State-Space and I/O Stability
for Nonlinear Systems

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On the occasion of George Zames' 60th birthday

Abstract

This paper surveys several alternative but equivalent definitions of "input to state stability" (ISS), a property which provides a natural framework in which to formulate notions of stability with respect to input perturbations. Relations to classical Lyapunov as well as operator theoretic approaches, connections to dissipative systems, and applications to stabilization of several cascade structures are mentioned. The particular case of linear systems subject to control saturation is singled-out for stronger results.

1 Introduction

George Zames has long been a proponent of *input/output* approaches to the analysis of control systems. Among his many deep contributions, he pioneered the use of operator techniques for determining the stability of feedback configurations. These techniques focus on the estimation of bounds on solutions, expressed in terms of bounds on forcing functions, and allow powerful tools to be applied, such as small-gain theorems. Conceptually, the main competing variants of the notion of stability are based on *state-space* ideas, which concentrate on the asymptotic stability of equilibria (or of more general attractors) in the absence of —or subject to only small— external "disturbance" inputs. It is the purpose of this paper to briefly survey various links between these two alternative paradigms of stability, through the systematic use of the notion of "input to state stability" (ISS).

Mathematically, the state-space theory is grounded on classical dynamical systems; Lyapunov functions and geometric methods play a central role. In contrast, input/output stability has classically had a more operator-theoretic flavor and developed independently. The latter notion is arguably the most

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useful in many control applications, since it permits the natural quantification of performance bounds and it is well-behaved under operations such as cascading of systems. In addition, i/o stability provides a framework in which to study the classification and parameterization of dynamic controllers. It is also the most natural notion to consider in the context of building observer-based controllers.

Based on linear systems intuition, where all notions coincide, it is perhaps surprising that state-space and i/o stability are not automatically related. Even for feedback linearizable systems, this relation is more subtle than might appear: if one first linearizes a system and then stabilizes the equivalent linearization, in terms of the original system one does not in general obtain a closed-loop system that is input/output stable in any reasonable sense. However, it is always possible to make a choice of a —usually different— feedback law that achieves such stability, in the linearizable case as well as for all other smoothly stabilizable systems. This paper presents a brief and informal survey of such results, and discusses precise definitions of input to state stability, nonlinear gains, and stability margins which lend themselves to useful theoretical analysis.

One important source of inspiration for our approach is the pioneering work of Willems ([27]), who introduced an abstract concept of energy dissipation in order to unify i/o and state space stability, and in particular with the purpose of understanding conceptually the meaning of Kalman-Yakubovich positive-realness (passivity), and frequency-domain stability theorems such as those due to Zames, in a more general nonlinear context. His work was continued by many authors, most notably Hill and Moylan (see e.g. [5, 6]).

However, although extremely close in spirit, technically our work does not make much contact with the existing dissipation literature. Mathematically it is grounded instead in more classical converse Lyapunov arguments in the style of Massera, Kurzweil, and Zubov,

The results reported here regarding equivalences between different notions of input to state stability originate with the paper [14], but the definitive conclusions were obtained in recent work jointly carried out with with Yuan Wang in [20], which in turn built upon research with Wang and Yuandan Lin in [9] and [18]; the input-saturated results are based on joint papers with Wensheng Liu and Yacine Chitour ([10]) as well as Sussmann and Yang ([21]). Some recent and very relevant results by Jiang, Praly, and Teel ([7]) are also mentioned.

In the interest of exposition, the style of presentation in this survey is informal. The reader should consult the references for more details and, in some cases, for precise statements.

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Preliminaries

This paper deals with continuous time systems of the standard form

$$\dot{x} \;=\; f(x,u)\,, \tag{1}$$

where $x(t)\in\mathbb{R}^n$ and $u(t)\in\mathbb{R}^m$. (Since global asymptotical stability will be of interest, there is no reason to consider systems evolving in more general manifolds than Euclidean space. For undefined terminology from control theory see [17].) It is assumed that $f:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is locally Lipschitz and satisfies $f(0,0)=0$. *Controls* or *inputs* are measurable locally essentially bounded functions $u:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$. The set of all such functions is denoted by $L^m_{\infty,e}$, and one denotes $\|u\|_\infty = (\text{ess})\sup\{|u(t)|, t\geq 0\} \leq \infty$; when this is finite, one obtains the usual space L^m_∞, endowed with the (essential) supremum norm. (Everywhere, $|\cdot|$ denotes Euclidean norm in the appropriate space of vectors, and $\|\cdot\|$ induced norm for matrices, while $\|\cdot\|_\infty$ is used for sup norm.) For each $x_0\in\mathbb{R}^n$ and each $u\in L^m_\infty$, $x(t,x_0,u)$ denotes the trajectory of the system (1) with initial state $x(0)=x_0$ and input u. This is a priori defined only on some maximal interval $[0,T_{x_0,u})$, with $T_{x_0,u}\leq+\infty$. If the initial state and input are clear from the context, one writes just $x(\cdot)$ for the ensuing trajectory. The system is *(forward-) complete* if $T_{x_0,u}=+\infty$ for all x_0 and u.

The questions to be studied relate to the "stability," understood in an appropriate sense, of the input to state mapping $(x_0,u(\cdot))\mapsto x(\cdot)$ (or, in the last section, when an output is also given, of the input to output mapping $\mapsto y(\cdot)$). To appreciate the type of problem that one may encounter, consider the following issue. Suppose that in the absence of inputs the trivial solution $x\equiv 0$ of the differential equation

$$\dot{x} \;=\; f_0(x) = f(x,0) \tag{2}$$

is globally asymptotically stable (for simplicity, in such a situation, we'll simply say that (1), or equivalently the zero-input restricted system (2), is GAS). Then one would like to know if, for solutions of (1) associated to *non*zero controls, it holds that

$$u(\cdot)\underset{t\to\infty}{\longrightarrow}0 \quad\Rightarrow\quad x(\cdot)\underset{t\to\infty}{\longrightarrow}0$$

(the "converging input converging state" property) or that

$$u(\cdot)\text{ bounded} \quad\Rightarrow\quad x(\cdot)\text{ bounded}$$

(the "bounded input bounded state" property). Of course, for linear systems $\dot{x}=Ax+Bu$ these implications are always true. Not only that, but one has explicit estimates

$$|x(t)| \;\leq\; \beta(t)|x_0| + \gamma\,\|u_t\|_\infty$$

where

$$\beta(t) = \left\|e^{tA}\right\| \to 0 \;\text{ and }\; \gamma = \|B\|\int_0^\infty \left\|e^{sA}\right\|ds$$

for any Hurwitz matrix A, where u_t is the restriction of u to $[0,t]$, though of as a function in L_∞^m which is zero for $s>t$. From these estimates both properties can be easily deduced.

These implications fail in general for nonlinear systems, however, as has been often pointed out in the literature (see for instance [26]). As a trivial illustration, take the system

$$\dot{x} = -x + (x^2+1)u \tag{3}$$

and the control $u(t) = (2t+2)^{-1/2}$. With $x_0 = \sqrt{2}$ there results the unbounded trajectory $x(t) = (2t+2)^{1/2}$. This is in spite of the fact that the system is GAS. Thus, the converging input converging state property does not hold. Ever worse, the bounded input $u \equiv 1$ results in a finite-time explosion. This example is not artificial, as it arises from the simplest case of feedback linearization design. Indeed, given the system

$$\dot{x} \;=\; x + (x^2+1)u\,,$$

the obvious stabilizing control law (obtained by first cancelling the nonlinearity and then assigning dynamics $\dot{x}=-x$) is

$$u := \frac{-2x}{x^2+1} + v$$

where v is the new external input. In terms of this new control (which might be required in order to meet additional design objectives, or may represent the effect of an input disturbance), the closed-loop system is as in (3), and thus is ill-behaved. Observe, however, that if instead of the obvious law just given one would use:

$$u := \frac{-2x}{x^2+1} - x + v\,,$$

then the closed-loop system becomes instead

$$\dot{x} = -2x - x^3 + (x^2+1)u\,.$$

This is still stable when $u \equiv 0$, but in addition it tolerates perturbations far better, since the term $-x^3$ dominates $u(x^2+1)$ for bounded u and large x. The behavior with respect to such u is characterized qualitatively by the notion of "ISS" system, to be discussed below. More generally, it is possible to show that up to *feedback equivalence*, GAS always implies (and is hence equivalent) to the ISS property to be defined. This is one of many motivations for the study of the ISS notion, and will be reviewed after the precise definitions have been given.

Besides being mathematically natural and providing the appropriate framework in which to state the above-mentioned feedback equivalence result, there are several other reasons for studying the ISS property, some of which are briefly mentioned in this paper. See for instance the applications to observer design and new small gain theorems in [24], [25], [7], and [12]; the construction of coprime stable factorizations was the main motivation in the original paper [14] which introduced the ISS concept, and the stabilization of cascade systems using these ideas was briefly discussed in [15].

2 The Property ISS

Next, four natural definitions of input to state stability are proposed and separately justified. Later, they turn out to be equivalent. The objective is to express the fact that states remain bounded for bounded controls, with an ultimate bound which is a function of the input magnitude, and in particular that states decay when inputs do.

2.1 From GAS to ISS — A First Pass

The simplest way to introduce the notion of ISS system is as a generalization of GAS, global asymptotic stability of the trivial solution $x \equiv 0$ for (2). The GAS property amounts to the requirements that the system be complete and the following two properties hold:

1. *(Stability):* the map $x_0 \mapsto x(\cdot)$ is continuous at 0, when seen as a map from $\mathbb{R}^n$ into $C^0\left([0,+\infty),\mathbb{R}^n\right)$, and
2. *(Attractivity):* $\lim_{t\to+\infty} |x(t,x_0)| = 0$.

Note that, under the assumption that 1. holds, the convergence in the second part is automatically uniform with respect to initial states x_0 in any given compact. By analogy, one defines the system (1) to be *input to state stable* (ISS) if the system is complete and the following properties, which now involve nonzero inputs, hold:

1. the map $(x_0, u) \mapsto x(\cdot)$ is continuous at $(0,0)$ (seen as a map from $\mathbb{R}^n \times L^m_\infty$ to $C^0\left([0,+\infty),\mathbb{R}^n\right)$, and
2. there exists a "nonlinear asymptotic gain" $\gamma \in \mathcal{K}$ so that

$$\overline{\lim_{t\to+\infty}} |x(t,x_0,u)| \;\leq\; \gamma(\|u\|_\infty) \tag{4}$$

uniformly on x_0 in any compact and all u.

(The class $\mathcal{K}$ consists of all functions $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which are continuous, strictly increasing, and satisfy $\gamma(0) = 0$. The uniformity requirement means, explicitly: for each r and ε positive, there is a $T > 0$ so that $|x(t,x_0,u)| \leq \varepsilon + \gamma(\|u\|_\infty)$ for all u and all $|x_0| \leq r$ and $t \geq T$.)

In the language of robust control, the inequality (4) is an "ultimate boundedness" condition. Note that this is a direct generalization of attractivity to the case $u \not\equiv 0$; the "lim sup" is now required since the limit need not exist.

2.2 From Lyapunov to Dissipation — A Second Pass

A potentially different concept of input to state stability arises when generalizing classical Lyapunov conditions to certain classes of dissipation inequalities.

A *storage* or energy function is a $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ which is continuously differentiable, proper (that is, radially unbounded) and positive definite (that is, $V(0)=0$ and $V(x)>0$ for $x \neq 0$). A (classical) *Lyapunov function* for the zero-input system (2) is a storage function for which there exists some function α of class $\mathcal{K}_\infty$ —that is, of class $\mathcal{K}$ and so that also $\alpha(s) \to +\infty$ as $s \to +\infty$— so that

$$\nabla V(x) \cdot f_0(x) \;\leq\; -\alpha(|x|)$$

holds for all $x \in \mathbb{R}^n$. This means that $dV(x(t))/dt \leq -\alpha(|x(t)|)$ along all trajectories.

By analogy, when nonzero inputs must be taken into account, it is sensible to define an ISS-*Lyapunov* function as a storage function for which there exist two class $\mathcal{K}_\infty$ functions α and θ such that

$$\nabla V(x) \cdot f(x,u) \;\leq\; \theta(|u|) - \alpha(|x|) \tag{5}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$. Thus, along trajectories one now obtains the inequality $dV(x(t))/dt \leq \theta(|u(t)|) - \alpha(|x(t)|)$.

A *smooth* ISS-Lyapunov function is a V which satisfies these properties and is in addition infinitely differentiable. Smoothness is an extremely useful property in this context, as one may then use iterated derivatives of V along trajetories for various design as well as analysis questions, in particular in so-called "backstepping" design techniques.

In the terminology of [27, 6], (5) is a *dissipation inequality* with storage function V and *supply* function $w(u,x) = \theta(|u|) - \alpha(|x|)$. (In the context of dissipative systems one often postulates the equivalent integral form $V(x(t,x_0,u)) - V(x_0) \leq \int_0^t w(u(s),x(s))ds$, which must hold along all trajectories, and no differentiability is required of V. Moreover, outputs $y=h(x)$ are used instead of states in the estimates, so the present setup corresponds to the case $h(x)=x$.) The estimate (5) is a generalization of the one used by Brockett in [1] when defining "finite gain at the origin;" in that paper, the function θ is restricted to be quadratic, and the concepts are only defined locally, but the ideas are very similar.

2.3 Gain Margins — A Third Pass

Yet another possible approach to formalizing input to state stability is motivated both by the classical concept of total stability and as a generalization of the usual gain margin for linear systems.

In [20], a *(nonlinear) stability margin* for system (1) is defined as any function $\rho \in \mathcal{K}_\infty$ with the following property: for each admissible —possibly nonlinear and/or time-varying— feedback law $\|$ bounded by ρ, that is, so that

$$|\,\|(t,x)| \;\leq\; \rho(|x|)$$

for all (t,x), the closed-loop system

$$\dot{x} \;=\; f(x, \|(t,x)) \tag{6}$$

is GAS, uniformly on $\|$. (More precisely, an admissible feedback law is a measurable function $\| : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$ for which (6) is well-posed; that is, for each initial state $x(0)$ there is an absolutely continuous solution, defined at least for small times, and any two such solutions coincide on their interval of existence. Uniformity in $\|$ means that all limits in the definition of GAS are independent of the particular $\|$, as long as the inequality $|\,\|(t,x)| \leq \rho(|x|)$ holds.) A system is said to be *robustly stable* if there exists some such ρ.

Observe that for arbitrary nonlinear GAS systems, in general only small perturbations can be tolerated (cf. total stability results). The requirement that $\rho \in \mathcal{K}_\infty$ is thus highly nontrivial: it means that for large states relatively large perturbations should not affect stability.

2.4 Estimates — Fourth Pass

A final proposed notion of input to state stability can be introduced by means of an estimate similar to that which holds in the linear case:

$$|x(t,x_0,u)| \;\leq\; \left\|e^{tA}\right\| |x_0| \,+\, \left(\|B\| \int_0^\infty \left\|e^{sA}\right\| ds\right) \|u_t\|_\infty \,.$$

It is first necessary to review an equivalent —if somewhat less widely known— definition of GAS. This is a characterization in terms of comparison functions. Recall that a function of class $\mathcal{KL}$ is a

$$\beta \,:\, \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

so that $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each $s \geq 0$ (example of relevance to the linear case: $ce^{-at}s$, with $a>0$ and a constant c). It is not difficult to prove (this is essentially in [4]; see also [14]) that the system (2) is GAS if and only if there exists a $\beta \in \mathcal{KL}$ so that

$$|x(t,x_0)| \;\leq\; \beta(|x_0|,t) \tag{7}$$

for all t, x_0. (Note that sufficiency is trivial, since forward completeness follows from the fact that trajectories stay bounded, the estimate $|x(t,x_0)| \leq \beta(|x_0|,0)$ provides stability, and $|x(t,x_0)| \leq \beta(|x_0|,t) \to 0$ shows attractivity. The converse is established by formulating and solving a differential inequality for $|x(t,x_0)|$.)

In this context, it is then natural to consider the following "$\beta+\gamma$" property: There exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ so that, for all initial states and controls, and all $t \geq 0$:

$$|x(t,x_0,u)| \leq \beta(|x_0|,t) + \gamma(\|u_t\|_\infty)\,. \tag{8}$$

(One could use a "max" instead of the sum of the two estimates, but the same concept would result. Also, it makes no difference to write $\|u\|_\infty$ instead of the norm of the restriction $\|u_t\|_\infty$.) This is a direct generalization of both the linear estimate and the characterization of GAS in terms of comparison functions.

2.5 All Are Equivalent

The following result was recently proved by Yuan Wang and the author:

Theorem 1 ([20]) *For any system (1), the following properties:*

1. ISS *(nonlinear asymptotic gain),*
2. *there is an* ISS*-Lyapunov function (dissipativity),*
3. *there is a smooth* ISS*-Lyapunov function,*
4. *there is a nonlinear stability margin (robust stability), and*
5. *there is some* $\beta+\gamma$ *estimate,*

are all equivalent. ∎

The proof is heavily based on a result obtained by Wang, Lin, and the author in [9], which states essentially that a parametric family of systems $\dot{x} = f(x,d)$, with arbitrary time-varying "disturbances" $d(t)$ taking values on a compact set D, is uniformly globally asymptotically stable if and only if there exists a smooth storage function V and an $\alpha \in \mathcal{K}_\infty$ so that

$$\nabla V(x) \cdot f(x,d) \;\leq\; -\alpha(|x|)$$

for all $x \in \mathbb{R}^n$ and values $d \in D$. Note that the construction of a smooth V is not entirely trivial (this subsumes as particular cases several standard converse Lyapunov theorems).

2.6 Checking the ISS Property

Of course, verifying the ISS property is in general very hard —after all, in the particular case of systems with no inputs, this amounts to checking global asymptotic stability. Nonetheless, the dissipation inequality (5) provides in principle a good tool, playing the same role as Lyapunov's direct method for asymptotic stability. Actually, even more useful is the following variant, which is the original definition of "ISS-Lyapunov function" in [14]. Consider a storage function with the property that there exist two class $\mathcal{K}$ functions α and χ so that the implication

$$|x| \geq \chi(|u|) \;\;\Rightarrow\;\; \nabla V(x) \cdot f(x,u) \;\leq\; -\alpha(|x|) \tag{9}$$

holds for each state $x \in \mathbb{R}^n$ and control value $u \in \mathbb{R}^m$. It is shown in [20] that the existence of such a V provides yet another necessary and sufficient characterization of the ISS property. (Other variants are also equivalent, for instance, asking that α be of class $\mathcal{K}_\infty$.)

As an illustration, consider the following system, which will appear again later in the context of an example regarding the stabilization of the angular momentum of a rigid body. The state space is $\mathbb{R}$, the control value space is $\mathbb{R}^2$, and dynamics are given by:

$$\dot{x} \;=\; -x^3 + x^2 u_1 - x u_2 + u_1 u_2 \,. \tag{10}$$

This system is GAS when $u \equiv 0$, and for large states the term $-x^3$ dominates, so it can be expected to be ISS. Indeed, using the storage function $V(x) = x^2/2$ there results

$$\nabla V(x) \cdot f(x,u) \;\leq\; -\left(\frac{2}{9}\right) x^4$$

provided that $3|u_1| \leq |x|$ and $3|u_2| \leq x^2$. A sufficient condition for this to hold is that $|u| \leq \nu(|x|)$, where $\nu(r) := \min\{r/3, r^2/3\}$. Thus V is an ISS-Lyapunov function as above, with $\alpha(r) = (2/9)r^4$ and $\chi = \nu^{-1}$.

Another example is as follows. Let $\mathrm{SAT} : \mathbb{R} \to \mathbb{R}$ be the standard saturation function: $\mathrm{SAT}[r] = r$ if $|r| \leq 1$, and $\mathrm{SAT}[r] = \mathrm{sign}(r)$ otherwise. Consider the following one-dimensional one-input system:

$$\dot{x} \;=\; -\,\mathrm{SAT}[x+u] \,. \tag{11}$$

This is an ISS system, as will be proved next by showing that

$$V(x) \;:=\; \frac{|x|^3}{3} + \frac{x^2}{2} \tag{12}$$

is an ISS-Lyapunov function. Observe that V is once differentiable, as required. This is a very particular case of a more general result dealing with linear systems with saturated controls, treated in [10]; more will be said later about the general case (which employs a straightforward generalization of this V).

To prove that V satisfies a dissipation inequality, first note that, since $|r - \mathrm{SAT}[r]| \leq r\,\mathrm{SAT}[r]$ for all r,

$$|x - \mathrm{SAT}[x+u]| \leq |x+u-\mathrm{SAT}[x+u]| + |u| \leq (x+u)\,\mathrm{SAT}[x+u] + |u| \tag{13}$$

for all values $x \in \mathbb{R}$ and $u \in \mathbb{R}$. It follows that

$$\begin{aligned} -x\,\mathrm{SAT}[x+u] \;&=\; x(-x) + x(x - \mathrm{SAT}[x+u]) \\ &\leq\; -x^2 + |x|(x+u)\,\mathrm{SAT}[x+u] + |x||u| \end{aligned}$$

for all x, u. On the other hand, using that $\mathrm{SAT}[r] \leq 1$ for all r,

$$\begin{aligned} -|x|x\,\mathrm{SAT}[x+u] \;&=\; |x|\,[-(x+u)\,\mathrm{SAT}[x+u] + u\,\mathrm{SAT}[x+u]] \\ &\leq\; -|x|(x+u)\,\mathrm{SAT}[x+u] + |x||u| \,. \end{aligned}$$

Adding the two inequalities, it holds that

$$-(1+|x|)x\,\mathrm{SAT}[x+u] \;\leq\; -x^2 + 2|x||u| \tag{14}$$

so that indeed

$$\nabla V(x) \cdot f(x,u) \;\leq\; -\frac{x^2}{2} + 2u^2$$

as desired (note that $\nabla V(x) = x(1+|x|)$).

2.7 Relations Among Estimates, Zero-State Responses, and Linear Gains

There are many relationships among the various estimates which appear in the alternative characterizations of ISS. Two of them are as follows.

Assume that V is a storage function satisfying the estimates in Equation (5):

$$\nabla V(x) \cdot f(x,u) \;\leq\; \alpha_4(|u|) - \alpha_3(|x|) \tag{15}$$

for some $\mathcal{K}_\infty$ functions α_3 and α_4. Since V is proper, continuous, and positive definite, there are as well two other class $\mathcal{K}_\infty$ functions α_1 and α_2 such that

$$\alpha_1(|x|) \;\leq\; V(x) \;\leq\; \alpha_2(|x|) \tag{16}$$

for all $x \in \mathbb{R}^n$. It then holds that one may pick an asymptotic gain γ in Equation (4) of the form:

$$\gamma \;=\; \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \circ \alpha_4 \;. \tag{17}$$

Moreover, if instead of (15) there holds a slightly stronger estimate of the form

$$\nabla V(x) \cdot f(x,u) \;\leq\; \alpha_4(|u|) - \alpha_3(|x|) - \alpha(|x|)$$

where α is any class $\mathcal{K}$ function, then the γ function in the "β+γ" property (8) can also be picked as in Equation (17). These conclusions are implicit in the proofs given in [14] and [20].

For trajectories starting at the particular initial state $x_0 = 0$, for any input function u, and assuming only that V satisfies (15)-(16), it holds that $|x(t,0,u)| \leq \gamma(\|u\|_\infty)$ for all $t \geq 0$, not merely asymptotically, for the same γ as in (17), that is,

$$\|x(\cdot,0,u)\|_\infty \;\leq\; \gamma(\|u\|_\infty) \,.$$

Thus the zero-state response has a "nonlinear gain" bounded by this γ.

A particular case of interest is when both of (α_1,α_2) and (α_3,α_4) are *convex estimate pairs* in the following sense: a pair of class $\mathcal{K}$ functions (α,β) is a convex estimate pair if α and β are convex functions and there is some real number $k \geq 1$ such that $\beta(r) \leq k\alpha(r)$ for all $r \geq 0$. Note that for any convex function α in $\mathcal{K}$ and any $k \geq 1$ it holds that $\alpha^{-1}(k\alpha(r)) \leq kr$ for all nonnegative r, from which it follows that $\alpha^{-1}(\beta(r)) \leq kr$ if k is as in this definition. One concludes that if each of (α_1,α_2) and (α_3,α_4) is a convex estimate pair, then the gain γ can be taken to be bounded by a linear function. In other words, the input to state operator, starting from $x_0 = 0$, is bounded as an operator with respect to sup norms:

$$\|x(\cdot,0,u)\|_\infty \;\leq\; g\,\|u\|_\infty \;.$$

This is the standard situation in linear systems theory, where V is quadratic (and hence admits estimates in terms of α_1 and α_2 of the form $c_i r^2$, where c_1 and c_2 are respectively the smallest and largest singular values of the associated form) and the supply function can likewise be taken of the form $c_4|u|^2 - c_3|x|^2$.

So finiteness of linear gain, that is, operator boundedness, follows from convexity of the estimation functions. Somewhat surprinsingly, for certain linear systems subject to actuator saturation, convex (but not quadratic) estimates are also possible, and this again leads to finite linear gains. For example, this applies to the function V in Equation (12), as an ISS-Lyapunov function for system (11): there one may pick $\alpha = \alpha_1 = \alpha_2 = r^3/3 + r^2/2$, which is convex since $\alpha''(r) = 2r + 1 > 0$, while α_3 and α_4 can be taken quadratic (cf. Equation (14)).

As an additional remark, note that, just from the fact that V is nonnegative and $V(0) = 0$, and integrating the dissipation inequality (5), for $x_0 = 0$ there results the inequality $\int_0^{+\infty} \alpha(|x(t, 0, u)|)dt \leq \int_0^{+\infty} \theta(|u(t)|)dt$. In this manner, it is routine to use dissipation inequalities for proving operator boundedness in various pth norms (in particular, when $\alpha(r) = c_1 r^2$ and $\theta(r) = c_2 r^2$ one is estimating "H^∞" norms). But in the current context, more general nonlinearities than powers are being considered.

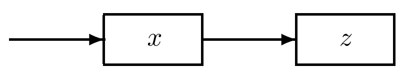
It is also interesting to note that, if V and α are so that the estimate (9) is satisfied, then there is some θ so that the dissipation estimate (5) also holds, with these same V and α.

3 Interconnections

It is by now well known, and easy to prove, that the cascade of two ISS systems is again ISS (in particular, a cascade of an ISS and a GAS system is GAS). It is interesting to observe that this statement can be understood very intuitively in terms of the dissipation formalism, and it provides further evidence of the naturality of the ISS notion. In addition, proceeding in this manner, one obtains a Lyapunov function (with strictly negative derivative along trajectories) for the cascade.

Theorem 2 *Consider the system in cascade form*

$$\begin{aligned} \dot{z} &= f(z, x) \\ \dot{x} &= g(x, u) \end{aligned}$$



where $f(0,0) = g(0,0) = 0$, the second equation is ISS*, and the first equation is* ISS *when x is seen as an input. Then the composite system is* ISS. ∎

The proof can be based on the following argument. First one shows that it is possible to obtain storage functions V_1 and V_2 so that V_1 satisfies a dissipation estimate

$$\nabla V_1(z) \cdot f(z, x) \leq \theta(|x|) - \alpha(|z|)$$

for the first subsystem, while V_2 is a storage function for the x-subsystem so that

$$\nabla V_2(x) \cdot g(x, u) \leq \tilde{\theta}(|u|) - 2\theta(|x|).$$

Then $V_1(z)+V_2(x)$ is a storage function for the composite system, which satisfies the dissipation inequality with derivative bounded by $\tilde{\theta}(|u|)-\theta(|x|)-\alpha(|z|)$.

A beautiful common generalization of both the cascade result and the usual Small-Gain Theorem was recently obtained by Jiang, Teel, and Praly. We write $\tilde{\gamma}\succ\gamma$ for two functions of class $\mathcal{K}$ if there is some $\rho\in\mathcal{K}_\infty$ so that $\tilde{\gamma}=(I+\rho)\circ\gamma$.

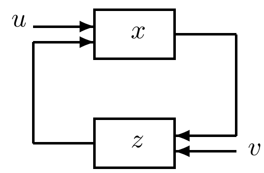


Figure 1: *Composite Feedback Form*

Theorem 3 ([7]) *Consider a system in composite feedback form (cf. Figure 1):*

$$\begin{aligned}\dot{z} &= g(z,x,v)\\ \dot{x} &= f(x,z,u)\end{aligned}$$

where u,v are the inputs to the composite system. Assume:

- *Each of $\dot{x}=f(x,z,u)$ and $\dot{z}=g(z,x,v)$ is an ISS system, when (z,u) and (x,v) are considered as inputs respectively; let γ_1 and γ_2 denote the gains for the x and z subsystems, in the sense of the estimate of type (8).*
- *The following small-gain condition holds: there are $\tilde{\gamma}_1\succ\gamma_1$ and $\tilde{\gamma}_2\succ\gamma_2$ so that $(\tilde{\gamma}_1\circ\tilde{\gamma}_2)(r)\le r$ and $(\tilde{\gamma}_2\circ\tilde{\gamma}_1)(r)\le r$ for all $r\ge 0$.*

Then, the composite system is ISS. ∎

(Note that in the special case in which the $\gamma_i(r)=g_ir$, the small gain condition is satisfied iff $g_1g_2<1$, thus generalizing the usual case.) It is important to note that the result in [7] is far more general; for instance, it deals with partially observed systems and with "practical stability" notions. Also, the small gain condition can be stated just in terms of the gains with respect to the z and x variables. Related to these results is previous work on small-gain conditions, also relying on comparison functions, in [13, 11].

A different cascade form, with an input feeding into both subsystems, is of interest in the context of stabilization of saturated linear systems (using an approach originally due to Teel, cf. [22]) and in other applications. This provides yet another illustration of the use of ISS ideas. The structure is (cf. Figure 2):

$$\begin{aligned}\dot{z} &= f(z,x,u)\\ \dot{x} &= g(x,u)\,.\end{aligned}$$

First assume that a (locally Lipschitz) feedback law k can be found which makes the system $\dot{z}=f(z,x,k(z))$ GAS *uniformly on* x, that is, $f(0,x,k(0))=0$

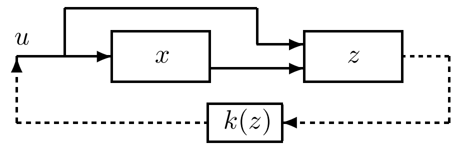


Figure 2: *Special Cascade Configuration*

for all x and an estimate as in (7), $|z(t)| \leq \beta(|z(0)|, t)$ holds, which is independent of $x(t)$. Suppose also that the x subsystem is ISS. Then, the feedback law $u = k(z)$ gives closed-loop equations $\dot{z} = f(z, x, k(z))$, $\dot{x} = g(x, k(z))$; because x is essentially irrelevant in the first equation, these equations behave just as a cascade of a GAS system (the z-system) and an ISS one, so the GAS property results as before. (More precisely, this is because it is still possible to find a Lyapunov function which depends only on z for the z-subsystem, due to the assumed uniformity property; see [9].) The interesting fact is the same global conclusions hold under more local assumptions on the z-subsystem. Assume:

- The z-subsystem is stabilizable with small feedback, uniformly on x small, meaning that for each $0 < \varepsilon \leq \varepsilon_0$ there is a (locally Lipschitz) feedback law k_ε with $|k_\varepsilon(z)| \leq \varepsilon$ for all z so that $\dot{z} = f(z, x, k_\varepsilon(z))$ is GAS uniformly on $|x| \leq \varepsilon_0$; further, under the feedback law $u = k_\varepsilon(z)$ the composite system is forward complete (solutions exist for all $t > 0$).
- The x subsystem is ISS.

(Later we discuss an interesting class of examples where these properties are verified.) Then, the claim is that, for any small enough $\varepsilon > 0$, the composite system under the feedback law $u = k_\varepsilon(z)$ is GAS. Stability is clear: for small x and z, trajectories coincide with those that would result if uniformity would hold globally on x (cf. the previous case). We are left to show that every solution $(x(\cdot), z(\cdot))$ satisfies $x(t) \to 0$ and $z(t) \to 0$ as $t \to +\infty$.

To establish this fact, pick any ε as follows. Let γ be a "nonlinear asymptotic gain" as in Equation (4), so that $\overline{\lim}_{t \to +\infty} |x(t, x_0, u)| \leq \gamma(\|u\|_\infty)$ for all inputs and initial conditions. Now take any $0 < \varepsilon < \varepsilon_0$ so that $\gamma(\varepsilon) < \varepsilon_0$. Pick any k_ε so that $|k_\varepsilon(z)| \leq \varepsilon$ for all z. Consider any solution $(x(\cdot), z(\cdot))$. Seeing $v(t) = k_\varepsilon(z(t))$ as an input to the x-subsystem, with $\|v\|_\infty \leq \varepsilon$, the choice of γ means that for some T, $t \geq T$ implies $|x(t)| < \varepsilon_0$. It follows that $z(t) \to 0$. Now the second equation is an ISS system with an input $v(t) \to 0$, so also $x(t) \to 0$, as required.

4 An Example

As a simple illustration of the use of the ISS concept, we may consider the oft-studied problem of globally stabilizing to zero the angular momentum of a rigid body which is controlled by means of two external torques applied along principal axes, and suggest an alternative way of achieving this objective using

ISS ideas. (This may represent a model of a satellite under the action of a pair of opposing jets.) The components of the state variable $\omega = (\omega_1, \omega_2, \omega_3)$ denote the angular velocity coordinates with respect to a body-fixed reference frame with origin at the center of gravity and consisting of the principal axes. Letting the positive numbers I_1, I_2, I_3 denote the respective principal moments of inertia (positive numbers), this is a system on $\mathbb{R}^3$, with controls in $\mathbb{R}^2$ and equations:

$$I\dot{\omega} \;=\; S(\omega)I\omega + Bu\,, \tag{18}$$

where I is the diagonal matrix with entries I_1, I_2, I_3 and where B is a matrix in $\mathbb{R}^{3\times 2}$ whose columns describe the axes on which the control torques apply. Since it is being assumed that the two torques act along two principal axes, without loss of generality the columns of B are $(0,1,0)'$ and $(0,0,1)'$ respectively. The matrix $S(\omega)$ is the rotation matrix

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

Dividing by the I_j's, and applying the obvious feedback and coordinate transformations, there results a system on $\mathbb{R}^3$ of the form:

$$\begin{aligned} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= u_2 \end{aligned}$$

where u_1 and u_2 are the controls.

To globally stabilize this system, and following the ideas of [1] for the corresponding local problem, one performs first a change of coordinates into new coordinates (x, z_1, z_2), where $x = x_1$ and

$$x_2 = -x_1 + z_1\,, \;\; x_3 = x_1^2 + z_2\,.$$

The system is now viewed as a cascade of two subsystems. One of these is described by the x variable, with z_1 and z_2 now thought of as inputs, and the second one is the z_1, z_2 subsystem. The first subsystem is precisely the one in example (10), and it is therefore ISS. Since a cascade of an ISS and a GAS system is again GAS, it is only necessary to stabilize the z_1, z_2 subsystem. In other words, looking at the system in the new coordinates:

$$\begin{aligned} \dot{x} &= -x^3 + x^2 z_1 - x z_2 + z_1 z_2 \\ \dot{z}_1 &= u_1 + (-x + z_1)\left(x^2 + z_2\right) \\ \dot{z}_2 &= u_2 - 2x_1 (-x + z_1)\left(x^2 + z_2\right)\,, \end{aligned}$$

any feedback that stabilizes the last two equations will also make the composite system GAS. One may therefore use

$$u_1 = -x_1 - x_2 - x_2 x_3\,, \;\; u_2 = -x_3 + x_1^2 + 2x_1 x_2 x_3\,,$$

which renders the last two equations $\dot{z}_1 = -z_1$ and $\dot{z}_2 = -z_2$. As a remark, note that a conceptually different approach to the same problem can be based upon zero dynamics techniques ([2, 23]). In that context, one uses Lie derivatives of a Lyapunov function for the x-subsystem in building a global feedback law; see the discussion in [17], Section 4.8. For the present rigid body stabilization problem, the feedback stabilizing law obtained using that approach would be as follows ([2]):

$$u_1 = -x_1 - x_2 - x_2 x_3 - 2x_1 x_3 \ , \quad u_2 = -x_3 + 3x_1^2 + 2x_1 x_2 x_3 \, .$$

5 Linear Systems with Actuator Saturation

For linear systems subject to actuator saturation, more precise results regarding stabilization can be obtained. The objective is to study control problems for plants P that can be described as in Figure 3, where W indicates a linear transfer matrix. For simplicity, we consider here just the state-observation

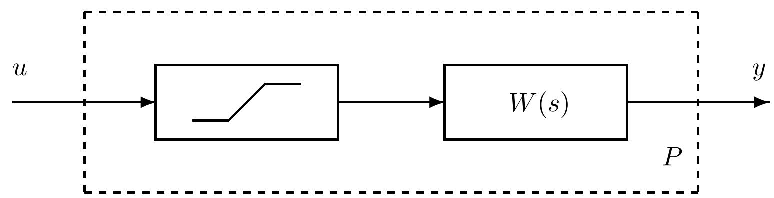


Figure 3: Saturated-Input Linear System

case, that is, systems of the type

$$\dot{x} \;=\; Ax + B\,\mathrm{SAT}[u] \, . \tag{19}$$

By an L_p-stable system one means that the zero-initial state response induces a bounded operator $L_p \to L_p$. The following result was recently obtained by W. Liu, Y. Chitour, and the author (see also [3] for related results on input to state dependence for such systems):

Theorem 4 ([10]) *Assume that the pair (A, B) is controllable and that A is neutrally stable (i.e., there is some symmetric positive definite Q so that $A^T Q + QA \leq 0$). Then, there exists a matrix F so that the system*

$$\dot{x} \;=\; Ax + B\,\mathrm{SAT}[Fx + u]$$

is L_p-stable for each $1 \leq p \leq \infty$. ∎

The fact that GAS can be achieved for such systems is a well-known and classical application of dissipation ideas, and a quadratic Lyapunov function suffices; obtaining the ISS property, and in particular operator stability, is far harder. Not surprisingly, the proof involves establishing a dissipation inequality involving a suitable storage function. What is perhaps surprising is that the storage function that is used is only of class C^1, in general not smooth: V is of the form $x'Px + |x|^3$, for some positive definite P. One establishes by means of such a V that the system is ISS. Since the used V admits convex estimates (in the sense discussed in Section 2.7), stronger operator stability conclusions can be obtained. The second example given in Section 2.6 (system (11) and storage function (12)) illustrates the detailed calculations in a very simple case.

The hypotheses in Theorem 4 can be relaxed considerably. For instance, controllability can be weakened, and the result is also valid if instead of SAT one uses a more general bounded saturation function σ which satisfies: (1) near the origin, σ is in a sector $[\kappa_1, \kappa_2]$: $0 < \kappa_1 \leq \frac{\sigma(r)}{r} \leq \kappa_2$ for all $0 < |r| \leq 1$, and (2) $\text{sign}\,(r)\sigma(r) > \kappa > 0$ if $|r| > 1$.

A different line of work concerns linear systems subject to control saturation in the case in which the matrix A is not stable, but still has no eigenvalues with positive real part. This is the case, for instance, if A has a Jordan block of size at least two corresponding to an eigenvalue at the origin (the multiple integrator). In that case, L_p stabilization is not possible, but, since the system is open-loop null-controllable (assuming as in Theorem 4 that the pair (A, B) is controllable, or at least stabilizable as a linear pair), it is realistic to search for a globally stabilizing feedback.

A first result showing that a smooth globally stabilizing feedback always exists was given in work by Sussmann and the author ([19]). A remarkable design in terms of combinations of saturations was supplied by Teel ([22]), for the particular case of single-input multiple integrators, and a general construction based on Teel's ideas was completed recently in work of Sussmann, Yang, and the author ([21]). For simplicity, call a function $\mathbb{R}^n \to \mathbb{R}^m$ each of whose coordinates has the form

$$\varphi_1 x + \alpha_1 \,\text{SAT}[\varphi_2 x + \alpha_2 \,\text{SAT}[\ldots \text{SAT}[\varphi_{s-1} x + \alpha_{s-1} \,\text{SAT}[\varphi_s x]] \ldots]]$$

for some s and some real numbers α_i and linear functionals φ_i a *cascade of saturations*, and one for which coordinates have the form

$$\alpha_1 \,\text{SAT}[\varphi_1 x] + \alpha_2 \,\text{SAT}[\varphi_2 x] + \ldots + \alpha_s \,\text{SAT}[\varphi_s x]$$

a *superposition of saturations*. (In the terminology of artificial neural networks, this last form is a "single hidden layer net.") There are two results, one for each of these controller forms:

Theorem 5 ([21]) *Consider the system (19), where the pair (A, B) is stabilizable and A has no eigenvalues with positive real part. Then there exist a cascade of saturations k and a superposition of saturations ℓ so that $\dot{x} = Ax + B\,\text{SAT}[k(x)]$ and $\dot{x} = Ax + B\,\text{SAT}[\ell(x)]$ are both* GAS. ∎

(The coefficients α_i in the second case can be chosen arbitrarily small, which means that the second result could also be stated as stability of $\dot{x} = Ax + B\ell(x)$ since the saturation is then irrelevant.)

For cascades of saturations, this design proceeds in very rough outline as follows (the superposition case is similar). A preliminary step is to bring the original system (19) to the following composite form:

$$
\begin{aligned}
\dot{z} &= A_1 z + B_1(-Fx + \mathrm{SAT}[u]) \\
\dot{x} &= A_2 x + B_2\, \mathrm{SAT}[u]\,,
\end{aligned}
$$

where F is a matrix which has the property that the system $\dot{x} = A_2 x + B_2\, \mathrm{SAT}[Fx + u]$ is ISS. (An example of such F is provided by the case $\dot{x} = -\mathrm{SAT}[x + u]$, shown earlier to be ISS, and more generally the case treated in Theorem 4.) Further, it is assumed that for each $\varepsilon > 0$ sufficiently small there is a (locally Lipschitz) feedback law k_ε with $|k_\varepsilon(z)| \leq \varepsilon$ for all z and so that $\dot{z} = A_1 z + B_1 k_\varepsilon(z)$ is GAS. Now the feedback law

$$
u = Fx + k_\varepsilon(z)
$$

is so that for small x and ε the z-equation is GAS independently of x (in fact, the x variable is completely cancelled out), and hence the discussion given in connection with Figure 2 applies. Thus the composite system is stabilized, assuming only that the z-subsystem can be stabilized with small feedback. Moreover, $Fx + k_\varepsilon(z)$ has a cascade form provided that k_ε be a saturation of a cascade. These assumptions can be in turn obtained inductively, by decomposing the z equation recursively into lower dimensional subsystems. (More precisely, instead of SAT one may use a scaled version with smaller lower bounds, $\mathrm{SAT}_\delta[r] = \delta\, \mathrm{SAT}[r/\delta]$, and the proof is the same. This provides the small feedback needed in the inductive step.) See [21] for details as well as a far more general result, which allows many other saturation functions σ instead of SAT.

6 Feedback Equivalence

As mentioned earlier, with the concept of ISS, it is possible to prove a general result on feedback equivalence. Consider two systems

$$
\dot{x} = f(x,\, u) \quad \text{and} \quad \dot{x} = g(x,\, u)
$$

with the same state and input value spaces (same n, m). These systems are *feedback equivalent* if there exist a smooth $\| : \mathbb{R}^n \to \mathbb{R}^m$, and an $m \times m$ matrix Γ consisting of smooth functions having $\det \Gamma(x) \neq 0$ for all x, such that

$$
g(x,\, u) = f(x,\, \|(x) + \Gamma(x) u)
$$

for all x and u (see Figure 4). The systems are *strongly* feedback equivalent if this holds with $\Gamma = I$ (see Figure 5).

Strong equivalence is the most interesting concept when studying actuator perturbations, while feedback equivalence is a natural concept in feedback linearization and other design techniques.

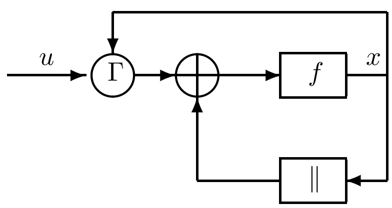


Figure 4: *Feedback Equivalence*

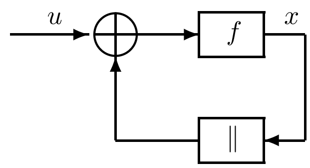


Figure 5: *Strong Feedback Equivalence*

The system (1) is *stabilizable* if there exists a smooth function $\|$ (with $\|(0){=}0$) so that

$$\dot{x} \;=\; f(x, \|(x))$$

is GAS. Equivalently, the system is strongly feedback equivalent to a GAS system. It is ISS-*stabilizable* if it is feedback equivalent to an ISS system.

Theorem 6 ([14, 16]) *The following properties are equivalent, for any system:*

- *The system is stabilizable.*
- *The system is* ISS*-stabilizable.*

For systems affine in controls u (that is, $f(x,u)$ is affine in u) the above are also equivalent to strong feedback equivalence to ISS*.* ∎

7 Input/Ouput Stability (IOS)

Until here, only input to state stability was discussed. It is possible to give an analogous definition for input/output operators. This will be done next, and a result will be stated which shows that this property is equivalent to internal stability under suitable reachability and observability conditions, just as with linear systems (cf. for instance Section 6.3 in [17]).

An *i/o operator* is a causal map $F : L^m_{\infty,e} \to L^p_{\infty,e}$. (More generally, partially defined operators can be studied as well, but since only the stable case will be considered, and since stability implies that F is everywhere defined, there is no need to do so here; see [14] for more details.)

The i/o operator F is *input/output stable* (IOS) if there exist two functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ so that

$$|F(u)(t)| \;\leq\; \beta(\|u_t\|_\infty, t-T) + \gamma(\left\|u^t\right\|_\infty)$$

for all pairs $0 \leq T \leq t$ (a.e.) and all $u \in L^m_{\infty,e}$. Here u_t denotes as earlier the restriction of the input u to $[0,t]$ and u^t denotes its restriction to $[t, +\infty)$, in both cases seen as elements of $L^m_{\infty,e}$ having zero value outside of the considered range. This notion is well-behaved in various senses; for instance, it is closed under composition (serial connection), and $u \to 0$ implies $F(u) \to 0$.

Consider now a control system $\dot{x} = f(x,u)$ with outputs

$$y \;=\; h(x) \tag{20}$$

where $h:\mathbb{R}^n\to\mathbb{R}^p$ is continuous and satisfies $h(0)=0$. With initial state $x_0=0$, this induces an operator

$$F(u)(t) \;:=\; h(x(t,0,u))$$

(a priori only partially defined). The system (1)-(20) is called IOS if this operator is.

The system with outputs (1)-(20) is *well-posed observable* ("strongly" observable in [14]) provided that the following property holds: there exist two functions α_1, α_2 of class $\mathcal{K}$ such that, for each triple of state, control, and output functions on $t\geq 0$

$$(x(\cdot), u(\cdot), y(\cdot))$$

satisfying the equations, the norms of these functions necessarily satisfy

$$\|x\|_\infty \;\leq\; \alpha_1(\|u\|_\infty) + \alpha_2(\|y\|_\infty). \tag{21}$$

Analogously, one has a notion of a *well-posed reachable* system (1). This is a system for which there is a function α_3 of class $\mathcal{K}$ with the following property: for each $x_0\in\mathbb{R}^n$ there exists a time $T>0$ and a control u so that

$$\|u\|_\infty \leq \alpha_3(|x_0|)$$

and so that $x(T,0,u)=x_0$.

For linear systems, these properties are equivalent to observability and reachability from zero respectively. In general, the first one corresponds to the possibility of reconstructing the state trajectory in a regular fashion —similar notions have been studied under various names such as "algebraic observability" or "topological observability" — and the second models the situation where the energy needed to control from the origin to any given state must be in some sense proportional to how far this state is from the origin. The proof of the following result is a routine argument, and is quite similar to the proofs of analogous results in the linear case as well as in the dissipation literature:

Theorem 7 ([14]) *If (1) is* ISS*, then (1)-(20) is* IOS*. Conversely, if (1)-(20) is* IOS*, well-posed reachable, and well-posed observable, then (1) is* ISS*.* ∎

Many variants of the notion of IOS are possible, in particular in order to deal with nonzero initial states ([8, 7]), or to study notions of practical stability, in which convergence to a small neighborhood of the origin is desired. Also of interest, is the study of the IOS (or even ISS) property relative to attracting invariant sets, not necessarily the origin and not even necessarily compact; see ([8, 9]) for instance.

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