ABSTRACT

This paper describes how notions of input-to-state stabilization are useful when stabilizing cascades of systems.

1 Introduction

Consider a cascade as follows:

$$\text{(CAS)} \quad \begin{cases} \dot{x} &= f(x,y) \\ \dot{y} &= g(y,u) \end{cases}$$

where f and g are smooth, x and y evolve in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and $f(0,0) = g(0,0) = 0$. The input u takes values in $\mathbb{R}^k$. It is natural to ask: If the system $\dot{x} = f(x,y)$ is stabilizable (with y thought of as a control) and the same is true for $\dot{y} = g(y,u)$, what can one conclude about the cascade? More particularly, what can be said if the "zero-input" system

$$\dot{x} = f(x,0) \tag{1}$$

is already known to be asymptotically stable? There are many reasons for studying these problems; see the tutorial paper [7] for motivations and references.

The simplest result along these lines is local, and it states that a cascade of locally asymptotically stable systems is again asystable. More precisely, if (1) has the origin as a locally asystable point, and if in (CAS) the second equation is independent of u, $g = g(y)$, and

$$\dot{y} = g(y) \tag{2}$$

also has 0 as locally asystable, then the same is true for $(0,0)$ in (CAS). This follows from classical "total stability" theorems, and was proved for instance in [11] and in a somewhat different manner in [6] using Lyapunov techniques. (An elementary proof is indicated below.) The local stabilization of (CAS) can then be achieved if (1) is already stable and the y-subsystem of (CAS) is stabilizable.

It is easy to see that the same result is not true for *global* stability, since there is no reason for the first system in (CAS), seen as a system with input y, now denoted "u":

$$\dot{x} = f(x,u) \tag{3}$$

to satisfy the following "converging input converging state" condition:

CICS: For each control $u(\cdot)$ on $[0,+\infty)$ such that $\lim_{t\to\infty} u(t) = 0$ and for each initial state x_0, the solution of (3) with $x(0) = x_0$ exists for all $t \geq 0$ and converges to 0.

This research was supported in part by US Air Force Grant AFOSR-88-0235.

"Control" means any measurable function into $\mathbb{R}^m$, though for our application just smooth controls would be enough. Clearly if CICS holds then $(0,0)$ is a global attractor for the cascade, so together with the local result one can conclude global asymptotic stability.

For instance, consider the one-dimensional system

$$\dot{x} = -x + (x^2 + 1)u$$

which is globally asymptotically stable when $u \equiv 0$. With

$$u(t) = \frac{1}{\sqrt{2t+2}}, \quad x_0 = \sqrt{2}$$

there results the unbounded trajectory $x(t) = \sqrt{2t+2}$. This is not at all a "pathological" example; it is in fact one of the simplest systems that are linearizable by feedback. This example does not even satisfy the following weaker "converging input bounded state" property:

CIBS: For each control $u(\cdot)$ on $[0, +\infty)$ such that $\lim_{t\to\infty} u(t) = 0$ and for each initial state x_0, the solution of (3) with $x(0) = x_0$ exists for all $t \geq 0$ and is bounded.

Nor, of course, does it satisfy the stronger property that the system be "bounded-input/bounded-output" or "BIBS" stable, where the boundedness property is required to hold for each bounded control u.

Interestingly, property CIBS is the only obstruction. If it is assumed, there is indeed a global version:

Theorem. Assume that both (1) and (2) have the origin as a globally asymptotically stable state and that property CIBS holds. Then the cascade (CAS) has $(0,0)$ as a globally asymptotically stable state.

Thus, if one can in any way insure that solutions of the first system do not blow up for small controls, then the cascade's global asystability follows from that of its components. This was proved in [6] using a stronger property than CIBS, "input to state stability (ISS)" (see [5] for the definition of the concept of ISS). Comparisons between ISS and BIBS stability are discussed in [6]. Here we show that the same result follows under the weaker hypothesis. The proof is basically the same as in that reference, but it seems appropriate to provide a separate proof that does not involve the terminology used there ("functions of class $\mathcal{KL}$") and the use of Lyapunov inverse theorems, but only elementary notions. Observe also that under extra (and fairly restrictive) hypotheses on the first system, such as that f be globally Lipschitz, the CIBS conditions can be relaxed –the paper [3] provides a detailed discussion of this issue, which was previously considered in [10] and [4].

If the system (3) does not satisfy the CIBS property, it can be modified under feedback so that this becomes true. We will discuss that result and its implications for the stabilization of (CAS) in the last section.

2 Proof of the Theorem

It will be enough to establish the following lemma.

Proposition. If 0 is a globally asymptotically stable state for (1) then CIBS and CICS are equivalent for (3).

Proof. We will denote by single bars "$|\cdot|$" norms on $\mathbb{R}^n$ and $\mathbb{R}^m$, and we will use double bars "$\|\cdot\|$" for the sup norm on the spaces of controls (essentially bounded measurable functions from intervals into $\mathbb{R}^m$) and state trajectories (absolutely continuous functions into $\mathbb{R}^n$). We also let $\phi(\cdot, x_0, u)$, or just $x(\cdot)$ if x_0 and u are clear from the context, be the solution of (3) with control $u(\cdot)$ and initial condition x_0. Note that the CIBS property insures that this solution is defined for all $t \geq 0$, because on each finite interval $[0, T]$ it coincides with the solution for a control that converges to zero (namely, truncate $u(\cdot)$ at T, and continue with zero).

Assuming CIBS, we first show that for each $\varepsilon > 0$ there are some $\delta_0, \mu > 0$ with the following property:

(P) If $|x_0| < \delta_0$ and if $\|u\| < \mu$ is a control on $[0, +\infty)$ then $|x(t)| < \varepsilon$ for all $t \geq 0$.

This will be needed later, but it is also of interest in that it trivially implies the local stability theorem for cascades mentioned in the introduction.

Take any $\varepsilon > 0$. From the assumption that (1) is (at least locally) asymptotically stable, there are some $T, \delta_0 > 0$ so that

$$\xi \in \overline{B}_{\delta_0} \;\Rightarrow\; |\phi(t,\xi,0)| < \varepsilon/2 \;\; \forall t \in [0,T] \text{ and } |\phi(T,\xi,0)| < \delta_0/2.$$

Necessarily, $\delta_0 < \varepsilon$. Consider the map

$$\alpha_T : \overline{B}_{\delta_0} \times L^\infty([0,T], \mathbb{R}^m) \to C^0([0,T], \mathbb{R}^n), (x,u) \mapsto \phi(\cdot, x, u)$$

where $\overline{B}_c$ denotes the closed ball of radius c. This map is continuous (and defined on the whole domain as remarked above). For each $\xi \in \overline{B}_{\delta_0}$ there is then an open neighborhood V_ξ of ξ and a $\mu_\xi > 0$ such that

$$z \in V_\xi \text{ and } \|\omega\| < \mu_\xi \;\Rightarrow\; \|\alpha_T(z,\omega) - \alpha_T(\xi,0)\| < \delta_0/2$$

where we use ω to denote controls on the interval $[0,T]$. We cover $\overline{B}_{\delta_0}$ with the V_ξ's and extract a finite subcover; let μ be smaller than all the corresponding μ_ξ's. Then, for each $x_0 \in \overline{B}_{\delta_0}$,

$$|\phi(T, x_0, \omega)| < \delta_0$$

and also

$$|\phi(t, x_0, \omega)| < \varepsilon$$

for each $t \in [0,T]$, provided that $\|\omega\| < \mu$.

We now prove (P). With these δ_0, μ, assume that x_0 and u are as in the property. Then, applying the above arguments using the restriction ω of u to $[0,T]$ we know that $x(t)$ remains in the ε-ball. But at time T, $x(T)$ is again in $\overline{B}_{\delta_0}$, so we can repeat the argument on $[T, 2T]$, and an induction gives the desired fact.

We now prove the Proposition. Pick any trajectory $x(\cdot)$ corresponding to a control $u(t) \to 0$. Pick any $\varepsilon > 0$. We wish to show that there is some T_ε so that $\|x(t)\| < \varepsilon$ for all $t \geq T_\varepsilon$.

Let K be a compact set so that $x(t) \in K$ for all $t \geq 0$ (CIBS assumption) and let δ_0, μ be as in property (P), for the given ε.

Using the global asymptotic stability of the origin for (1), there is some $T > 0$ so that $|\phi(T, \xi, 0)| < \delta_0$ for all $\xi \in K$. By a compactness argument identical to the one used to prove (P), we know that there is some $\nu > 0$ so that

$$|\phi(T, \xi, \omega)| < \delta_0$$

whenever $\xi \in K$ and ω is a control on $[0, T]$ with $\|\omega\| < \nu$. Pick T' such that $|u(t)| < \min\{\nu, \mu\}$ for all $t \geq T'$.

Consider the original trajectory $x(\cdot)$. At time

$$T_\varepsilon := T + T'$$

this equals $\phi(T, x(T'), \omega)$ where ω is the restriction of u to $[T', T_\varepsilon]$, and therefore $|x(T_\varepsilon)| < \delta_0$. For any $t > T_\varepsilon$ the control remains bounded in norm by μ, so $x(t)$ remains with $|x(t)| < \varepsilon$, as desired. ∎

Remark. It is sometimes of interest to study asymptotic stability of compact invariant sets different from the origin. The exact same proof, but with distance to the set S used in place of the norm of states, shows that if S is any such set with respect to (1), and if the CIBS property holds, then also every trajectory corresponding to convergent controls will converge to S. In terms of cascades, $S \times \{0\}$ is then a stable attractor for the system (CAS). ∎

3 Stabilization

In [6] we proved a stronger version (the one there is for "input to state stability") of this fact:

If the system (1) has the origin globally asymptotically stable, then there exists an everywhere nonzero smooth function $\beta : \mathbb{R}^n \to \mathbb{R}$ *such that the system*

$$\dot{x} = f(x, \beta(x)u) \tag{4}$$

is CIBS (and hence also CICS) stable.

A sketch of the proof is as follows. Let V be a Lyapunov function for (1), that is, a smooth positive definite and proper function

$$V : \mathbb{R}^n \to \mathbb{R}$$

such that $\nabla V(\xi).f(\xi, 0) < 0$ for all nonzero ξ. Then there exists some function $\beta : \mathbb{R}^n \to \mathbb{R}$, smooth and everywhere nonzero, such that, for each $\mu > 0$ there is some K so that

$$\nabla V(\xi).f(\xi, \beta(\xi)v) < 0$$

whenever $|v| \leq \mu$ and $|\xi| > K$. (The existence of β follows from a simple continuity argument, because

$$\nabla V(\xi).f(\xi, w) < 0$$

for all $|w| < \eta(\xi)$ for some function η; see Lemmas 3.1 and 3.2 in [6].)

If $u(\cdot)$ is any control converging to 0 and $x(0) = x_0$ is given, we need to prove that $x(t)$, the trajectory for (4), cannot diverge. Let μ be a bound on u, and pick K as above. Without loss of generality, assume that $|x_0| < K$.

Let c be the largest value of V on the closed ball of radius K, and consider the absolutely continuous function $V(x(t))$. For those t so that $x(t)$ is outside that closed ball, $V(x(t))$ has derivative (which exists almost everywhere) negative, and when

$x(t)$ is inside, the value of V is bounded by c. It follows that $V(x(t))$ is everywhere bounded by c, and thus that $x(t)$ remains bounded, since V is proper. Thus the system (4) is CIBS stable, as claimed.

Now we show how this can be used to stabilize systems that appear in what is called the "relative degree one" situation in zero-dynamics studies. Here we assume that $g(y,u) = u$ in (CAS) and that the system (3) is smoothly stabilizable, that is, that there exists some $k : \mathbb{R}^n \to \mathbb{R}^m$ smooth, with $k(0) = 0$, so that

$$\dot{x} = f(x, k(x))$$

has the origin globally asystable. The claim is that the cascade is again smoothly stabilizable; in other words, integrating the control does not destroy smooth stabilizability. This result was proved before by [2], in the context of "PD control" of mechanical systems, as well as [9] and [1]. In [8], an application to rigid body control is given. Here we wish to show how the existence of β gives a very natural, and alternative, stabilization technique.

Applying the above-cited fact (applied to $\dot{x} = f(x, k(x)+u)$), there is then a smooth everywhere nonvanishing β so that

$$\dot{x} = f(x, k(x) + \beta(x)u)$$

is CIBS. We make in (CAS) the change of variables

$$y = k(x) + \beta(x)z$$

so we can write

$$\begin{aligned}\dot{x} &= f(x, k(x) + \beta(x)z) \\ \dot{z} &= \frac{1}{\beta(x)}[h(x,z) + u]\end{aligned}$$

where h is some smooth function. Then the control law

$$u := -\beta(x)z - h(x,z)$$

stabilizes the z-subsystem, and thus also the cascade, because the first system is a CICS-stable system, which is asymptotically stable when $z \equiv 0$.

Take for instance the system

$$\begin{aligned} \dot{x} &= x^2 y \\ \dot{y} &= u \end{aligned}$$

for which $k(x) = -x$ and $\beta \equiv 1$ can be used (since $\dot{x} = -x^3 + x^2 u$ is BIBS, as the cubic term dominates for large x). Then with the change of variables $y = -x + z$, the equation for z becomes $\dot{z} = u + x^2 y$, and therefore

$$u := -z - x^2 y = -x - y - x^2 y$$

stabilizes the cascade globally.

The fact that in the example we could choose $\beta \equiv 1$ is no accident. In general for systems linear in controls that will be possible, as follows from the main result in [5].

For another example, take the system

$$\begin{aligned} \dot{x} &= -x^5 + y^2 \\ \dot{y} &= x^2 + u \end{aligned}$$

considered in [1]. The first system is already BIBS and asystable for $y \equiv 0$, so the control law $u := -x^2 - y$ stabilizes the cascade. (This is simpler than the solution given in that reference. Yet another illustration is the satellite with two controls in that same reference, which can easily be brought into a cascade using the same k as there; a simpler control law again results.)

4 References

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