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The effect of sampling on linear equivalence and feedback linearization

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Abstract: We investigate the effect of sampling on linearization for continuous time systems. It is shown that the discretized system is linearizable by state coordinate change for an open set of sampling times if and only if the continuous time system is linearizable by state coordinate change. Also, it is shown that linearizability via digital feedback imposes highly nongeneric constraints on the structure of the plant, even if this is known to be linearizable with continuous-time feedback. For $n = 2$, we show, under the assumption of completeness of $\mathrm{ad}_F G$, that if the discretized system is lineariable by state coordinate change and feedback, then the continuous time affine complete analytic system is linearizable by state coordinate change only. Also, we suggest a method of proof when $n \geq 3$.

Keywords: Feedback linearization; sampling; nonlinear systems; discrete-time systems; discretization.

1. Introduction

An area of research that has attracted some recent interest is that of determining the effect of digital implementation of nonlinear control laws for continuous-time smooth systems. Specifically, there have been characterizations of the preservation of controllability (Sontag [10,12]) and observability (Sontag [11]) under zero-th order hold sampled control, and questions have been raised concerning the effect of such sampling on linear equivalence and feedback linearization techniques. This paper deals with the latter topic.

The problem of linearization of a nonlinear system is that of finding a (nonlinear) state coordinate change (and feedback) such that the resulting closed-loop system behaves as a linear system under the new

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coordinates. Once such a linearizing transformation and feedback are obtained, we can then use linear system theory in order to control the original system. Thus, linearization when applicable is an extremely powerful technique for the development of efficient control laws for nonlinear systems (e.g., see Tarn et al. [16]).

The linearization problem for continuous time systems has been studied extensively (see, e.g., the references in Lee et al. [7]). However, the complexity of the control laws makes almost mandatory the use of computers to perform the necessary on-line calculation. The effect of the discretization on the linearization process has not been fully researched. For example, suppose that a continuous time system (e.g., a robot manipulator) is linearizable by state coordinate change and feedback. Then if a desired feedback is applied, the resulting discretized closed-loop system is no longer a linear one under the new coordinates. This is because the control input is a constant between the sampling times.

Necessary and sufficient conditions for linearizability of the discrete-time system by state coordinate change (and feedback) can be found in Grizzle [2], Jakubzyk [5], Lee et al. [7], and Lee and Marcus [8]. Some work on the effects of sampling on linearizability has been reported in Grizzle [2] and Grizzle and Kokotovic [3]. Much of the research in this area has been stimulated by the work of J. Grizzle. In particular, he has conjectured the following [2].

Conjecture [2]. *Let* Σ: $\dot{x}=f(x, u)$ *be a single-input analytic control system on* $\mathbb{R}^n$, $n \geq 2$, *such that* $f(0, 0)=0$, *and let* Σ_h: $x_{k+1}=f_h(x_k, u_k)$ *be its sampled data representation for a sampling interval* h. *Then* Σ_h *is locally feedback linearizable for an open set of sampling times* (*i.e., it is sampled feedback linearizable*) *if and only if* Σ *is state-equivalent to a controllable linear system.*

This conjecture is incorrect if we consider a general nonlinear system of the form $\dot{x}=f(x, u)$. (Suppose that $\dot{x}=F(x)+G(x)u$ is state-equivalent to a controllable linear system. Let $f(x, u)=F(x)+G(x)(u+u^3)$.) Thus we consider an affine control system of the form $\dot{x}=F(x)+G(x)u$ and furthermore, we assume that $F(x)$ and $G(x)$ are analytic vector fields.

In Section 2, we define our notation and review some background material. In Section 3, it is shown that the discretized system is linearizable by state coordinate change for an open set of sampling times if and only if the continuous time system is linearizable by state coordinate change. In Section 4, we show that sampled feedback linearizability implies that the continuous time system is also lineariable by state coordinate change and feedback, and we derive an interesting necessary condition for sampled feedback linearizability. Finally, under the assumption of the completeness of $\mathrm{ad}_F G$, we show that J. Grizzle's conjecture is true for affine control systems when $n=2$ and suggest a method of proof when $n \geq 3$. Some preliminary aspects of this work were reported in [22].

2. Preliminaries and definitions

Consider a nonlinear continuous time control system of the form

$$\dot{x}(t)=F(x(t))+G(x(t))u(t) \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $F(x)$, $G(x)$ are analytic vector fields with $F(0)=0$. Then the discretized system of (2.1) with sampling interval $h>0$ is as follows:

$$x(t+h)=f_h(x(t), u(t)) \tag{2.2}$$

where $f_h(x, u)=\Phi_h^{F+uG}(x)$, with Φ^X denoting the flow of X. Notice that $f_h(0, 0)=0$.

Definition 2.1. The discrete time system (2.2) (or the continuous time system (2.1)) is *linearizable by state coordinate change* if there exists a smooth state coordinate change around $0 \in \mathbb{R}^n$ which transforms it into a reachable linear system.

Definition 2.2. The discrete time system (2.2) (or the continuous time system (2.1)) is *linearizable by state coordinate change and feedback* if there exists a smooth nonsingular feedback $u = \gamma(x, v)$ such that the closed-loop system is linearizable by state coordinate change.

Definition 2.3. The system (2.1) is *sampled linearizable by state coordinate change* if there exists $\delta > 0$ such that for every $h \in (0, \delta)$, the system (2.2) is linearizable by state coordinate change.

Definition 2.4. The system (2.1) is *sampled feedback linearizable* if there exists $\delta > 0$ such that for every $h \in (0, \delta)$, the system (2.2) is linearizable by state coordinate change and feedback.

Since the problem of linearization is essentially local in nature, the results presented here will be primarily in terms of local coordinates. Define, for $k \geq 1$,

$$V_h^k(x, u) = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \left(\Phi^F_{(k-1)h} \Phi_h^{F+(u+\epsilon)G} \Phi_{-h}^{F+uG} \Phi^F_{-(k-1)h} \right)(x).$$

Since $f_h(\cdot, u)$ is a diffeomorphism for each h, u, $V_h^k(\cdot, u)$ is a (locally) well-defined vector field on $\mathbb{R}^n$ for each h, u (similar vector fields are also used in Sontag [11] and Jacubczyk and Sontag [13]). Now (see Goodman [1], Normand-Cyrot [9], Sontag [12], Varadarajan [17], Jakubczyk and Normand-Cyrot [18], Monaco and Normand-Cyrot [19–21]),

$$V_h^1(\cdot, u) = \sum_{i=0}^{\infty} \frac{(-1)^i h^{i+1}}{(i+1)!} \mathrm{ad}^i_{F+uG} G = \sum_{j=0}^{\infty} u^j \beta_j(h). \tag{2.3}$$

The vector fields $\beta_j(h)$ take the form

$$\beta_j(h) = \sum_{i=j+1}^{\infty} (-1)^{i+1} h^i \beta_j^i, \; j \geq 0,$$

where $\beta_0^i = (1/i!)\mathrm{ad}_F^{i-1} G$, $i \geq 1$, and $\beta_j^i = (1/i)(\mathrm{ad}_F \beta_j^{i-1} + \mathrm{ad}_G \beta_{j-1}^{i-1})$, $j \geq 1$, $i \geq j+2$, with $\beta_j^{j+1} = 0$ for $j \geq 1$.

From the characterization of linear equivalence and feedback linearizability for discrete time systems given in Grizzle [2], Jakubczyk [5], and Lee et al. [7], we have the following basic results.

Theorem 2.1. *The system* (2.1) *is sampled linearizable by state coordinate change if and only if there exists* $\delta > 0$ *such that, for every* $h \in (0, \delta)$,
(i) $\{V_h^1(0, 0), V_h^2(0, 0), \ldots, V_h^n(0, 0)\}$ *are linearly independent*;
(ii) $V_h^1(x, u_1) = V_h^1(x, u_2)$ *for all* (x, u_1), (x, u_2) *in a neighborhood of* (0, 0);
(iii) $[V_h^i, V_h^j] = 0$ *for* $1 \leq i, j \leq n+1$.

Theorem 2.2. *The system* (2.1) *is sampled feedback linearizable if and only if there exists* $\delta > 0$ *such that, for every* $h \in (0, \delta)$,
(i) $\{V_h^1(0, 0), V_h^2(0, 0), \ldots, V_h^n(0, 0)\}$ *are linearly independent*;
(ii) $V_h^1(\cdot, u_1)$ *and* $V_h^1(\cdot, u_2)$ *are parallel for all* u_1, u_2 *near* 0;
(iii) $\mathrm{span}\{V_h^1, V_h^2, \ldots, V_h^i\}$ *are involutive distributions for* $1 \leq i \leq n-1$.

Remark 2.1. Since $\Phi_h^F(\cdot)$ is a diffeomorphism, the following follows easily: if $V_h^1(x, u)$ is not a function of u in a neighborhood of (0, 0), then the same is true of $V_h^p(x, u)$. Similarly, if $V_h^1(\cdot, u)$ are parallel for every u near 0, then for $p \leq 2$, $V_h^p(\cdot, u)$ are also parallel for every u near 0.

Remark 2.2. Proofs of Theorems 2.1 and 2.2 follow easily from the results of [7] once we notice that
(i) $\Phi_h^{F+uG}(x)|_*(\partial/\partial u)$ is a well-defined vector field if and only if $V_h^1(x, u)$ is not a function of u;
(ii) $\Phi_h^{F+uG}(x)|_*(\mathrm{sp}\{\partial/\partial u\})$ is a well-defined distribution if and only if $V_h^1(x, u_1)$ and $V_h^1(x, u_2)$ are parallel for all u_1 and u_2.

3. Sampled linear equivalence

It is evident that if the system (2.1) is linearizable by state coordinate change only, then the discretized system (2.2) is linearizable by the same state coordinate change. The following theorem establishes a converse to this fact.

Theorem 3.1. *If the system* (2.1) *is sampled linearizable by state coordinate change, then the system* (2.1) *is linearizable by state coordinate change.*

Proof. Suppose that (2.1) is sampled linearizable by state coordinate change. Then, by Theorem 2.1 and Remark 2.1, there exists $\delta > 0$ such that, for every $h \in (0, \delta)$, $V_h^p(x, u)$ is not a function of u in a neighborhood of (0, 0) and is thus a well-defined vector field on $\mathbb{R}^n$ for $p \geq 1$. Define the vector field, for $p \geq 1$,

$$Y_h^p := \sum_{i=1}^{\infty} \frac{h^i}{i!}(-1)^{i+1}\{p^i - (p-1)^i\}\mathrm{ad}_F^{i-1}G. \tag{3.1}$$

Then it is easily seen (Varadarajan [17]), that for $p \geq 1$, $V_h^p = Y_h^p$. Thus for $p \geq 1$,

$$X^p := \sum_{j=1}^{p} V_h^j = \sum_{i=1}^{\infty} \frac{h^i}{i!}(-1)^{i+1} p^i \mathrm{ad}_F^{i-1}G.$$

Since the coefficients of $\{X^p,\ p = 1,\dots,n\}$ in terms of $\{(h^p/p!)\mathrm{ad}_F^{p-1}G,\ p = 1,\dots,n\}$ form a Vandermonde matrix, Theorem 2.1(i) implies that $\{X^p,\ p = 1,\dots,n\}$, and hence $\{G(0), (\mathrm{ad}_F G)(0),\dots, (\mathrm{ad}_F^{n-1}G)(0)\}$ are linearly independent. Also, from (2.3) we see that Theorem 2.1(ii) implies that $\beta_1(h) = 0$ for every $h \in (0, \delta)$. Since

$$\beta_1^3 = \frac{1}{3!}\mathrm{ad}_G(\mathrm{ad}_F G),$$

it follows $\mathrm{ad}_G(\mathrm{ad}_F G) = 0$. Now assume that $\mathrm{ad}_G(\mathrm{ad}_F^i G) = 0$ for $1 \leq i \leq k$. Then

$$\beta_1^{k+3} = \frac{1}{(k+3)!}\mathrm{ad}_G\left(\mathrm{ad}_F^{k+1}G\right)$$

Hence $\mathrm{ad}_G(\mathrm{ad}_F^i G) = 0$ for $i \geq 1$. This and the linear independence of $\{G(0), (\mathrm{ad}_F G)(0),\dots,(\mathrm{ad}_F^{n-1}G)(0)\}$ imply that (2.1) is locally linearizable by state coordinate change (Sussmann [15]). □

However, if the system (2.1) is not locally linearizable by state coordinate change but is locally linearizable by state coordinate change and feedback, then the discretized system (2.2) is no longer guaranteed to be locally linearizable by state coordinate change and feedback. This is because the control $u(t)$ must be a constant between the sampling times. When $n = 1$, (2.2) is locally linearizable by state coordinate change and feedback if $G(0) \neq 0$. Thus, in the next section we assume $n \geq 2$ and investigate the effect of sampling on linearizability by state coordinate change and feedback.

4. Necessary conditions for sampled feedback linearizability

Theorem 4.1. *If the system* (2.1) *is sampled feedback linearizable, then the system* (2.1) *is feedback linearizable.*

Proof. For the controllability condition, see Theorem 3.1. Consider $V_h^p(x, u)$, for $p \geq 1$; Theorem 2.2(ii) and Remark 2.1 do not imply that this is a well-defined vector field on $\mathbb{R}^n$, but that it is a well defined distribution, and it is easily seen that, for (x, u) in a neighborhood of $(0, 0)$ and $p \geq 1$,

$$\mathrm{span}\{V_h^p(x, u)\} = \mathrm{span}\{Y^p(x)\},$$

where $\{Y^p\}$ is the vector field defined in (3.1). Now, let $\overline{X^p} = \Sigma_{j=1}^p Y^j$ for $1 \leq p \leq n$. Then it follows, as in the proof of Theorem 3.1, that Theorem 2.2 (iii) implies that $\{G, \mathrm{ad}_F G, \ldots, \mathrm{ad}_F^{n-2} G\}$ is an involutive distribution. Hence, the system (2.1) is feedback linearizable (Hunt, Su, and Meyer [4], Jakubczyk and Respondek [6], and Su [14]). □

Condition (ii) of Theorem 2.2 has not been used fully in deriving the necessity of the conditions in Theorem 4.1. In the following, we obtain stronger necessary conditions by further using condition (ii) of Theorem 2.2.

Theorem 4.2. *If the system* (2.1) *is sampled feedback linearizable, then* $\mathrm{ad}_G \mathrm{ad}_F G = \alpha_1 G$ *for some analytic scalar function* α_1, *and* $\mathrm{ad}_G^2 \mathrm{ad}_F G = 0$.

Proof. By (ii) of Theorem 2.2 and (2.3), the vector fields $\{\beta_p(h),\ p \geq 0\}$ are parallel for every $h \in (0, \delta)$. In this theorem, we use only the fact that β_0, β_1, and β_2 should be parallel. Using the fact that β_0 and β_1 are parallel, it follows that $\mathrm{ad}_G \mathrm{ad}_F G = \alpha_1 G$ for some analytic scalar function α_1. Note that

$$\beta_1^3 = \frac{1}{3!}\mathrm{ad}_G \mathrm{ad}_F G = \frac{1}{3!}(\alpha_1, G), \qquad \beta_1^4 = \frac{1}{4}(\mathrm{ad}_F \beta_1^3 + \mathrm{ad}_G \beta_0^3) = \frac{1}{4!}(2L_F(\alpha_1)G + 2\alpha_1 \mathrm{ad}_F G),$$

$$\beta_2^4 = \frac{1}{4}\mathrm{ad}_G \beta_1^3 = \frac{1}{4!}L_G(\alpha_1)G,$$

$$\beta_2^5 = \frac{1}{5}(\mathrm{ad}_F \beta_2^4 + \mathrm{ad}_G \beta_1^4) = \frac{1}{5!}\{(L_F L_G(\alpha_1) + 2L_G L_F(\alpha_1) + 2\alpha_1^2)G + 3L_G(\alpha_1)\mathrm{ad}_F G\}.$$

Now using the fact that β_0 and β_2 are parallel, and the fact that the determinant of a matrix is a multilinear function of the columns, it follows that

$$\begin{aligned} 0 &= \det[\beta_0\ \beta_2\ \mathrm{ad}_F^2 G\ \mathrm{ad}_F^3 G \cdots \mathrm{ad}_F^{n-1} G] \\ &= -h^5 \det[\beta_0^1\ \beta_2^4\ \mathrm{ad}_F^2 G \cdots \mathrm{ad}_F^{n-1} G] + h^6 \sum_{i=1}^{2} \det[\beta_0^i\ \beta_3^{6-i}\ \mathrm{ad}_F^2 G \cdots \mathrm{ad}_F^{n-1} G] + \mathrm{O}(h^7) \end{aligned}$$

for every $h \in (0, \delta)$. Thus

$$0 = \sum_{i=1}^{2} \det[\beta_0^i\ \beta_2^{6-i}\ \mathrm{ad}_F^2 G \cdots \mathrm{ad}_F^{n-1} G] = \frac{1}{5!2}L_G(\alpha_1)\det[G\ \mathrm{ad}_F G\ \mathrm{ad}_F^2 G \cdots \mathrm{ad}_F^{n-1} G].$$

Since $\det[G\ \mathrm{ad}_F G \cdots \mathrm{ad}_F^{n-1} G] \neq 0$, $L_G(\alpha_1) = 0$, which implies that $\mathrm{ad}_G^2 \mathrm{ad}_F G = 0$. □

By using a simple induction argument and the Jacobi identity, we can conclude the following.

Corollary 4.3. *If the system* (2.1) *is sampled feedback linearizable, then every Lie bracket with two more G's than F's must vanish identically.*

5. A special case

In this section, we assume that $\mathrm{ad}_F G$ is a complete vector field and show that Grizzle's conjecture is true for affine control systems when $n = 2$ and suggest a method of proof when $n \geq 3$. As a by-product, we obtain stronger necessary conditions than those in Section 4 for sampled feedback linearizability.

Lemma 5.1. *Suppose that* (2.1) *is sampled feedback linearizable, and assume that* $\mathrm{ad}_F G$ *is a complete vector field. Then* $\mathrm{ad}_G \mathrm{ad}^i_F G = 0$ *for* $1 \le i \le 2$.

Proof. By Theorem 4.2, $\beta_2^4 = 0$ and $\beta_2^5 = (1/5!)(2L_G L_F(\alpha_1) + 2\alpha_1^2)G$. By using the Jacobi identity, it can be easily shown that

$$\begin{aligned}\mathrm{ad}_G \mathrm{ad}_G \mathrm{ad}^3_F G &= \mathrm{ad}_G \mathrm{ad}_F \mathrm{ad}_G \mathrm{ad}^2_F G - \mathrm{ad}_G \mathrm{ad}_{\mathrm{ad}_F G} \mathrm{ad}^2_F G \\ &= \mathrm{ad}_G \mathrm{ad}_F \mathrm{ad}_G \mathrm{ad}^2_F G - \mathrm{ad}_{\mathrm{ad}_F G} \mathrm{ad}_G \mathrm{ad}^2_F G + \mathrm{ad}_{\mathrm{ad}^2_F G} \mathrm{ad}_G \mathrm{ad}_F G.\end{aligned}$$

Thus $\mathrm{ad}_G \mathrm{ad}_G \mathrm{ad}^3_F G = c_1 G + 3L_G L_F(\alpha_1)\mathrm{ad}_F G$ for some scalar function c_1. Now, note that

$$\beta_1^5 = \frac{1}{5}\left(\mathrm{ad}_F \beta_1^4 + \mathrm{ad}_G \beta_0^4\right) = \frac{1}{5!}\left(2L_F^2(\alpha_1)G + 4L_F(\alpha_1)\mathrm{ad}_F G + 2\alpha_1 \mathrm{ad}^2_F G + \mathrm{ad}_G \mathrm{ad}^3_F G\right).$$

Thus $\beta_2^6 = (1/6)(\mathrm{ad}_F \beta_2^5 + \mathrm{ad}_G \beta_1^5) = c_2 G + (1/6!)(9L_G L_F(\alpha_1) + 4\alpha_1^2)\mathrm{ad}_F G$ for some scalar function c_2. Now consider

$$\begin{aligned}0 &= \det\left[\beta_0 \ \beta_2 \ \mathrm{ad}^2_F G \cdots \mathrm{ad}^{n-1}_F G\right] \\ &= h^6 \det\left[\beta_0^1 \ \beta_2^5 \ \mathrm{ad}^2_F G \cdots \mathrm{ad}^{n-1}_F G\right] - h^7 \sum_{i=1}^{2} \det\left[\beta_0^i \ \beta_2^{7-i} \ \mathrm{ad}^2_F G \cdots \mathrm{ad}^{n-1}_F G\right] + O(h^8).\end{aligned}$$

Thus

$$0 = \sum_{i=1}^{2} \det\left[\beta_0^i \ \beta_2^{7-i} \ \mathrm{ad}^2_F G \cdots \mathrm{ad}^{n-1}_F G\right] = \frac{1}{6!}\left(3L_G L_F(\alpha_1) - 2\alpha_1^2\right) \det\left[G \ \mathrm{ad}_F G \cdots \mathrm{ad}^{n-1}_F G\right].$$

Therefore, $L_G L_F(\alpha_1) = \frac{2}{3}\alpha_1^2$. Since $L_{\mathrm{ad}_F G}(\alpha_1) = L_F L_G(\alpha_1) - L_G L_F(\alpha_1)$ and $L_G(\alpha_1) = 0$, $L_{\mathrm{ad}_F G}(\alpha_1) = -\frac{2}{3}\alpha_1^2$. By assumption, $\mathrm{ad}_F G$ is a complete vector field. Therefore, $\alpha_1(x) = 0$, which implies $\mathrm{ad}_G \mathrm{ad}_F G = 0$. By the Jacobi identity,

$$\mathrm{ad}_G \mathrm{ad}^2_F G = \mathrm{ad}_F \mathrm{ad}_G \mathrm{ad}_F G - \mathrm{ad}_{\mathrm{ad}_F G} \mathrm{ad}_F G = \mathrm{ad}_F \mathrm{ad}_G \mathrm{ad}_F G = 0. \qquad \square$$

Remark 5.1. By using the Jacobi identity, it can be easily shown that $\mathrm{ad}_G \mathrm{ad}^i_F G = 0$ for $1 \le i \le 2$ implies that every bracket with the same number or more G's than F's must vanish identically.

In order to show that sampled feedback linearizability implies feedback linearizability of (2.1), we must prove that $\mathrm{ad}_G \mathrm{ad}^j_F G = 0$ for $j \ge 1$; Lemma 5.1 shows this for $j = 1, 2$. We would like to prove this for all $j \ge 1$ by induction; so we assume that it holds for $j \ge k$, and derive the consequences.

Lemma 5.2. *Suppose that* (2.1) *is sampled feedback linearizable. Let* $k \ge 2$ *and assume* $\mathrm{ad}_G \mathrm{ad}^i_F G = 0$ *for* $i \le k$. *Then*

(i) $\beta_1^i = 0$ *for* $i \le k+2$ *and* $\beta_1^{k+3} = (1/(k+3)!)\mathrm{ad}_G \mathrm{ad}^{k+1}_F G$;
(ii) $\mathrm{ad}_G \mathrm{ad}^{k+1}_F G = \alpha_{k+1} G$ *for some analytic function* α_{k+1};
(iii) $L_G(\alpha_{k+1}) = 0$.

Proof. (i) and (ii) are obvious. By using the Jacobi identity, it is easy to show that $\mathrm{ad}_G \mathrm{ad}^{k+2}_F G = \frac{1}{2}(k+2)\mathrm{ad}_F \mathrm{ad}_G \mathrm{ad}^{k+1}_F$. Note that

$$\beta_1^{k+3} = \frac{1}{(k+1)!}\alpha_{k+1} G, \qquad \beta_1^{k+4} = \frac{1}{(k+4)!}\left\{\frac{k+4}{2} L_F(\alpha_{k+1})G + \frac{k+4}{2}\alpha_{k+1}\mathrm{ad}_F G\right\},$$

$$\beta_2^{k+i} = 0 \quad \text{for } i \le 3, \qquad \beta_2^{k+4} = \frac{1}{k+4}\mathrm{ad}_G \beta_1^{k+3} = \frac{1}{(k+4)!} L_G(\alpha_{k+1})G,$$

$$\beta_2^{k+5} = \frac{1}{k+5}\{\mathrm{ad}_F\beta_2^{k+4} + \mathrm{ad}_G\beta_1^{k+4}\}$$
$$= \frac{1}{(k+5)!}\left\{\left(L_FL_G(\alpha_{k+1}) + \frac{k+4}{2}L_GL_F(\alpha_{k+1})\right)G + \frac{k+6}{2}L_G(\alpha_{k+1})\mathrm{ad}_FG\right\}.$$

Now consider

$$0 = \det[\beta_0\ \beta_2\ \mathrm{ad}_F^2G \cdots \mathrm{ad}_F^{n-1}G]$$
$$= h^6\det[\beta_0^1\ \beta_2^5\ \mathrm{ad}_F^2G \cdots \mathrm{ad}_F^{n-1}G] - h^7\sum_{i=1}^{2}\det[\beta_0^i\ \beta_2^{7-i}\ \mathrm{ad}_F^2G \cdots \mathrm{ad}_F^{n-1}G] + \mathrm{O}(h^8).$$

Thus

$$0 = \sum_{i=1}^{2}\det[\beta_0^i\ \beta_2^{k+6-i}\ \mathrm{ad}_F^2G \cdots \mathrm{ad}_F^{n-1}G] = \frac{1}{(k+5)!2}L_G(\alpha_{k+1})\det[G\ \mathrm{ad}_FG \cdots \mathrm{ad}_F^{n-1}G].$$

Hence $L_G(\alpha_{k+1}) = 0$. □

Remark 5.2. Under the assumption of Lemma 5.2, if we can show that $\alpha_{k+1} = 0$, then (2.1) is locally linearizable by state coordinate change. Now we show that $\alpha_{k+1} = 0$ when $n = 2$ and $k = 2$.

Theorem 5.3. *Let $n = 2$, and assume that ad_FG is a complete vector field. The system (2.1) is sampled feedback linearizable if and only if it is locally linearizable by state coordinate change only.*

Proof. (Sufficiency) Obvious by Theorem 3.1.

(Necessity) By Lemma 5.1, $\mathrm{ad}_G\mathrm{ad}_F^iG = 0$ for $1 \le i \le 2$. By Lemma 5.2, $\mathrm{ad}_G\mathrm{ad}_F^3G = \alpha_3G$ and $L_G(\alpha_3) = 0$. Therefore $\beta_2^i = 0$ for $i \le 6$ and $\beta_2^7 = (1/7!)\{3L_GL_F(\alpha_3)G\}$. Let $\beta_0^q = \Sigma_{i=1}^2\epsilon_i^q\mathrm{ad}_F^{i-1}G$ and $\beta_1^q = \Sigma_{i=1}^2\lambda_i^q\mathrm{ad}_F^{i-1}G$. By considering $0 = \Sigma_{i=1}^3\det[\beta_0^i\ \beta_1^{8-i}]$ we obtain

$$\beta_1^7 = \frac{1}{7!}\left\{c_3G + \left(\frac{21}{2}L_F(\alpha_3) + 42\alpha_3\epsilon_2^3\right)\mathrm{ad}_FG\right\}$$

for some scalar function c_3. Thus

$$\beta_2^8 = \frac{1}{8!}\left\{c_4G + \frac{27}{2}L_GL_F(\alpha_3)\mathrm{ad}_FG\right\}$$

for some scalar function c_4, because $L_G(\alpha_3) = 0$ and $L_G(\epsilon_2^3) = 0$. By considering

$$0 = \sum_{i=1}^{2}\det[\beta_0^i\ \beta_2^{q-i}] = \frac{3}{8!2!}L_GL_F(\alpha_3)\det[G\ \mathrm{ad}_FG],$$

it follows $L_GL_F(\alpha_3) = 0$. Since $L_G(\alpha_3) = 0$, $L_{\mathrm{ad}_FG}(\alpha_3) = 0$. Therefore α_3 is a constant. Using the Jacobi identity, it can be easily shown that

$$\beta_2^8 = \frac{1}{8!}\left\{\sum_{i=0}^{2}\mathrm{ad}_G\mathrm{ad}_F^i\mathrm{ad}_G\mathrm{ad}_F^{5-i}G\right\} = 0, \qquad \beta_2^9 = \frac{1}{9!}\left\{\sum_{i=0}^{3}\mathrm{ad}_G\mathrm{ad}_F^i\mathrm{ad}_G\mathrm{ad}_F^{6-i}G\right\} = \frac{1}{9!}\{18\alpha_3^2G\}.$$

Consider

$$0 = \sum_{i=1}^{5}\det[\beta_0^i\ \beta_1^{10-i}] = \sum_{\lambda=1}^{5}\{\epsilon_1^i\lambda_2^{10-i} - \epsilon_2^i\lambda_1^{10-i}\}\det[G\ \mathrm{ad}_FG].$$

Note that $\epsilon_1^1 = 1$, $\epsilon_2^1 = 0$, $\epsilon_1^2 = 0$, and $\epsilon_2^2 = \frac{1}{2}$. Thus

$$0 = \sum_{i=1}^{5} \{\epsilon_1^i \lambda_2^{10-i} - \epsilon_2^i \lambda_1^{10-i}\} = \lambda_2^9 - \tfrac{1}{2}\lambda_1^8 + \sum_{i=3}^{5} \epsilon_1^i \lambda_2^{10-i} - \sum_{i=3}^{5} \epsilon_2^i \lambda_1^{10-i}.$$

Thus,

$$\lambda_2^9 = \tfrac{1}{2}\lambda_1^8 + \sum_{i=3}^{5} \epsilon_2^i \lambda_1^{10-i} - \sum_{i=3}^{5} \epsilon_1^i \lambda_2^{10-i}$$

and

$$L_G(\lambda_2^9) = \tfrac{1}{2}L_G(\lambda_1^8) + \sum_{i=3}^{5} L_G(\epsilon_2^i)\lambda_1^{10-i} + \sum_{i=3}^{5} \epsilon_2^i L_G(\lambda_1^{10-i}) - \sum_{i=3}^{5} L_G(\epsilon_1^i)\lambda_2^{10-i} - \sum_{i=3}^{5} \epsilon_1^i L_G(\lambda_2^{10-i}).$$

Since $\beta_2^q = 0$ for $q \le 8$, $L_G(\lambda_1^{10-i}) = L_G(\lambda_2^{10-i}) = 0$ for $3 \le i \le 5$. Also, since $\mathrm{ad}_G \mathrm{ad}_F^2 G = 0$, $\mathrm{ad}_G \mathrm{ad}_F^3 G = \alpha_3 G$, and $\mathrm{ad}_G \mathrm{ad}_F^4 G = 2\alpha_3 \mathrm{ad}_F G$, it follows $L_G(\epsilon_1^3) = L_G(\epsilon_2^3) = 0$, $L_G(\epsilon_1^4) = \alpha_3/4!$, $L_G(\epsilon_2^4) = 0$, $L_G(\epsilon_1^5) = 0$, and $L_G(\epsilon_2^5) = 2\alpha_3/5!$. Also note that $\lambda_1^5 = 2\alpha_3/5!$ and $\lambda_2^6 = \alpha_3/5!2$. Thus $L_G(\lambda_2^9) = \frac{1}{2}L_G(\lambda_1^8) - \alpha_3^2/5!5!2$. Since $\beta_2^9 = \frac{1}{9}\mathrm{ad}_G \beta_1^8 = 18/9!\alpha_3^2 G$, it follows $L_G(\lambda_1^8) = (18/8!)\alpha_3^2$. Therefore,

$$\begin{aligned}
\beta_2^{10} &= \tfrac{1}{10}(\mathrm{ad}_F \beta_2^9 + \mathrm{ad}_G \beta_1^9) \\
&= \frac{1}{10}\left\{c_5 G + \left(\frac{18}{9!}\alpha_3^2 + \frac{18}{8!2}\alpha_3^2 - \frac{1}{5!5!2}\alpha_3^2\right)\mathrm{ad}_F G\right\} \\
&= \frac{1}{10}c_5 G + \frac{432}{10!5}\alpha_3^2 \mathrm{ad}_F G,
\end{aligned}$$

$$0 = \sum_{i=1}^{2} \det[\beta_0^i \ \beta_2^{11-i}] = \frac{-18}{10!5}\alpha_3^2 \det[G \ \mathrm{ad}_F G].$$

Therefore $\alpha_3 = 0$, i.e. $\mathrm{ad}_G \mathrm{ad}_F^3 G = 0$. Hence, (2.1) is locally linearizable by state coordinate change (Sussmann [15]). □

Remark 5.3. We conjecture that when $n \ge 3$ (and with the assumption of completeness of $\mathrm{ad}_F^j G$, $j = 1, \ldots, n-1$), Theorem 5.3 is true and that the same method of proof works.

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