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The Lattice of Minimal Realizations of Response Maps Over Rings

Eduardo D. Sontag*

Department of Mathematics, Rutgers University, New Brunswick, N.J. 08903

Abstract. A lattice characterization is given for the class of minimal-rank realizations of a linear response map defined over a (commutative) Noetherian integral domain. As a corollary, it is proved that there are only finitely many nonisomorphic minimal-rank realizations of a response map over the integers, while for delay-differential systems these are classified by a lattice of subspaces of a finite-dimensional real vector space.

1. Definitions and Notations

The following notational conventions hold throughout the paper:

R is a fixed (commutative) Noetherian integral domain, Q its quotient field.

"Module" means R-module, "linear" means R-linear.

For any module M, M' is the module $\mathrm{Hom}_R(M,R)$;
rank $M := \dim_Q(M \otimes Q)$.

Definition 1.1. *Let m,p be positive integers. A* response map *(over R) is an infinite sequence $f=(A_1,A_2,A_3,\ldots)$ of $p\times m$ matrices over R. The* rank *of f is the Q-rank of the (block) "behavior" or Hankel matrix*

$$\underline{H}(f) := \begin{bmatrix} A_1 & A_2 & A_3 & \cdots \\ A_2 & A_3 & A_4 & \cdots \\ A_3 & A_4 & A_5 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}.$$

Let f be an arbitrary but fixed response map of *finite* rank.

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Definition 1.2. $\Sigma=(F,G,H)$ *is a* minimal-rank realization (*over* R) of f *iff* X *is a torsion-free R-module with* $\operatorname{rank} X=\operatorname{rank} f$, $F:X\to X$, $G:R^m\to X$ *and* $H:X\to R^p$ *are linear, and* $A_i=HF^{i-1}G$ *for all* i.

Note that, for any Σ as above, $\Sigma\otimes Q:=(X\otimes Q, F\otimes Q, G\otimes Q, H\otimes Q)$ is a minimal-rank realization (over Q) of f, when f is seen as a response map over Q. It follows from standard facts on realization theory over fields (see for instance Kalman, Falb and Arbib [1969, Ch. 10]) that $\operatorname{rank} f$ is the minimal possible value for rank X_Σ, for *any* realization of f. This justifies the above terminology.

For each Σ as in (1.2), $\Sigma':=(X',F',H',G')$ is a minimal-rank realization of $f':=(A_1',A_2',A_3',\ldots)$. Dual systems Σ' appear here for purely technical purposes, but they are of fundamental importance in studying questions of regulation (duality of reachability and observability); see Ching and Wyman [1978] and Sontag [1978] for further discussions of duality.

Minimal-rank realizations of a fixed f form a category when morphisms $T:\Sigma_1\to\Sigma_2$ are defined as linear maps $T:X_1\to X_2$ with $TG_1=G_2$, $TF_1=F_2T$, and $H_2T=H_1$. Denote by $\mathfrak{M}\mathfrak{R}(f)$ the set of isomorphism classes of minimal-rank factorizations of f. By a slight abuse of notation, Σ will denote both a realization and its corresponding isomorphism class in $\mathfrak{M}\mathfrak{R}(f)$.

2. Results

The sets $\mathfrak{M}\mathfrak{R}(f)$ are characterized in this section as lattices of submodules of finitely generated torsion modules. When R is a principal-ideal domain, a minimal-rank realization is free (i.e., $X_\Sigma=R^n$ for some n), so elements of $\mathfrak{M}\mathfrak{R}(f)$ correspond to minimal-size matrix realizations (modulo changes of basis in R^n). Of particular applied interest are the cases $R=$integers and $R=$polynomial ring in one variable over the reals. The former corresponds to the modelling of linear systems in a digital device, where all parameters involved are necessarily integral. In this case, since a finitely generated torsion module is finite, one concludes from the results presented here that there are only finitely many nonisomorphic minimal realizations of any given f. On the contrary, when R is a polynomial ring such modules are always infinite (unless trivial), since they are finite-dimensional vector spaces. This case, exemplified in the next section, corresponds to the modelling of systems described by delay-differential equations; Theorem (2.5) gives a characterization of the (possibly infinite) class $\mathfrak{M}\mathfrak{R}(f)$. Other rings of system-theoretic interest are described in Sontag [1976].

Lemma 2.1. *Let* $T:\Sigma_1\to\Sigma_2$ *be a morphism. Then,*

(i) $T:X_1\to X_2$ *is one-to-one, and*

(ii) T *is the unique morphism from* Σ_1 *to* Σ_2.

Proof. Consider $T\otimes Q:\Sigma_1\otimes Q\to\Sigma_2\otimes Q$. Since $\Sigma_1\otimes Q$ and $\Sigma_2\otimes Q$ are both minimal over the *field* Q, they are both canonical (=reachable and observable) realizations of the response f over Q. Thus (see Eilenberg [1974, Cor. XVI.5.7], where "minimal" means our "canonical") $T\otimes Q$ is a unique isomorphism. Since both X_i are torsion-free, X_i is included in $X_i \otimes Q$, $i=1,2$, and $T \otimes Q$ extends T. Thus T is one-to-one and unique also. □

Corollary 2.2. *$\mathfrak{M}\mathfrak{R}(f)$ is a partially-ordered set under:*

$$\Sigma_1 \leqslant \Sigma_2 \qquad \textit{iff there is a } T: \Sigma_1 \to \Sigma_2.$$

Let $\tau_f: \mathfrak{M}\mathfrak{R}(f) \to \mathfrak{M}\mathfrak{R}(f'): \Sigma \mapsto \Sigma'$. Then τ_f is an order-reversing map, and the pair $(\tau_f, \tau_{f'})$ constitutes a Galois connection (Kurosh [1963, par. 51]) between the posets $\mathfrak{M}\mathfrak{R}(f)$ and $\mathfrak{M}\mathfrak{R}(f')$. In other words, (i) $\Sigma \leqslant \Sigma''$ for each Σ, and (ii) $\Sigma_1 \leqslant \Sigma_2$ implies $\Sigma_2' \leqslant \Sigma_1'$. Statement (i) follows from the existence of a canonical map $X \to X'': x \mapsto$ evaluation at x. Statement (ii) follows from the fact that a linear map $T: X_1 \to X_2$ gives rise (by transposition) to $T': X_2' \to X_1'$; if T defines a morphism $\Sigma_1 \to \Sigma_2$, then T' induces a morphism $\Sigma_2' \to \Sigma_1'$.

Notation 2.3. Σ_f is the canonical realization of f; Σ^f denotes $(\Sigma_{f'})'$.

By Eilenberg [1974, Theorem XVI.5.6], Σ_f is the unique smallest element of $\mathfrak{M}\mathfrak{R}(f)$. Since $\Sigma_{f'}$ is smallest in $\mathfrak{M}\mathfrak{R}(f')$ and $(\tau_f, \tau_{f'})$ is a Galois connection, Σ^f is the unique largest element of $\mathfrak{M}\mathfrak{R}(f)$: indeed, given any Σ in $\mathfrak{M}\mathfrak{R}(f)$ one has Σ' in $\mathfrak{M}\mathfrak{R}(f')$, so $\Sigma_{f'} \leqslant \Sigma'$, thus concluding $\Sigma \leqslant \Sigma'' \leqslant (\Sigma_{f'})' = \Sigma^f$.

Remark 2.4. For any Σ in $\mathfrak{M}\mathfrak{R}(f)$, X_Σ is *finitely generated*. This is immediate from Bourbaki [1965, VII.4.1, Corollary to Proposition 1].

Note that the above remark constitutes in particular a simple proof of the result in Rouchaleau, Wyman and Kalman [1972], Rouchaleau and Wyman [1975], that finite rank implies finite realizability over a Noetherian domain (for other proofs see Eilenberg [1974, Theorem XVI.12.1] and Sontag [1976, Appendix 1]).

Theorem 2.5. *The poset $\mathfrak{M}\mathfrak{R}(f)$ is isomorphic to the lattice of L-invariant submodules of M, for some finitely generated torsion module M and linear $L: M \to M$. Conversely, any such lattice is of the form $\mathfrak{M}\mathfrak{R}(f)$, for some f.*

Proof. Let $\Sigma^f = (X^f, F^f, G^f, H^f)$. For each Σ in $\mathfrak{M}\mathfrak{R}(f)$, let T_Σ be the unique morphism from Σ to Σ^f. It follows from Lemma (2.1) that the assignment $\Sigma \to T_\Sigma(X_\Sigma)$ is an order-preserving isomorphism between the poset $\mathfrak{M}\mathfrak{R}(f)$ and the lattice of those F^f-invariant submodules of X^f which contain $T_{\Sigma_f}(X_f)$ (itself an F^f-invariant submodule, since T_{Σ_f} is a morphism). This lattice is isomorphic to the lattice of L_f-invariant submodules of M_f, where $M_f := X^f / T_{\Sigma_f}(X_f)$ and $L_f: M_f \to M_f$ is the map induced canonically by F^f. Since $\operatorname{rank} X_f = \operatorname{rank} f = \operatorname{rank} X^f$, it follows that M_f is a torsion module. Since X^f is finitely generated by (2.4), M_f is also finitely generated.

Conversely, let M be a finitely generated torsion module and $L: M \to M$. There is then some integer p such that M can be expressed as R^p/X, with X a finitely generated module of rank p. Let $F: R^p \to R^p$ be any linear map inducing L on R^p/X, let $G: R^m \to X$ be onto, for a suitable m, and let $H: X \to R^p$ be the inclusion map. Define $A_i := HF^{i-1}G$, $i = 1, 2, \ldots$. An easy calculation shows that $M = M_f$ and $L = L_f$.

Remark 2.6. All of the above definitions and results can be extended trivially to the study of representations of power series in a finite number of variables (Fliess [1974]), applying thus to certain classes of nonlinear systems over rings.

In this problem, a finite alphabet $a_1, \ldots, a_s$ is given, and a "response map" consists of an assignment of a p by m matrix A_w for each word $w = a_{i_1} \ldots a_{i_r}$, $r \geqslant 0$. The Hankel matrix and rank are defined in a way analogous to (1.1) (see Fliess [1974]), and a realization of minimal rank is a $\Sigma = (F_1, \ldots, F_s, G, H)$, where the $F_i : X \to X$, $G : R^m \to X$, and $H : X \to R^p$ are linear maps, X is torsion-free with $\operatorname{rank} X = \operatorname{rank} f$, and $A_w = HF_{i_1} \ldots F_{i_r} G$ for each $w = a_{i_1} \ldots a_{i_r}$. For each response map f, $\mathfrak{M}\mathfrak{R}(f)$ will now be represented by the lattice of those submodules of an M as in (2.5) *jointly* invariant under a finite set of linear maps $L_i : M \to M$, $i = 1, \ldots, s$. We restricted our attention to linear systems for notational simplicity and because this appears to be the most interesting case in applications.

3. Examples

We illustrate the representation theorem (2.5) with three easy examples using $R = \mathbf{R}[\sigma]$, the ring of polynomials in one variable with real coefficients. Systems over R can be interpreted via delay-differential systems, as explained by Kamen [1975, 1977]. An exposition of such facts is given in Sontag [1976]; the essential point consists in viewing the indeterminate σ as representing a shift operator on time-functions. This will be clear from the examples below.

We take $m = p = 2$ in all examples; I will denote the 2 by 2 identity matrix.

Example 3.1. Let

$$f := (\sigma I, 0, 0, \ldots).$$

In delay-differential terms, this corresponds to the completely decoupled input/output map, with two input and two output channels,

$$\begin{cases} \dot{y}_1(t) = u_1(t-1) \\ \dot{y}_2(t) = u_2(t-1) \end{cases}.$$

Simply checking reachability and observability, it is clear that the canonical realization of f is $\Sigma_f = (0, I, \sigma I)$, with $X_f = R^2$. Since $f = f'$, it follows that $\Sigma_{f'} = \Sigma_f$, and its dual is then $\Sigma^f = (0, \sigma I, I)$. The (unique) morphism $T : \Sigma_f \to \Sigma^f$ is multiplication by σ. Thus $T_{\Sigma_f}(X_f)$ is the submodule σR^2 of $X_f = R^2$. So $M_f = R^2 / \sigma R^2$; this is isomorphic to Euclidean 2-space $\underline{R}^2$ via $(P(\sigma), Q(\sigma)) \mapsto (P(0), Q(0))$. Since the induced endomorphism L is zero, $\mathfrak{M}\mathfrak{R}(f)$ is the set of *all* subspaces of $\mathbf{R}^2$.

Thus $\mathfrak{M}\mathfrak{R}(f)$ consists of two elements (the zero subspace, and the entire space $\mathbf{R}^2$) plus a projective line (i.e., the set of lines through the origin in the plane). More concretely, Σ_f corresponds to the zero subspace, Σ^f to the plane, and for each line $\mathfrak{L}$ there is a minimal-rank realization $\Sigma_{\mathfrak{L}}$ defined as follows. Either $\mathfrak{L} = \langle 1, a \rangle$ = set of multiples of some (unique) vector $(1, a)$, or $\mathfrak{L} = \langle 0, 1 \rangle$ = line$(x_1 = 0)$. If $\mathfrak{L} = \langle 1, a \rangle$ then the state-space $X_{\mathfrak{L}}$ of $\Sigma_{\mathfrak{L}}$ is the submodule of R^2

containing σR^2 whose image under the canonical map $R^2 \to \mathbf{R}^2$ (evaluation at 0) is $\mathcal{V}$, i.e., $X_{\mathcal{V}}$ is generated by $(1,a)$ and $(0,\sigma)$. When $\mathcal{V}=\langle 0,1\rangle$, $X_{\mathcal{V}}$ is generated by $(\sigma,0)$ and $(0,1)$. In any of these cases, there are isomorphisms $T: R^2 \to X_{\mathcal{V}}$, so the systems $\Sigma_{\mathcal{V}}$ are (isomorphic to) the following systems with state-spaces R^2:

$$\Sigma_{\langle 0,1\rangle}=\left(0,\begin{bmatrix} 1 & 0 \\ 0 & \sigma \end{bmatrix},\begin{bmatrix} \sigma & 0 \\ 0 & 1 \end{bmatrix}\right)$$

and

$$\Sigma_{\langle 1,a\rangle}=\left(0,\begin{bmatrix} \sigma & 0 \\ -a & 1 \end{bmatrix},\begin{bmatrix} 1 & 0 \\ a & \sigma \end{bmatrix}\right),$$

for each a in $\mathbf{R}$. Translating into delay-differential terms, one concludes that the nonisomorphic two-dimensional realizations of (3.1) are represented by:

$$\Sigma_f: \{\ \dot{x}_i(t)=u_i(t), \quad y_i(t)=x_i(t-1), \quad i=1,2$$

$$\Sigma^f: \{\ \dot{x}_i(t)=u_i(t-1), \quad y_i(t)=x_i(t), \quad i=1,2$$

$$\Sigma_{\langle 0,1\rangle}: \begin{cases} \dot{x}_1(t)=u_1(t) & y_1(t)=x_1(t-1) \\ \dot{x}_2(t)=u_2(t-1) & y_2(t)=x_2(t) \end{cases}$$

$$\Sigma_{\langle 1,a\rangle}: \begin{cases} \dot{x}_1(t)=u_1(t-1) & y_1(t)=x_1(t) \\ \dot{x}_2(t)=-au_1(t)+u_2(t) & y_2(t)=ax_1(t)+x_2(t-1) \end{cases}$$

Example 3.2. Let C be the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In this example,

$$f:=(\sigma I,\sigma C,\sigma I,\sigma C,\dots).$$

In delay-differential terms, f corresponds to an input/output equation

$$\begin{cases} \dot{y}_1(t)=u_1(t-1)-y_2(t) \\ \dot{y}_2(t)=u_2(t-1)-y_1(t) \end{cases}$$

Proceeding as in the previous example, with $X=R^2$ one has

$$\Sigma_f=(C,I,\sigma I),$$

$$\Sigma^f=(C,\sigma I,I).$$

Thus M_f is again Euclidean 2-space $\mathbf{R}^2$. But in the present example the induced map L is not zero, but a rotation. The only invariant lines $\mathcal{V}$ are now $\langle 1,1\rangle$ and $\langle 1,-1\rangle$. This corresponds to the two submodules of R^2 generated by $(1,1),(0,\sigma)$

and $(1,-1),(0,\sigma)$ respectively. Thus, with $X=R^2$ the only two other realizations are

$$\Sigma_3=\left(C,\begin{bmatrix}\sigma & 0\\ -1 & 1\end{bmatrix},\begin{bmatrix}1 & 0\\ 1 & \sigma\end{bmatrix}\right)$$

and

$$\Sigma_4=\left(C,\begin{bmatrix}\sigma & 0\\ 1 & 1\end{bmatrix},\begin{bmatrix}1 & 0\\ -1 & \sigma\end{bmatrix}\right).$$

In delay-differential terms, this means that the possible minimal realizations are the following four, up to isomorphism:

$$\Sigma_f=\begin{cases}\dot x_1(t)=x_2(t)+u_1(t) & y_1(t)=x_1(t-1)\\ \dot x_2(t)=x_1(t)+u_2(t) & y_2(t)=x_2(t-1)\end{cases}$$

$$\Sigma^f=\begin{cases}\dot x_1(t)=x_2(t)+u_1(t-1) & y_1(t)=x_1(t)\\ \dot x_2(t)=x_1(t)+u_2(t-1) & y_2(t)=x_2(t)\end{cases}$$

$$\Sigma_3=\begin{cases}\dot x_1(t)=x_2(t)+u_1(t-1) & y_1(t)=x_1(t)\\ \dot x_2(t)=x_1(t)-u_1(t)+u_2(t) & y_2(t)=x_1(t)+x_2(t-1)\end{cases}$$

$$\Sigma_4=\begin{cases}\dot x_1(t)=x_2(t)+u_1(t-1) & y_1(t)=x_1(t)\\ \dot x_2(t)=x_1(t)+u_1(t)+u_2(t) & y_2(t)=-x_1(t)+x_2(t-1)\end{cases}.$$

Example 3.3. Denote

$$A:=\begin{bmatrix}\sigma+1 & \sigma\\ 1 & 1\end{bmatrix}$$

and consider $f:=(A,A,A,\dots)$. In delay-differential terms,

$$\begin{cases}\dot y_1(t)=u_1(t-1)+u_1(t)+u_2(t-1)-y_1(t)\\ \dot y_2(t)=u_1(t)+u_2(t)-y_2(t)\end{cases}$$

With $X_f=R^2$, $\Sigma_f=(I,I,A)$, or in equations:

$$\begin{cases}\dot x_1(t)=x_1(t)+u_1(t) & y_1(t)=x_1(t-1)+x_1(t)+x_2(t-1)\\ \dot x_2(t)=x_2(t)+u_2(t) & y_2(t)=x_1(t)+x_2(t)\end{cases}$$

Since clearly $\Sigma_{f'}=(I,I,A')$, it follows that $\Sigma^f=(\Sigma_{f'})'=(I,A,I)$. But $T:\Sigma_f\to\Sigma^f$ defined by $T=A:R^2\to R^2$ is an isomorphism, since $\det A=1=$unit in R. Thus $M_f=0$, so the lattice $\mathfrak{M}\mathfrak{R}(f)$ is in this case trivial, consisting just of Σ_f (f "splits", in the sense of Sontag [1978]).

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