Stabilization of polynomially parametrized families of linear systems. The single-input case.

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Abstract

Given a continuous-time family of finite dimensional single input linear systems, parametrized polynomially, such that each of the systems in the family is controllable, there exists a continuously parametrized control law making each of the systems in the family stable.

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1. Introduction.

There has been considerable interest lately in questions dealing with the solution of synthesis problems for linear systems depending on parameters; see for instance [1,4-8], and the references there. Typically, the questions asked involve a local-global passage: if a given problem is solvable for each value of the parameter(s), does there also exist a similarly parametrized family of solutions? Take for instance a family

(1.1) $\dot{x}(t) = A_\lambda x(t) + b_\lambda u(t)$,

where A_λ, b_λ are matrices ($n\times n$ and $n\times 1$ respectively) whose entries are functions of $\lambda = (\lambda_1,...\lambda_r) \in \Re^r$, (we restrict attention here to scalar-input systems,) and consider the *stabilization* problem: to find a parametrized control law $u(t) = k_\lambda x(t)$ such that, for each λ, all solutions of

(1.2) $\dot{x}(t) = (A_\lambda + b_\lambda k_\lambda)x(t)$

converge asymptotically to zero. The problem becomes interesting when an algebra of functions A is specified, with all entries of A and b belonging to A, and it is required that the solution k_λ again have entries over A.

In this note, we shall be especially interested in the case $A = \Re[\lambda] = \Re[\lambda_1,...,\lambda_r]$, although our results will also apply to many other algebras A. Besides being mathematically natural, this problem has in principle a computational interest: once the "off-line" computation of k_λ has been carried out, it is only necessary to store its coefficients, the calculation of the precise k_λ being essentially trivial when a particular λ is given. (If a good polynomial approximation to a given family can be found, this kind of approach povides an alternative to conventional gain-scheduling methods.)

2. Some algebras of functions.

Consider the set $C[\Lambda,\Re]$ of all continuous maps $\Lambda\to\Re$, where Λ is a fixed connected topological space. If f,g are in $C[\Lambda,\Re]$, f>g will mean that $f(\lambda)>g(\lambda)$ *for every* λ, and f>= g that $f(\lambda)>= g(\lambda)$ for each λ; 0, 1 will be used to denote the functions constantly equal to 0, 1 respectively. Thus $C[\Lambda,\Re]$ is an algebra with identity 1.

All results will refer to a fixed but arbitrary subalgebra *A* of $C[\Lambda,\Re]$ which satisfies the following property:

(*) $g\in A, g>0 \Rightarrow (\exists k\in A)\ kg>= 1$.

Typically, $\Lambda\subseteq\Re^r$ for some r; *A* may then be a set of real-analytic, or smooth, or rational, or just continuous functions, in which cases the inequality can be satisfied exactly, with $k:=g^{-1}$ -and our results will be basically trivial in that case. Our interest lies however in the case $\Lambda=\Re^n$, $A = \Re[\lambda] = \Re[\lambda_1,...,\lambda_r]$. These also satisfy (*): by the *reelnullstellensatz* (see [2,3],) g real polynomial >0 implies that there exists a real polynomial k such that $kg = 1+\sum u_i^2$, for some real rational functions u_i, and this implies (*).

It is clear that (*) should be the desired property in the context of stabilization: the one-dimensional system $\dot{x}=x+gu$ ($g=/=0$) is stabilizable with u=kx if and only if 1+gk<0, i.e. if (-k)g>1. Existence of such a stabilizer for every one-dimensional reachable system then implies (*).

(2.1) LEMMA. Assume that c, b_{n-1}, ..., b_0 are in *A*, with c>0. There exists then a $\Psi>0$ in *A* such that, whenever $b_n>= c$ and $\psi>= \Psi$:

(2.2) $\sum_{i=0}^{n} b_i\psi^i >= 1$.

PROOF. Consider first the case n=1. Let c, b_0 be given. Pick $k\in A$ such that kc>= 1, and let

(2.2) $\Psi := [(1-b_0)^2+1]k > 0$.

Take now any $b_1>= c$ and any $\psi>= \Psi$. Since $\Psi b_1 >= \Psi c >= (1+b_0)^2+1 >= 1-b_0$, it follows that $\psi b_1+b_0 >= 1$, as required. The proof is completed by induction on n. Let c, b_{k-1}, ..., b_0 be given, and assume the lemma true for n<= k-1. By the case n=k-1 applied to the subsequence c, b_{k-1}, ..., b_1, there exists a $\Psi'>0$ in *A* such that

(2.3) $b_k\psi^{k-1} + b_{k-1}\psi^{k-2} + ... + b_1 >= 1$

whenever $b_k>= c$ and $\psi>= \Psi'$. Consider now the case n=1 applied to the data 1, b_0: there is then a $\Psi''>0$ in *A* such that

(2.4) $\psi d + b_0 >= 1$

whenever $\psi>= \Psi''$ and d>= 1. Let $\Psi:=\Psi'+\Psi''$. If $\psi>\Psi$, (2.4) holds with d = the left term of (2.3), and the proof is completed.n

3. Hurwitz polynomials.

A polynomial $p = p_\lambda(s) = b_n s^n+b_{n-1}s^{n-1}+...+b_0 \in A[s]$ will be said to be a *Hurwitz* polynomial if $b_n>0$ and, for each $\lambda\in\Lambda$, the polynomial $b_n(\lambda)s^n+...+b_0(\lambda)$ has all its roots with negative real parts. Given any elements b_n, ..., b_0 in *A* we consider the n Hurwitz minors corresponding to the polynomial $p=\sum b_i s^i$:

$$\text{(3.1)} \quad H_i(b_n,b_{n-1},...,b_0) = \begin{matrix} b_{n-1} & b_{n-3} & ... & b_{n-(2i-1)} \\ b_n & b_{n-2} & ... & . \\ 0 & b_{n-1} & ... & . \\ 0 & b_n & ... & . \\ 0 & 0 & ... & . \\ . & . & ... & . \\ . & . & ... & b_{n-i} \end{matrix}$$

where $b_j:=0$ if j<0, i=1,...n. The elements H_i are in *A*. (Strictly speaking, we should include explicitely the order n in the notation for H_i; we omit it for notational simplicity.) By the Hurwitz stability test, a polynomial p as above, with $b_n>0$, is Hurwitz if and only if $H_i(b_n,...,b_0)>0$ for all i=1,...,n. The following lemma is suggested by classical root-locus techniques:

(3.2) LEMMA. Let $p,q\in A[s]$, with q Hurwitz and deg(p) <= deg(q) = n-1. Then, there exists a $\Psi>0$ in *A* such that $s^n+p+\psi q$ is Hurwitz whenever $\psi>= \Psi$.

PROOF. Let $p=\sum_{i<= n-1}a_is^i$, $q=\sum_{i<= n-1}b_is^i$. Fix any i=1,...,n. The main observation is that there exist elements $d_j^{(i)}\in A$, j=0,...,i-1, such that, for each ψ,

(3.4) $H_i(1,a_{n-1}+\psi b_{n-1},...,a_0+\psi b_0)=$

$b_{n-1}H_{i-1}(b_{n-1},b_{n-2},...,b_0)\psi^i + \sum_{j<i}d_j^{(i)}\psi^j$.

(We denote H_0:=1.) To establish the result, we need to prove that all the expressions in (3.4) can be made simultaneously positive for large ψ. Since $\sum b_is^i$ is Hurwitz, $b_{n-1}>0$, so also $c_i:= b_{n-1}H_{i-1}(b_{n-1},...,b_0)>0$ for each i. Applying lemma (2.1) to each set of data $(c_i,d_{i-1}^{(i)},...,d_0^{(i)})$, we obtain $\Psi_1,...,\Psi_n$, all >0 and in *A*, such that, for each i,

(3.5) $c_i\psi^i + \sum_{j<i}d_j^{(i)}\psi^j > 0$

whenever $\psi>\Psi_i$. Let now Ψ be the sum of all the Ψ_i; this satisfies all the requirements.n

4. Stabilization.

Let $A\in A^{n\times n}$, $b\in A^{n\times 1}$. The pair (A,b) is *A-stabilizable with arbitrary convergence rates* if for each $\alpha\in\mathfrak{R}$ there exists a $k\in A^{1\times n}$ such that, for each $\lambda\in\Lambda$, $A_\lambda+b_\lambda k_\lambda$ has all its eigenvalues with real part $<\alpha$. The pair (A,b) is *pointwise controllable* iff (A_λ,b_λ) is controllable for each $\lambda\in\Lambda$. The main result is:

(4.1) THEOREM. (A,b) is *A*-stabilizable with arbitrary convergence rates if and only if it is pointwise controllable.

Necessity follows by elementary system theory. The sufficiency proof will involve a sequence of simplifying arguments. First note that it is enough to prove stabilizability with $\alpha=0$. Indeed, given any α, assume that the resut is known for the case $\alpha=0$. Let $A':=A-\alpha I$. Since (A',b) is again controllable, there is a $k\in A$ such that all eigenvalues of $C_\lambda:=A'_\lambda+b_\lambda k_\lambda$ have negative real parts (i.e., $\chi_C:=\det(sI-C)$ is Hurwitz). Then, all eigenvalues of $A_\lambda+b_\lambda k_\lambda = C_\lambda+\alpha I$ have real part $<\alpha$, as required.

For two pairs of the same dimension n, denote (A,b) < (A’,b’) iff there exists a matrix $T \in A^{n \times n}$ such that AT=TA’, b=Tb’, and $\det(T_\lambda)$=/=0 for all $\lambda \in \Lambda$.

(4.2) LEMMA. Assume that (A,b)<(A’,b’) and that there exists a $k \in A^{1\times n}$ such that χ_{A+bk} is Hurwitz. Then, the same conclusion holds for (A’,b’).

PROOF. Let $k' := kT \in A^{1\times n}$. Since T is pointwise invertible, A+bk and A’+b’k’ have the same characteristic polynomial for each λ, and a fortriori as elements of A.n

(4.3) LEMMA. For any pointwise reachable (A’,b’), there exists an (A,b) < (A’,b’) of the particular form:

$$\textbf{(4.4)}\qquad A \;=\; \begin{array}{ccccc} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & & 1 \\ \alpha_0 & . & . & . & \alpha_{n-1} \end{array} \qquad b \;=\; \begin{array}{c} 0 \\ 0 \\ . \\ 0 \\ \Delta \end{array} \qquad ,$$

with $\Delta := \det(b', A'b', \ldots, (A')^{n-1}b')$.

Assume for a moment that (4.3) has been proved. By (4.2), it is then enough to prove the theorem for (A,b) of the form (4.4). By reachability, Δ_λ=/=0 for all λ. Since $\Delta : \Lambda \to \Re$ is continuous, either $\Delta > 0$ or $\Delta < 0$. The problem is then reduced to proving that, for any $a_{n-1}, \ldots, a_0$ in A, and Δ as above, there exist $k_0, \ldots, k_{n-1}$ in A such that

(4.5) $s^n + (a_{n-1} + \Delta k_{n-1})s^{n-1} + (a_0 + \Delta k_0)$

is Hurwitz. Without loss of generality, we may assume that $\Delta > 0$. Let $g \in A$ be such that $g\Delta >= 1$. Now pick *any* Hurwitz polynomial $b_{n-1}s^{n-1} + \ldots + b_0 \in A[s]$. Apply lemma (3.2) to obtain a $\Psi \in A$ such that the property there is satisfied. Since $\psi := g\Delta\Psi >= \Psi$, this means that (4.5) can be made to be Hurwitz with the choice $k_i := g\Psi b_i$.n

Thus we are only left to prove (4.3). But this is basically what results when one tries to reduce (A’,b’) pointwise to the controllability canonical form, with care not to perform any of the required inversions.

More precisely, assume that A' has characteristic polynomial $s^n-\sum\alpha_i s^i$. Now let S be the matrix whose i-th column, i=1,...,n, is:

(4.6) $-\alpha_i b'-\alpha_{i+1}A'b'-...-\alpha_{n-1}A'^{n-i-1}b'+A'^{n-i}b'$.

For each fixed λ, S_λ is invertible. (However, in general S is not invertible over *A*, unless (A',b') is *ring reachable*.) Arguing pointwise as usual, $S_\lambda{}^{1}A'_\lambda S_\lambda = A_\lambda$ and $b' = S_\lambda(0,...,0,1)^T$ for all λ, where A is as in (4.4). Let T' be the cofactor matrix of S, and let $T:=(-1)^kT'$ if n=2k or n=2k+1. Then, AT=TA' and b=Tb', as desired.n

5. Remarks.

We now describe the relation between the problem studied here and analogous ones considered in the references.

For polynomial families, the main difference lies in the controllability assumptions made on (A_λ,b_λ). If this would be ring reachable, i.e., (A_λ,b_λ) is reachable for every *complex* value $\lambda\in\mathbf{C}^r$, then one can achieve arbitrary characteristic polynomials for $A_\lambda+b_\lambda k_\lambda$ (clear from the above arguments: Δ is a unit). In fact, more interesting results are known for that case, even in the multi-input problem (b is an n×m matrix, m>1): if r=1 one has arbitrary pole assignment ([9]); if r>1 this is still true, but one must employ *dynamic* feedback (see [5], and [6] for the dual, somewhat easier, observer problem). All these results apply also to *discrete time* systems $x(t+1) = A_\lambda x(t) + B_\lambda x(t)$, since the conclusions permit placing poles inside the unit circle. The result in this note, however, *does not* generalize to the discrete case: consider the example $A_\lambda=1$, $B_\lambda=\lambda^2+1$ (n=m=1); this is reachable for all real λ but is not polynomially stabilizable. (For no possible polynomial k_λ is $1+(\lambda^2+1)k_\lambda$ less than 1 for all λ.)

Another possible assumption on (A_λ,B_λ) if stabilization with *arbitrary* convergence rates is not required, is simply stabilizability pointwise. By the results in ([5,6]) the *dynamic* version of this problem is equivalent to the right invertibility of the matrix $[sI-A_\lambda,B_\lambda]$ with respect to the ring of stable transfer functions over $\Re[\lambda]$. When r=1, this property is equivalent to pointwise stabilizability (see [8]), but the problem is open in general. For stabilization over other algebras, see [7].

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