_x^T g u_R < 0 \\ \iff & V_x^T g u_R < -2V_x^T f \\ \iff & -\beta V_x^T g g^T V_x < -2V_x^T f \\ \iff & \beta > \frac{2V_x^T f}{V_x^T g g^T V_x} \end{aligned}$$

Combining these requirements proves sufficiency.

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Thus ∂S_R can also be visualized (see Figure 2) as an open hyper-plane that is perpendicular to $-g^T V_x$, and which lies inside S (except when $g^T V_x = 0$ at which point $S = S_R \triangleq 0$).

5.2 Performing the Projection

Since the boundaries of S and S_R are hyper-planes, performing the actual projection onto these sets is relatively simple.

Theorem 5.3 *If* u_{SDRE} *is an arbitrary SDRE control value and* β *is defined as*

$$\beta \triangleq \frac{-u_{SDRE}^T g^T V_x}{V_x^T g g^T V_x},$$

then $\beta > \frac{V_x^T f}{V_x^T g g^T V_x}$ *implies* $u_{SDRE} \in S$. *Otherwise, the augmented control:*

$$\hat{u}_{SDRE} = u_{SDRE} - \left(\frac{V_x^T f}{V_x^T g g^T V_x} - \beta + \epsilon\right) g^T V_x,$$

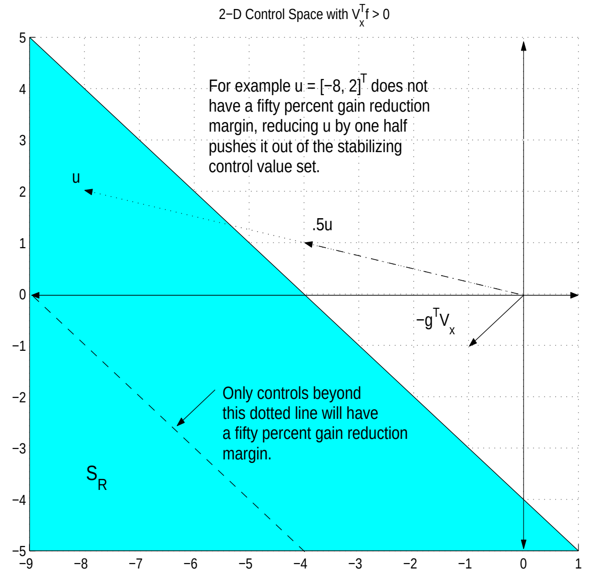

Figure 2: Gain Reduction Margins in the Control Space

where ϵ *is some very small number, is the projection of* u_{SDRE} *onto the satisficing set.*

Proof: The first statement follows directly from Theorem 5.1. The second follows from the Projection Theorem.

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Theorem 5.3 provides a constructive method for projecting an arbitrary SDRE control signal onto S (with some pre-determined error ϵ).

A similar operation can be performed (if robustness to input disturbances and inverse-optimality is desired) to project a control signal onto S_R.

Theorem 5.4 *If* u_{SDRE} *is an arbitrary SDRE control value and* β *is defined as*

$$\beta \triangleq \frac{-u_{SDRE}^T g^T V_x}{V_x^T g g^T V_x},$$

then $\beta > \max\left(0, \frac{2V_x^T f}{V_x^T g g^T V_x}\right)$ *implies* $u_{SDRE} \in S_R$. *Otherwise, the augmented control:*

$$\hat{u}_{SDRE} = u_{SDRE} - \left(\max\left(0, \frac{V_x^T f}{V_x^T g g^T V_x}\right) - \beta + \epsilon\right) g^T V_x,$$

where ϵ *is some very small number, is the projection of* u_{SDRE} *onto the robust satisficing set.*

Proof: The first statement follows directly from Theorem 5.2. The second follows from the Projection Theorem.

■

6 Example

We consider now the problem of regulating the attitude and angular velocities of a satellite to the origin (where the model used here is taken from [23]). Let $\omega \in \mathbb{R}^3$ be the angular velocities and let $\zeta \in \mathbb{R}^3$ be the Gibbs vector of angles. The equations of motion describing the satellite's attitude can be written as:

$$\begin{aligned} H\dot{\omega} &= p \times \omega + u \\ \dot{\zeta} &= Z(\zeta)\omega, \end{aligned}$$

where $H = H^T > 0$ is the inertia matrix, $u \in \mathbb{R}^3$ is the control vector of induced torques, $Z(\zeta) \triangleq \frac{1}{2}[I + \zeta\zeta^T + \zeta\times]$, $p = C(\zeta)p^I$ with p^I the constant angular momentum vector, and $C(\zeta) \triangleq 2(1+\zeta^T\zeta)^{-1}[I+\zeta\zeta^T - \zeta x] - I$. Note that $[p\times]$ denotes the vector product operation:

$$[p\times] \triangleq \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}.$$

We define the state x as $x \triangleq \begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix}$, and by differentiating we obtain the following state space representation:

$$\dot{x} = \begin{bmatrix} \dot{\zeta} \\ \ddot{\zeta} \end{bmatrix} = \begin{bmatrix} \dot{\zeta} \\ \left(\dot{Z}Z^{-1} + ZH^{-1}p \times Z^{-1}\right)\dot{\zeta} \end{bmatrix} + \begin{bmatrix} 0 \\ ZH^{-1} \end{bmatrix} u. \tag{7}$$

We also have a clf for system (7): $V(x) = \frac{1}{2}\left(\dot{x}^T Z^{-T} H Z^{-1}\dot{x} + x^T x\right)$. Note that though $\dot{V} \leq 0$ the invariance principle ensures asymptotic stability.

Equation (7) can be cast into SDRE form ($\dot{x} = A(x) + B(x)u$) by defining the state-dependent matrices $A(x)$ and $B(x)$ as follows:

$$\begin{aligned} A &\triangleq \begin{bmatrix} 0_3 & I_3 \\ 0_3 & \dot{Z}Z^{-1} + ZH^{-1}p \times Z^{-1} \end{bmatrix} \\ B &\triangleq \begin{bmatrix} 0_3 \\ ZH^{-1} \end{bmatrix} \end{aligned}$$

For our simulation, we chose the initial state as $x_0 = [-\pi/3,\ \pi/2,\ \pi/4,\ 1,\ -3,\ 2]^T$, and we set the inertia matrix to be

$$H = \begin{bmatrix} 2 & .5 & 1 \\ .5 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

An initial control, u_{SDRE}, was generated using the SDRE technique (with $R(x) = I$ and $Q(x) = .15\ \mathrm{diag}(1,\ 1,\ 1,\ 1,\ 1,\ 1)$: the SDRE control is obtained by solving the state-dependent Riccati equation for $P(x)$ at every state, and letting

$$u_{SDRE}(x) = -R^{-1}(x)B^T(x)P(x)x. \tag{8}$$

This control signal was minimally augmented at every x by the method described in Theorem 5.4 such that the resulting control law was a robustly satisficing control:

$$u_{Sat} = \arg\min_{u \in S_R(x)} \left\|u_{SDRE}(x) - u\right\|. \tag{9}$$

Figure 3 shows the system states, and the norm of the difference between the initial (SDRE) control and the augmented (satisficing) control. Notice that for most of the states $u_{SDRE} \in S_R$, but the projection onto S_R is non trivial at $t = .9$, 6.5, and 14.1 seconds. These results show that the performance benefits of the SDRE approach are retained while the analytical properties of satisficing have been added.

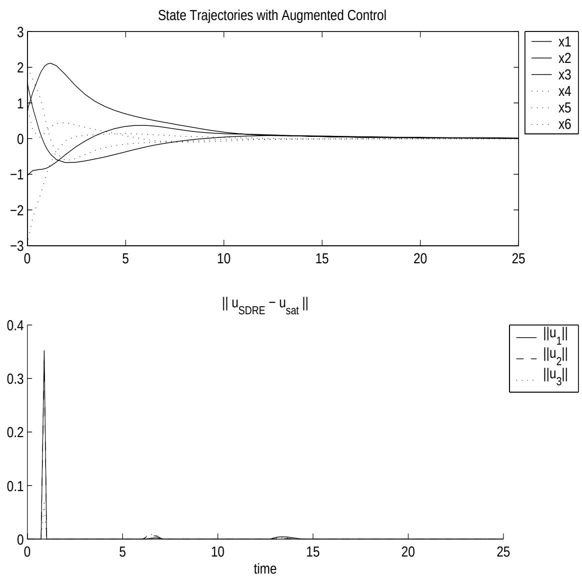

Figure 3: Augmented SDRE Spacecraft Attitude Control

7 Discussion and Conclusion

Lyapunov functions provide a powerful tool for analyzing the stability properties of nonlinear systems, and many formulas and techniques based on control Lyapunov functions have been proposed in the literature. Satisficing generates the state-dependent set of controls that render the closed-loop system stable (or inverse optimal) with respect to a known clf. By projecting an SDRE control point wise onto the satisficing set, the performance of an SDRE control is harnessed while guaranteeing desirable analytical properties. Because the satisficing set moves smoothly in x is is also clear that controls which are projections onto $S(x)$ will inherit this regularity.

Simulation of a non-trivial example provides a proof of concept. The simulation demonstrate that the SDRE approach can be conveniently combined with satisficing, and that the resulting controllers can inherit the performance of the SDRE strategy.

In the future, satisficing could be used to modify other control design techniques, such as model-predictive control, fuzzy controllers, neural networks, which do not have guaranteed stability properties. The resulting augmented controls would have guaranteed stability, optimality, and robustness properties.

Acknowledgments

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