Controllability of Dynamical Extensions with Bounded Control: Application to Robotics

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Abstract

In this paper, we consider dynamic extensions of controllable driftless systems, under bounded controls. Our main result is the proof that such systems are always controllable. An application to the motion planning problem for a class of two-driving-wheel mobile robots illustrates this property.

1 Introduction

The *motion planning problem* is one of the central problems in robotics. There is no general algorithm applicable to arbitrary nonlinear systems, and, moreover, most of the known controllability results are only applicable to the case when controls have no magnitude constraints. However, boundedness of controls is a practical requirement, as controls typically represent acceleration, temperature, and other physically constrained variables.

Typically, one describes robotics and other mechanical systems by providing a *configuration manifold* M, and a controlled differential equation $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)y_i$, where the vector fields f and g_i are assumed to be smooth. We may think of the vector y whose coordinates are the y_i's as a *velocity variable*, and x as a *configuration variable*. In this paper, we restrict ourselves to *driftless affine systems*, that is to say, we assume that f is identically zero. On the other hand, our object of study are *dynamic extensions* of these systems, in which accelerations (or forces) are the controls that can be manipulated (and are subject to magnitude constraints). In other words, the class of systems we consider here is of the form:

$$\dot{x}(t) = \sum_{i=1}^{n} g_i(x(t))y_i(t), \quad \dot{y}(t) = v(t)\,. \qquad (1)$$

These systems are defined on the state-space $M \times I\!R^m$ and v is the control. The state variable for this extended system is $z = (x, y)$. We will refer to $\dot{x}(t) = \sum_{i=1}^{n} g_i(x(t))u_i(t)$ as to the *original system* of the dynamic extension.

Let α_i, β_i be two collections of m real numbers satisfying $\alpha_i < 0 < \beta_i$ for each i. In the sequel, we assume the domain of control $\mathcal{U}$ to be:

$$\mathcal{U} = \{v \in I\!R^m;\ \alpha_i \leq v_i \leq \beta_i, i = 1, \cdots, m\}. \qquad (2)$$

An *admissible trajectory* for (1) is a trajectory z associated to a control v such that $v(t) \in \mathcal{U}$ for almost every t. The control v is called an *admissible control*.

Then, the controllability problem becomes: *given an initial and a final state, denoted respectively by z_0 and z_f, is there an admissible trajectory $z : [0, T] \to M \times I\!R^m$ such that $z(0) = z_0$ and $z(T) = z_f$?*

Of course, for the dynamic extension to have the controllability property, it is a necessary condition for the original system to be controllable. It turns out to that this is a sufficient condition as well. Sufficiency was proved in [9], when no bound on the control is assumed. More precisely, it is shown there that controllability for such a system can be achieved using smooth controls taking their values in $I\!R^m$. (The paper [8] extended this result to extensions of any controllable system, not just driftless systems, also under the assumption that the control space is $I\!R^m$.) Motivated by applications coming from robotics, we extend these results to the situation of bounded controls.

2 Controllability of dynamic extensions with bounded controls

Theorem 1 *The dynamic extension of a controllable driftless affine system is controllable with admissible controls.*

Proof: Le z_0 and z_f be two states in $M \times I\!R^m$. We have to show that there exists a control $v : [0, T] \to \mathcal{U}$ such that the corresponding solution of (1) satisfies $z(0) = z_0$ and $z(T) = z_f$.

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The first observation (clear by reversing time) is that a dynamic extension, as defined here, has the property that, if $v(t)$ is a control steering $z : [0,T] \to \mathcal{U}$ from $z(0) = \begin{pmatrix} x \\ y \end{pmatrix}$ to some state $z(T)$, then the control $\tilde{v}(t) = v(T-t)$ is an admissible control steering (1) from $z(T)$ to $\tilde{z}(0)$, where $\tilde{z}(0) = \begin{pmatrix} x \\ -y \end{pmatrix}$.

Thus, it is enough to prove that every state $z_0 \in M \times I\!R^m$ can be controlled to $z_f = 0$. (Since then, to go from an arbitrary z_0 to an arbitrary $z_f = \begin{pmatrix} x \\ y \end{pmatrix}$, we first steer z_0 to 0; then we find a control steering $\begin{pmatrix} x \\ -y \end{pmatrix}$ to 0 and reverse this control, producing one steering 0 to z_f; finally, we concatenate the two resulting controls.)

So we prove the result on controllability to the origin. If $z_0 = (x_0, y_0)$, due to its from it is obvious that there exists a constant admissible control $\hat{v}$ steering the system (1) from (x_0, y_0) to $(\hat{x}_0, 0)$ where $\hat{x}_0$ is some configuration variable. Indeed, if we choose $\hat{T}$ big enough such that $\alpha_i < -\frac{y_i(0)}{\hat{T}} < \beta_i$ for all i and define $v_i(t) = -\frac{y_i(0)}{\hat{T}}$, then we have an admissible constant control such that $y_i(\hat{T}) = 0$, $\forall i$. We now prove that there exists an admissible control $\tilde{v}$ steering the dynamic extension from $(\hat{x}_0, 0)$ to $(0,0)$. By [9], there exists $\tilde{v}^* : [0,T] \to I\!R^m$ a smooth control such that the corresponding trajectory z satisfies $z(0) = (\hat{x}_0, 0)$ and $z(T) = (0,0)$. Using a time reparametrization we can transform $\tilde{v}^*$ into an admissible control. Indeed, let $\varepsilon > 0$ and introduce the new variables $x^\varepsilon(t) = x(\varepsilon t), y^\varepsilon(t) = \varepsilon y(\varepsilon t)$. Then, it is an easy verification that the trajectory $z^\varepsilon = (x^\varepsilon, y^\varepsilon)$ defined on $[0, \frac{T}{\varepsilon}]$ is solution of (1) with control $\tilde{v}(t) = \varepsilon^2 \tilde{v}^*(\varepsilon t)$. As $\tilde{v}^*$ is a smooth function, it is bounded on $[0,T]$ and we can choose ε such that $\tilde{v}(t) \in \mathcal{U}$ for all t. Moreover, we have $z^\varepsilon(0) = (\hat{x}_0, 0)$ and $z^\varepsilon(T/\varepsilon) = (0,0)$. Finally, if we define v as the concatenation of $\hat{v}$ and $\tilde{v}$ we have an admissible control steering (1) from z_0 to 0, as desired.

We remark that using the smooth function φ introduced in [9] the control $\hat{v}$, instead of being a constant control, can be taken such that the concatenation between $\hat{v}$ and $\tilde{v}$ is smooth.

In a forthcoming article [1], we extend the results presented here to a more general class of dynamic extensions. In [1], we define an *fH-dynamic extension* as a system of the form:

$$\dot{x}(t) = \sum_{i=1}^{n} g_i(x(t)) y_i(t)$$

$$\dot{y}(t) = f(x(t), y(t)) + H(x(t)) v(t) \qquad (3)$$

where $f : M \times I\!R^m \to I\!R^m$ is a smooth function and H is a $m \times m$ invertible matrix whose column vectors are smooth linearly independent functions. Our motivation comes from the fact that controlled mechanical systems belong to the class of fH-dynamic extension. We prove for instance that an fH-dynamic extension whose function f is quadratic with respect to the velocity variables is *equilibrium controllable* (meaning that for any given pair of configurations x_1, x_2 there exists a control steering the system from $(x_1, 0)$ to $(x_2, 0)$) with admissible controls if and only if its original system is controllable.

In this paper our goal is to solve the controllability problem for a classical example coming from robotics and to illustrate our results by computing, for this example, some trajectories obtained from our algorithm. It is why we restrict to systems of the form (1)

3 Application

The initial motivation for our work was to establish the controllability of a two-driving-wheel mobile robot. The motion planning problem for nonholonomic systems has been a very active research field for the past two decades, and the most popular examples of nonholonomic systems arise from robotics.

An important class of mobile robot is the one formed by mobile robots with two parallel driving wheels whose accelerations are controlled by independent motors. In particular, at the Laboratory for Analysis and Architecture of Systems, real experiments are conducted on such a robot (named Hilare, see Figure 1) on which a trailer has been hooked up, see [6].

Figure 1 - Hilare

Hilare's system has been extensively studied (see for instance [2, 3, 4, 5]) and as mentioned before is now the platform to run indoor experiments of motion planning in a cluttered environment. Despite this huge interest for such a mobile robot, no proof of its controllability using admissible controls appears to have been given in the literature. The result in this paper shows that, indeed, as it was thought, *a two-driving-wheel mobile robot is controllable with bounded controls.*

Let us now describe the dynamics of the system we consider. The configuration of the vehicle is $I\!\!R^2 \times S^1$ where $I\!\!R^2$ describes its position and S^1 its orientation. With respect to a fixed frame, we denote by (x, y) the position variable (the reference point being the midpoint of the two wheels) and by θ the orientation of the two driving wheels with respect to the x-axis, see Figure 2. Then, if we denote by d the distance between the two wheels and by v_l, v_r the respective velocity of the left and right wheels, we get:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v}_r \\ \dot{v}_l \end{pmatrix} = \begin{pmatrix} w_1 \cos\theta \\ w_1 \sin\theta \\ w_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2 \tag{4}$$

where $w_1 = \frac{1}{2}(v_l + v_r)$ and $w_2 = \frac{1}{d}(v_r - v_l)$. As u_1 and u_2 represent the respective acceleration for the left and right wheel, they must satisfy constraints of the form: $|u_i| \leq \alpha_i$ where α_i is a positive real number for $i = 1, 2$.

The system (4) is a dynamic extension as defined previously in this paper. The configuration variables are x, y, θ and the velocity variables are v_1, v_2. The vector fields g_1 and g_2 are given by

$$g_1 = \begin{pmatrix} \frac{\cos\theta}{2} \\ \frac{\sin\theta}{2} \\ 1 \end{pmatrix}, \; g_2 = \begin{pmatrix} \frac{\cos\theta}{2} \\ \frac{\sin\theta}{2} \\ -1 \end{pmatrix}. \tag{5}$$

The rank condition for controllability is a necessary and sufficient condition for driftless analytic systems (see e.g. [7]); hence, to verify that the original system corresponding to (4) is controllable we simply have to compute the rank of the Lie algebra generated by g_1 and g_2. It is an easy verification that

$$[g_1, g_2] = \begin{pmatrix} -\frac{\sin\theta}{2} \\ \frac{\cos\theta}{2} \\ 0 \end{pmatrix}. \tag{6}$$

The Lie brackets $g_1.g_2, [g_1, g_2]$ are linearly independent everywhere, hence the original system is controllable and it follows from Theorem 1 that the system describing the equations of motion of a two-driving -heel mobile robot is controllable using admissible controls. We summarize our result in the next theorem.

Theorem 2 *A two-driving-wheel mobile robot whose equations of motion are described by (4) is controllable with bounded controls.*

On Figures 3 and 4, we represent the two components of the control of a trajectory whose computation is based on the algorithm described in [9] and on the proof of Theorem 1. Along the computations, we assumed for simplicity that $d = 1$ in equation (4). The function φ we used to smooth our controls is:

$$\varphi(t) = \begin{cases} 0 & x \leq -1 \\ \dfrac{e^{-\frac{4}{(3x+3)^2}}}{e^{-\frac{4}{(3x+3)^2}} e^{-\frac{4}{(3x+1)^2}}} & -1 < x < -\frac{1}{3} \\ 1 & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ \dfrac{e^{-\frac{4}{(3x-3)^2}}}{e^{-\frac{4}{(3x-3)^2}} e^{-\frac{4}{(3x-1)^2}}} & \frac{1}{3} < x < 1 \\ 0 & x \geq 1\,. \end{cases}$$

The trajectory steers the system from the origin to a nonzero configuration with nonzero velocities. Following our algorithm, the trajectory is divided into two pieces: on the time interval $[0, 12]$, the system moves from the origin to a nonzero configuration with zero velocities and then on the interval $[12, 13]$ it moves from the nonzero configuration at rest to another nonzero configuration but with nonzero velocities (more precisely, $v_1 = 0.3137, v_2 = 0.1569$)). We remark that for the time reparametrization used along the portion of the trajectory linking the two positions at rest, we took $\varepsilon = \frac{1}{6}$ to make sure our control is an admissible one. Moreover, we also used the function φ on the constant control for the first part of the trajectory steering the system from its initial position to a position at rest, in order to get at the end a smooth control along the whole trajectory.

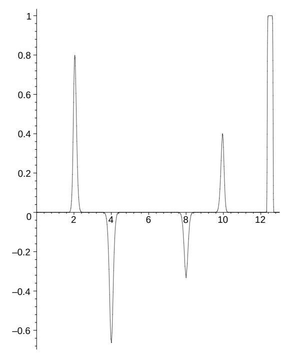

Figure 3: component u_1

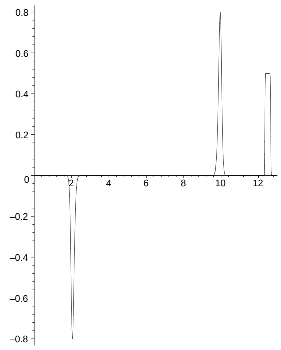

Figure 4: component u_2

For this specific example, there is an alternative way to compute the trajectories. It presents advantages and disadvantages with respect to the method which we just described. An advantage is that, given initial and final states, we can produce explicitly the path linking them, which was not the case with the previous algorithm. Indeed, the proof of our Theorem gives the existence of a path but is not a constructive proof (the implicit function theorem is involved in [9]). But, from a practical point of view, it has the disadvantage is that the computed path corresponds to a piecewise constant control (instantaneous changes in the acceleration are not possible to realize for the robot). Moreover, the following method cannot be generalized to arbitrary dynamic extension, and thus its applicability is limited.

Let us denote by $(x^0, y^0, \theta^0, v_r^0, v_l^0)$ and $(x^f, y^f, \theta^f, v_r^f, v_l^f)$ the given initial and final states. The first step consist to steer the initial state to a configuration at rest. As mentioned before, this can be done using admissible constant controls. Indeed, we have by definition that along a path

$$v_r(t) = \int_0^t u_1(s)ds + v_r(0),\ v_l(t) = \int_0^t u_2(s)ds + v_l(0).$$

Using that $v_r(0) = v_r^0, v_l(0) = v_l^0$ and that we want $v_r(T) = v_l(T) = 0$, we get the following two equations

$$\int_0^T u_1(s)ds = -v_r^0,\ \int_0^T u_2(s)ds = -v_l^0.$$

Assume T is fixed, then we can choose u_1 to be the constant $-\frac{v_r^0}{T}$ and $u_2 = -\frac{v_l^0}{T}$. Finally, if T is taken big enough, both components of the control can satisfy the constraints: $|u_i(t)| \leq \alpha_i$. Due to the simplicity of the chosen control, we can compute exactly the reached configuration. Indeed, as $v_r(t) = -\frac{v_r^0}{T}t + v_r^0$, $v_l(t) = -\frac{v_l^0}{T}t + v_l^0$, if we denote $\beta = v_r^0 - v_l^0$, $\gamma = v_r^0 + v_l^0$, easy integrations show that

$$\theta(t) = -\frac{\beta}{2T}t^2 + \beta t + \theta^0,$$

$$x(t) = \frac{\gamma}{2\beta}\sin(-\frac{\beta}{2T}t^2 + \beta t + \theta^0) - \frac{\gamma}{2\beta}\sin\theta^0 + x_0,$$

$$y(t) = -\frac{\gamma}{2\beta}\cos(-\frac{\beta}{2T}t^2 + \beta t + \theta^0) + \frac{\gamma}{2\beta}\cos\theta^0 + y_0.$$

Then, replacing t by T in the formulas above provides $\theta(T), x(T), y(T)$. At this point, we need to convert the real value obtained for $\theta(T)$ to a positive angle $\in [0, 2\pi[$ that we denote θ_1. Then, the configuration we reach at rest is (x_1, y_1, θ_1) where $x_1 = x(T)$ and $y_1 = y(T)$. The second step of our path planning is to do the exact same computations as above, but starting with the values $(x^f, y^f, \theta^f, v_r^f, v_l^f)$. The system then reach a configuration at rest that we denote (x_2, y_2, θ_2). The final step is to find an admissible control to steer the mobile robot from $(x_1, y_1, \theta_1, 0, 0)$ to $(x_2, y_2, \theta_2, 0, 0)$. Such a path can be realized as the concatenation of the following basic motions: *pure rotation* and *pure translation*. More precisely, let us denote by $\hat{\theta}$ the angle formed by the line in the plane (x, y) linking (x_1, y_1) and (x_2, y_2) and the x-axis. More precisely, if $m = \frac{y_2 - y_1}{x_2 - x_1}$, we have

$$\hat{\theta} = \begin{cases} \arctan m & \text{if } m \geq 0 \text{ and } y_2 \geq y_1 \\ \frac{\pi}{2} + \arctan|m| & \text{if } m \leq 0 \text{ and } y_2 \geq y_1 \\ \pi + \arctan m & \text{if } m \geq 0 \text{ and } y_2 \leq y_1 \\ 2\pi - \arctan|m| & \text{if } m \leq 0 \text{ and } y_2 \leq y_1 \end{cases}$$

To insure that we get an admissible control, let us introduce a new variable δ by $\delta = \min\{\alpha_1, \alpha_2\}$. Then, the control u defined by

$$u(t) = \begin{cases} (-\delta, +\delta) & \text{for } t \in [0, \sqrt{\frac{|\theta_1 - \hat{\theta}|}{2\delta}}] \\ (+\delta, -\delta) & \text{for } t \in [\sqrt{\frac{|\theta_1 - \hat{\theta}|}{2\delta}}, 2\sqrt{\frac{|\theta_1 - \hat{\theta}|}{2\delta}}] \end{cases}$$

if $\theta_1 - \hat{\theta} \geq 0$ (respectively $(+\delta, -\delta)$ and then $(+\delta, -\delta)$ if $\theta_1 - \hat{\theta} \leq 0$) steers system (4) from $(x_1, y_1, \theta_1, 0, 0)$ to $(x_1, y_1, \hat{\theta}, 0, 0)$ and is admissible. To go from $(x_1, y_1, \hat{\theta}, 0, 0)$ to $(x_2, y_2, \hat{\theta}, 0, 0)$ the robot can move along the straight line whose equation is given by $y - y_1 = m(x - x_1)$, m being as above, using the control

$$u(t) = \begin{cases} (+\delta, +\delta) & \text{for } t \in [0, \sqrt{\frac{x_2 - x_1}{2\cos\hat{\theta}}}] \\ (-\delta, -\delta) & \text{for } t \in [\sqrt{\frac{x_2 - x_1}{2\cos\hat{\theta}}}, 2\sqrt{\frac{x_2 - x_1}{2\cos\hat{\theta}}}]. \end{cases}$$

At the end, a pure rotation between $\hat{\theta}$ and θ_2 as previously will bring the robot to the desired state $(x_2, y_2, \theta_2, 0, 0)$. To summarize, by concatenating the three admissible controls we have computed, we have a trajectory steering the system from $(x^0, y^0, \theta^0, v_r^0, v_l^0)$

and $(x^f, y^f, \theta^f, v_r^f, v_l^f)$. We remark that this trajectory corresponds to a piecewise constant control. Figure 5 displays the projection in the plane (x, y) of such a path for the initial and final state values: $(1, 2, 0.5, -2, 1)$, $(-1, -3, -2.4, 2, 3)$, while the corresponding admissible piecewise constant controls are represented on Figures 6 and 7.

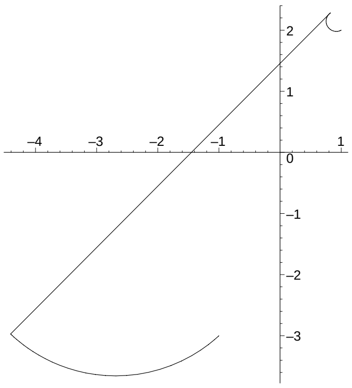

Figure 5

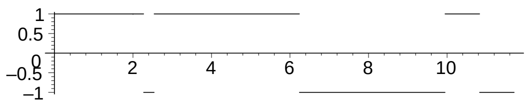

Figure 6: component u_1

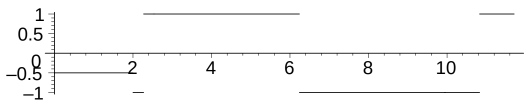

Figure 7: component u_2

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