Comments on Integral Variants of ISS

Eduardo D. Sontag

Dept. of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

This note discusses two integral variants of the input-to-state stability (ISS) property, which represent nonlinear generalizations of L^2 stability, in much the same way that ISS generalizes L^∞ stability. Both variants are equivalent to ISS for linear systems. For general nonlinear systems, it is shown that one of the new properties is strictly weaker than ISS, while the other one is equivalent to it. For bilinear systems, a complete characterization is provided of the weaker property. An interesting fact about functions of type $\mathcal{KL}$ is proved as well.

1 Introduction

We deal here with controlled systems of the general form

$$\dot{x} = f(x, u), \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and locally Lipschitz on x for bounded u, and inputs $u(\cdot) : [0, \infty) \to \mathbb{R}^m$ are assumed to be locally essentially bounded. The paper [7] introduced the notion of "input to state stability" (ISS), which roughly states that "no matter what is the initial state, if the inputs are uniformly small, then the state must eventually be small". Some results, applications, and further developments can be found in e.g. [4], [5], [11], [9], and [10], as well as many other recent papers. One frequently-remarked shortcoming of the ISS property is that provides no useful bounds in the situation in which inputs $u(\cdot)$ are unbounded but have in some sense finite energy. This note deals with two variants of ISS which take into account such "energy" information, and shows that one of them is in fact equivalent to ISS, while the other one is strictly weaker.

---

[1] Supported in part by US Air Force Grant F49620-95-1-0101.
E-mail: sontag@control.rutgers.edu

Notations: For a vector z in a space $\mathbb{R}^q$, $|z|$ denotes Euclidean norm. If $z : I \to \mathbb{R}^q$ is a measurable function defined on an interval containing $[0,t]$, $\|z_t\|$ denotes the (essential) supremum of $\{|z(s)|\,,\ s \in [0,t]\}$; for $t = \infty$, we write just $\|z\|$.

In order to motivate the definitions, we first review the classical case of linear systems

$$\dot{x} \;=\; Ax + Bu\,.$$

One often defines input/output stability of such a system in various ways, depending on the norms being used for state and input trajectories. The most common choices are L^2 and L^∞ norms. These can be used in various combinations, one of which ("L^∞ to L^2") is less interesting, being far too restrictive. The three possibilities that remain are defined by requiring the existence of constants c and λ, with $\lambda > 0$, so that, for each input $u(\cdot)$ and each initial state ξ, the solution $x(t)$ of $\dot{x} = Ax + Bu$, $x(0) = \xi$, satisfies the respective estimate:

$$|x(t)| \;\leq\; ce^{-\lambda t}\,|\xi| \,+\, c\|u_t\|_\infty \quad \text{for all } t \geq 0\,, \tag{2}$$

$$|x(t)| \;\leq\; ce^{-\lambda t}\,|\xi| \,+\, c\int_0^t |u(s)|^2\,ds \quad \text{for all } t \geq 0\,, \tag{3}$$

$$\int_0^t |x(s)|^2\,ds \;\leq\; c\,|\xi| \,+\, c\int_0^t |u(s)|^2\,ds \quad \text{for all } t \geq 0\,. \tag{4}$$

It is not difficult to see (and also follows from the more general results given later) that all these three properties are equivalent, and in fact they amount to simply asking that the matrix A be Hurwitz (all eigenvalues have negative real parts).

Our goal is to understand analogues of the above properties for nonlinear systems. We take the point of view that *good notions of stability should be invariant under (nonlinear) changes of variables*. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is a change of variables, by which we mean here that T is continuous, has a continuous inverse, and satisfies $T(0) = 0$. It is then easy to see that there are two functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(|x|) \;\leq\; |T(x)| \;\leq\; \alpha_2(|x|)$$

for all x. Under a coordinate change $x(t) = T(z(t))$ in states, $|x(t)|$ can be estimated below by $\alpha_1(|z(t)|)$ and above by $\alpha_2(|z(t)|)$, and similarly for any

* Recall, cf. [2], that $\mathcal{K}$ is the class of functions $[0,\infty) \to [0,\infty)$ which are zero at zero, strictly increasing, and continuous, $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded, and $\mathcal{KL}$ is the class of functions $[0,\infty)^2 \to [0,\infty)$ which are decreasing to zero on the second argument and of class $\mathcal{K}$ on the first argument.

power like $|x(t)|^2$. Therefore, in the new variables, each instance of a norm (or of a power of a norm) of the state becomes an estimate of the type "$\gamma(|x(t)|)$", where γ is some function of class $\mathcal{K}_\infty$. Similarly, input norms should be replaced by $\gamma(|u(t)|)$, as this is the only way to formulate estimates in a fashion that is independent of the choice of coordinates in the input space. Finally, a decay term of the type "$ce^{-\lambda t}|\xi|$" becomes, after a change of variables, $e^{-\lambda t}\gamma(|\xi|)$, which can also be written as $\beta(|\xi|, t)$, where the function $\beta \in \mathcal{KL}$ is $\beta(s,t) = e^{-\lambda t}\gamma(s)$. (In the definitions to follow, we use arbitrary functions of class $\mathcal{KL}$, in order to stay close to other current literature. See later, cf. Section 3, for a proof that the same definitions would result if had we insisted upon the special form $e^{-\lambda t}\gamma(s)$.)

In summary, we have the following nonlinear versions of the respective properties:

"L^∞ to L^∞" property (2): This gives rise, under coordinate changes, to:

There exist functions $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that, for all initial states $\xi \in \mathbb{R}^n$ and all inputs $u(\cdot)$,

$$\alpha(|x(t)|) \;\leq\; \beta(|\xi|, t) + \sup_{s\in[0,t]} \gamma(|u(s)|) \quad \text{for all } t \geq 0\,, \tag{5}$$

where $x(t)$ is the solution of (1) with this input and with $x(0) = \xi$.

Observe that, for any given input $u(\cdot)$, the solution $x(\cdot)$ is defined on some maximal interval $[0, a)$, where a priori a could be finite. As $\sup_{s\in[0,a]} \gamma(|u(s)|) < \infty$, however, the estimate (5) implies that the maximal trajectory stays bounded, and this, in turn, by standard facts from differential equations, see e.g. [8], Proposition 3.6 in Appendix C, implies that $a = +\infty$, that is to say, the solution is defined for all $t \geq 0$.

Applying the increasing function α^{-1} to both sides of the estimate (5), using the facts that $\alpha^{-1}(a+b) \leq \alpha^{-1}(2a)+\alpha^{-1}(2b)$ and that $\|\gamma(|u|)_t\|_\infty = \gamma(\|u_t\|_\infty)$, and redefining β and γ, there results the following simpler but equivalent statement: there exist $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that, for all initial states $\xi \in \mathbb{R}^n$ and all inputs $u(\cdot)$,

$$|x(t)| \;\leq\; \beta(|\xi|, t) + \gamma\left(\|u_t\|_\infty\right) \quad \text{for all } t \geq 0\,. \tag{5$'$}$$

In other words, this property is precisely the *input to state stability* (ISS) property from [7].

"L^2 to L^∞" property (3): This generalizes as follows:

There exist $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that the following estimate holds for *all initial states $\xi \in \mathbb{R}^n$ and all inputs $u(\cdot)$:*

$$\alpha(|x(t)|) \leq \beta(|\xi|, t) + \int_0^t \gamma(|u(s)|)ds \quad \text{for all } t \geq 0\,. \tag{6}$$

As before, the solution $x(\cdot)$ is necessarily defined for all $t \geq 0$. We will call a system which satisfies the property defined by estimate (6) an *integral-input to state stable (IISS)* system.

"L^2 to L^2" property (4): This becomes:

There exist $\alpha, \gamma, \kappa \in \mathcal{K}_\infty$ so that the following estimate holds for all initial states $\xi \in \mathbb{R}^n$ and all inputs $u(\cdot)$:

$$\int_0^t \alpha(|x(s)|)ds \leq \kappa(|\xi|) + \int_0^t \gamma(|u(s)|)ds \quad \text{for all } t \geq 0\,. \tag{7}$$

In this case, we *assume* that the solution is defined for all $t \geq 0$, as part of the definition of the property. We do not give a name to property (7), since we will prove as follows:

Theorem 1 *A system (1) satisfies the property defined by estimate (7) if and only if it is an ISS system.*

Regarding IISS, our objective is to compare it with the ISS property. In order to do this, we will prove the following dissipation-like sufficient condition (the recent work [1] provides a far more complete characterization):

Theorem 2 *Consider a system (1), and assume that there exists a positive definite proper smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a constant $q > 0$, and two class $\mathcal{K}_\infty$ functions γ_1 and γ_2, so that*

$$\nabla V(x).f(x,u) \leq (\gamma_1(|u|) - q)V(x) + \gamma_2(|u|) \tag{8}$$

for all vectors $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Then, the system is IISS

For comparison with the ISS property, we recall from [6]:

Theorem 3 *A system (1) is ISS if and only if there exist a positive definite proper smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}_\infty$ function γ so that*

$$\nabla V(x).f(x,u) \leq -V(x) + \gamma(|u|)$$

for all vectors $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. □

From the two results, we have (see also the "exp-ISS" property in [6]):

Corollary 4 *If the system (1) is ISS, then it is also IISS.* □

Consider the special case of bilinear systems, that is, systems of the following form:

$$\dot{x} = \left(A + \sum_{i=1}^{m} u_i A_i\right) x + Bu\,, \tag{9}$$

where $A, A_1, \ldots, A_m$ are $n \times n$ matrices and B is $n \times m$ (and u_i, $i = 1, \ldots, m$ are the coordinates of u).

Theorem 5 *The system (9) is IISS if and only if A is a Hurwitz matrix (all eigenvalues have negative real part).*

In contrast to linear systems, for which both notions coincide, the converse does not hold in general: there are systems that are IISS yet are not ISS. Indeed, the one-dimensional bilinear system (with Hurwitz A):

$$\dot{x} = -x + ux$$

is not ISS, since the constant input $u \equiv 2$ produces unbounded trajectories.

One consequence of the ISS property is that inputs u for which $u(t) \to 0$ as $t \to \infty$ induce state trajectories with $x(t) \to 0$ as $t \to \infty$. An analog for IISS systems is as follows:

Proposition 6 *If a system satisfies the estimate (6), then, for any control u such that $\int_0^\infty \gamma(|u(s)|)ds < \infty$, and for any initial state ξ, it holds for the corresponding trajectory that $x(t) \to 0$ as $t \to \infty$.*

2 Proofs

We prove here the various statements.

Proof of Theorem 2

Without loss of generality, we will assume that $\gamma_1 = \gamma_2 = \gamma$. (It is enough to find a $\gamma \in \mathcal{K}_\infty$ so that $\gamma_i(s) \le \gamma(s)$ for all s and $i = 1, 2$, and to observe that $(\gamma_1(s) - 1)V(x) \le (\gamma(s) - 1)V(x)$ for all s, x, because $V \ge 0$.)

Since V is positive definite, continuous, and proper, there exist two functions α_1, α_2 of class $\mathcal{K}_\infty$ so that $\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$ for all $x \in \mathbb{R}^n$. In terms of these functions, we introduce

$$\beta(s,t) := \theta_2^{-1}\left(\theta_1(e^{-qt}\alpha_2(s))\right)$$

and

$$\alpha(s) := \theta_2^{-1}\left(\frac{1}{2}\alpha_1(s)\right),$$

where

$$\theta_1(r) := r + \frac{1}{2}r^2, \quad \theta_2(r) := \frac{1}{2}\left(e^r - 1\right)^2 + e^r r$$

are both of class $\mathcal{K}_\infty$. Thus $\beta \in \mathcal{KL}$ and $\alpha \in \mathcal{K}_\infty$. We prove next that the integral-input to state estimate (6) holds with these functions β, α, and γ.

Pick any initial state ξ, input $u(\cdot)$, and the solution of the initial value problem $\dot{x} = f(x,u)$, $x(0) = \xi$. Consider the absolutely continuous scalar function $v(t) := V(x(t))$. We have, using (8):

$$\dot{v}(t) \leq (\gamma(|u(t)|) - q)v(t) + \gamma(|u(t)|)$$

for (almost all) $t \geq 0$. A standard comparison principle applied to the above differential inequality (see e.g. [3], Theorem 6; the assumption in that reference that the right-hand side is continuous in t is not necessary) provides the estimate:

$$v(t) \;\leq\; U(t)e^{-qt}v(0) + \int_0^t U(t,s)e^{-q(t-s)}\gamma(|u(s)|)ds$$

for all $t \geq 0$, where we are denoting

$$U(t,s) := e^{\int_s^t \gamma(|u(\tau)|)d\tau}$$

and $U(t) = U(t,0)$. Observe that $U(t,s) \leq U(t)$ and $e^{-q(t-s)} \leq 1$ for all $s \in [0,t]$, so we also have

$$v(t) \;\leq\; U(t)e^{-qt}v(0) + U(t)\int_0^t \gamma(|u(s)|)ds \tag{10}$$

for all $t \geq 0$. Writing

$$\begin{aligned} U(t)e^{-qt}v(0) &= e^{-qt}v(0) + (U(t)-1)e^{-qt}v(0) \\ &\leq e^{-qt}v(0) + \frac{1}{2}\left(e^{-qt}v(0)\right)^2 + \frac{1}{2}\left(U(t)-1\right)^2 \end{aligned}$$

we conclude from (10) that

$$v(t) \;\leq\; \theta_1\left(e^{-qt}v(0)\right) + \theta_2\left(\int_0^t \gamma(|u(\tau)|)d\tau\right).$$

From this, and recalling that $v(t) = V(x(t))$ and $v(0) = V(\xi)$, there follows the estimate:

$$\alpha_1(|x(t)|) \;\leq\; \theta_1\left(e^{-qt}\alpha_2(|\xi|)\right) + \theta_2\left(\int_0^t \gamma(|u(\tau)|)d\tau\right) .$$

Finally, dividing both sides of this inequality by 2 and applying θ_2^{-1}, the desired conclusion follows from the fact that $\theta_2^{-1}((a+b)/2) \leq \theta_2^{-1}(a) + \theta_2^{-1}(b)$ for all nonnegative a, b.

Proof of Theorem 5

The necessity part is clear from Proposition 6. Conversely, assume that A is Hurwitz, and pick any symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ so that $PA+A'P = -2I$ (see e.g. [8]), where we are using primes to denote transposes. Let $V(x) := x'Px$, and write $\dot{V}(x,u)$ for the directional derivative

$$\nabla V(x).\left[\left(A + \sum_{i=1}^{m} u_i A_i\right) x + Bu\right] .$$

Then:

$$\dot{V}(x,u) = -2x'PAx + \sum_{i=1}^{m} 2u_i x'PA_i x + 2x'PBu$$

for all $t \geq 0$. If p is the induced Euclidean norm of PB, then for all vectors x, u:

$$2x'PBu \leq 2\,|x|\,(p\,|u|) \leq |x|^2 + p^2\,|u|^2$$

and there is also a constant $c > 0$ so that

$$\left|\sum_{i=1}^{m} 2u_i x'PA_i x\right| \leq c\,|u|\,|x|^2 ,$$

so we conclude, using $2x'PAx = -2\,|x|^2$, that

$$\dot{V}(x,u) \;\leq\; -\,|x|^2 + c\,|u|\,|x|^2 + p^2\,|u|^2 \;\leq\; (cr\,|u| - q)V(x) + p^2\,|u|^2$$

for all $t \geq 0$, where $1/q$ is the largest eigenvalue of P and and $1/r$ is the smallest eigenvalue of P. The proof is completed by appealing to Theorem 2.

Proof of Proposition 6

Pick any $\varepsilon > 0$. We must find a T_ε so that $|x(t)| < \varepsilon$ for all $t \geq T_\varepsilon$. Pick $\varepsilon_0 := \alpha^{-1}(\varepsilon)$ and a $T > 0$ so that $\int_T^\infty \gamma(|u(s)|)ds < \varepsilon_0/2$. Such a T exists because $\int_T^\infty \gamma(|u(s)|)ds \to 0$ as $T \to \infty$. Consider the new initial state $\xi' := x(T)$ and the control u' which is the "tail" of u: $u'(t) := u(t+T)$ for $t \geq 0$. The ensuing trajectory x' satisfies $x'(t) = x(t+T)$. Applying the estimates to this new pair (ξ', u'), we conclude that, for all $t \geq 0$:

$$\alpha(|x(t+T)|) \;\leq\; \beta(|\xi'|\,,t) + \int_T^{t+T} \gamma(|u(s)|)ds \;<\; \beta(|\xi'|\,,t) + \varepsilon_0/2\,.$$

If T' is so that $\beta(|\xi'|\,,T') \leq \varepsilon_0/2$, we conclude that $\alpha(|x(t)|) < \varepsilon_0$, and hence $|x(t)| \leq \varepsilon$, for all $t \geq T_\varepsilon := T + T'$.

Proof of Theorem 1

One implication is routine: if the system is ISS, then from Theorem 3 we have a function V with $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ ($\alpha_i \in \mathcal{K}_\infty$) so that, for each initial state ξ and input $u(\cdot)$, $v(t) = V(x(t))$ satisfies $\dot{v}(t) \leq -v(t) + \gamma(|u(t)|)$ for (almost all) $t \geq 0$. Therefore

$$\int_0^t v(s)\,ds \;\leq\; v(0) - v(t) + \int_0^t \gamma(|u(s)|)\,ds$$

from which one has the estimate $\int_0^t \alpha_1(|x(s)|)ds \leq \alpha_2(|\xi|) + \int_0^t \gamma(|u(s)|)ds\,.$

The converse is more interesting. We use a variation of one of the characterizations of ISS given in [10]: A system is ISS if and only if 0 is a stable equilibrium of the unforced system $\dot{x} = f(x,0)$ and the so-called "LIM" property holds, namely there exists a function $\theta \in \mathcal{K}_\infty$ so that, for all initial states ξ and inputs $u(\cdot)$, $\inf_{t\geq 0}|x(t)| \leq \theta(\|u\|_\infty)$. (To be precise, in [10] it is stated that ISS is equivalent to LIM together with *asymptotic stability* of the origin for $\dot{x} = f(x,0)$. However, as pointed out to the author by Gene Ryan, the LIM property applied with $u \equiv 0$ means that every trajectory of $\dot{x} = f(x,0)$ gets arbitrarily close to the origin, so stability implies asymptotic stability when the LIM property holds. This variant should have been stated in [10], and it was an oversight not to have done so.)

Assume system (1) satisfies the property defined by the estimate (7). Then, the LIM property holds with $\theta = \alpha^{-1} \circ \gamma$. Indeed, consider any ξ and $u(\cdot)$. If it were the case that $\inf_{t\geq 0}|x(t)| > \theta(\|u\|_\infty)$ then there is some $\varepsilon > 0$ so that $\alpha(x(t)) \geq \varepsilon + \gamma(u(t))$ for (almost) all $t > 0$, and thus $\int_0^t \alpha(x(s))ds \geq \varepsilon t + \int_0^t \gamma(u(s))ds$, which gives the contradiction $\kappa(|\xi|) > \varepsilon t$ for all t. So we are only left to show that 0 is a stable equilibrium of the unforced system $\dot{x} = f(x,0)$. Pick any $\varepsilon > 0$. Let K be the set $\{x \,|\, \varepsilon/2 \leq |x| \leq \varepsilon\}$ and take $c := \sup_{x\in K}|f(x,0)|$. Finally, choose $0 < \delta < \varepsilon/2$ so that $r < \delta$ implies $\kappa(r) < s_0$, where $s_0 := \frac{\varepsilon\alpha(\varepsilon/2)}{2c}$. Let $|x(0)| < \delta$. Then $|x(t)| < \varepsilon$ for all $t \geq 0$. Indeed, suppose that there exists some $t > 0$ so that $|x(t)| \geq \varepsilon$. Then there is some interval $[t_1,t_2]$ so that $|x(t_1)| = \varepsilon/2$, $|x(t_2)| = \varepsilon$, and $x(t) \in K$ for all

$t \in [t_1, t_2]$. Thus

$$\int_0^\infty \alpha(|x(s)|)\, ds \;\geq\; \int_{t_1}^{t_2} \alpha(|x(s)|)\, ds \;\geq\; (t_2 - t_1)\alpha(\varepsilon/2)\,.$$

On the other hand,

$$\varepsilon/2 \;\leq\; |x(t_2) - x(t_1)| \;\leq\; \int_{t_1}^{t_2} |f(x(s), 0)|\; ds \;\leq\; c(t_2 - t_1)$$

so we conclude

$$s_0 \leq \int_0^\infty \alpha(|x(s)|) ds \leq \kappa(|x(0)|)\,,$$

a contradiction.

3 A Lemma on $\mathcal{KL}$ Functions

This section has the purpose of showing that, if we were to define the estimates in Equations (5) and (6) using only those functions of class $\mathcal{KL}$ which have the particular form $\beta(s,t) = e^{-\lambda t}\gamma(s)$, with $\lambda > 0$ and $\gamma \in \mathcal{K}_\infty$, or even functions of this form with $\lambda = 1$, we would arrive at precisely the same notions. The proof relies upon a lemma involving $\mathcal{KL}$ functions which seems not to have been remarked in the literature, and which is of independent interest:

Proposition 7 *Assume that $\beta \in \mathcal{KL}$. Then, there exist $\theta_1, \theta_2 \in \mathcal{K}_\infty$ so that*

$$\beta(s,t) \;\leq\; \theta_1\left(\theta_2(s)e^{-t}\right) \quad \forall\, s \geq 0, t \geq 0\,. \tag{11}$$

Let us first see how, on the basis of this result, we can show that equivalent stability concepts result. Let β be given, and find θ_1 and θ_2 as in the above statement. Let σ be any function of class $\mathcal{K}_\infty$ with the property that, for all $a \geq 0$, $\sigma(a) \geq \max\{2\theta_1(a), 2a\}$. Since $\sigma^{-1} \in \mathcal{K}_\infty$, we have, for all $p, q \geq 0$:

$$\sigma^{-1}(\theta_1(p) + q) \;\leq\; \sigma^{-1}(2\theta_1(p)) \;+\; \sigma^{-1}(2q) \;\leq\; p + q\,,$$

or, equivalently, $\theta_1(p) + q \leq \sigma(p+q)$. It follows that, for all $s, t, q \geq 0$:

$$\beta(s,t) + q \;\leq\; \sigma\left(\theta_2(s)e^{-t} + q\right)\,. \tag{12}$$

Now suppose that a system satisfies the estimate in Equation (6), namely $\alpha\left(|x(t)|\right) \;\leq\; \beta(|\xi|\,, t) + \int_0^t \gamma(|u(s)|) ds$ for all $t \geq 0$. Finding, for this β, functions

$\theta_1, \theta_2, \sigma$ as just discussed, we conclude that solutions must also satisfy

$$\widehat{\alpha}\left(|x(t)|\right) \;\leq\; e^{-t}\theta_2(|\xi|) \;+\; \int_0^t \gamma(|u(s)|)ds\,,$$

where $\widehat{\alpha}(r) := \sigma^{-1}(\alpha(r))$. In other words, we have proved that there is also an estimate involving the special form $\beta(s,t) = e^{-t}\gamma(s)$ for the $\mathcal{KL}$ function. A similar remark applies to the estimate (5), the ISS property, though in that case, an alternative proof is easy from Theorem 3.

The keys to proving Proposition 7 are two lemmas.

Lemma 8 *Assume that $\beta \in \mathcal{KL}$. Then, there exist $\gamma \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}_\infty$ so that*

$$\beta(s,t) \;\leq\; \gamma(s)\sigma\left(e^{-t}\right) \;\; \forall\, s \geq 0, t \geq 0\,. \tag{13}$$

PROOF. Let $\beta \in \mathcal{KL}$. Define

$$\alpha_s(t) \;:=\; \frac{\beta(s,t)}{\beta(s,0)^{1/2} + \beta(s,0)^2}$$

for all $t \geq 0$ and all $s > 0$. Note that

$$\alpha_s(t) \;\leq\; \frac{\beta(s,0)}{\beta(s,0)^{1/2} + \beta(s,0)^2} \;\leq\; 1 \tag{14}$$

for all $t \geq 0$ and all $s > 0$, and that $\alpha_s(t)$ decreases to zero as $t \to \infty$, for each fixed s. Let

$$\alpha(t) \;:=\; \sup_{s>0} \alpha_s(t)$$

for each $t \geq 0$. Since each α_s is decreasing, α is too, and $\alpha(t) \leq 1$ for all t.

Claim: $\alpha(t) \to 0$ as $t \to \infty$.

Indeed, pick any $\varepsilon > 0$. Since $\frac{\beta(s,0)}{\beta(s,0)^{1/2}+\beta(s,0)^2} = \rho(\beta(s,0))$, where $\rho(x) = \frac{x}{x^{1/2}+x^2}$, $\beta(\cdot,0) \in \mathcal{K}_\infty$, and $\rho(x) \to 0$ as $x \to \infty$ or $x \to 0^+$, it follows that

$$\frac{\beta(s,0)}{\beta(s,0)^{1/2} + \beta(s,0)^2} \;\to\; 0$$

when $s \to \infty$ and when $s \to 0^+$. Thus, there are $0 < a < b < \infty$ with the property that

$$s \notin [a,b] \;\Rightarrow\; \alpha_s(t) < \varepsilon \tag{15}$$

for all $t \geq 0$ (using Equation (14) and the above limit). Pick now any $s \in [a, b]$. For such an s,

$$\alpha_s(t) \;\leq\; \frac{\beta(b,t)}{\beta(a,0)^{1/2} + \beta(a,0)^2}\;.$$

Pick a t_0 so that $t > t_0$ implies

$$\beta(b,t) < \varepsilon \left(\beta(a,0)^{1/2} + \beta(a,0)^2\right)\,.$$

Then for all $t > t_0$ we have that $\alpha_s(t) < \varepsilon$, if $s \in [a, b]$. Together with Equation (15), we conclude that $\alpha_s(t) < \varepsilon$ for all $t > t_0$ and all s. Thus $\alpha(t) < \varepsilon$ for such t, and the claim is proved.

Since α is decreasing to zero, there is some function $\theta : [0,\infty) \to [0,\infty)$ which is continuous and strictly decreasing to zero, so that, for all $s > 0$ and all $t \geq 0$,

$$\frac{\beta(s,t)}{\beta(s,0)^{1/2} + \beta(s,0)^2} \;=\; \alpha_s(t) \;\leq\; \alpha(t) \;\leq\; \theta(t)$$

and therefore

$$\beta(s,t) \;\leq\; \gamma(s)\theta(t)\,, \tag{16}$$

where we defined $\gamma(s) := \left(\beta(s,0)^{1/2} + \beta(s,0)^2\right)$, and $\gamma \in \mathcal{K}_\infty$. Note that Equation (16) holds also at $s = 0$.

Finally, we let $\sigma_0(r) := \theta(-\ln r)$ for $x \in (0,1]$, and $\sigma_0(0) := 0$. We have that $\sigma_0 : [0,1] \to \mathbb{R}$ is strictly increasing and continuous at zero, so there exists some $\sigma \in \mathcal{K}_\infty$ so that $\sigma_0(r) \leq \sigma(r)$ for all $r \in [0,1]$. Therefore Equation (13) holds with this choice of γ and σ. ∎

For the next Lemma, we let $\mathcal{I}$ be the set of continuous functions $k : \mathbb{R} \to \mathbb{R}$ which are strictly increasing and satisfy that $\lim_{r\to+\infty} k(r) = +\infty$ and $\lim_{r\to-\infty} k(r) = -\infty$.

Lemma 9 *Assume that $f \in \mathcal{I}$. Then, there exist two functions $g, h \in \mathcal{I}$ so that*

$$f(x+y) \;\leq\; g(x) + h(y) \tag{17}$$

for all $x, y \in \mathbb{R}$.

PROOF. We start by defining a function $\varphi \in \mathcal{I}$ with the following three properties:

– $\varphi(x) \geq 2x$ for all $x \geq 0$;
– $\varphi(x) \geq \frac{x}{2}$ for all $x \leq 0$;
– $f(x) - f(\varphi(x)) \to -\infty$ as $x \to +\infty$ and as $x \to -\infty$.

One way to define φ is by means of the following formula, for all $x \in \mathbb{R}$:

$$\varphi(x) \;=\; \max\left\{f^{-1}\left(\frac{1}{2}f(x)\right), f^{-1}(2f(x)), 2x, \frac{x}{2}\right\} \,.$$

Observe that each of the four functions in the maximum is in $\mathcal{I}$, so $\varphi \in \mathcal{I}$. Furthermore, $f(\varphi(x)) \geq f(f^{-1}(f(x)/2)) = f(x)/2$, and similarly $f(\varphi(x)) \geq 2f(x)$, for all x. So

$$f(x) - f(\varphi(x)) \leq f(x) - \frac{1}{2}f(x) = \frac{1}{2}f(x) \to -\infty$$

as $x \to -\infty$, and

$$f(x) - f(\varphi(x)) \leq f(x) - 2f(x) = -f(x) \to -\infty$$

as $x \to +\infty$.

We let $g(x) := f(\varphi(x))$ and

$$h_0(y) \;:=\; \sup_{x\in\mathbb{R}} \left[f(x+y) - g(x)\right]$$

for $y \in \mathbb{R}$.

We claim that $h_0(y) < \infty$ for all y. To prove this, we consider the cases $y \leq 0$ and $y > 0$ separately. Take first $y \leq 0$. So $f(x+y) \leq f(x)$ for all x. If $x \leq 0$, then

$$f(x) \leq f(x/2) \leq f(\varphi(x)) = g(x),$$

and, if $x \geq 0$, then also

$$f(x) \leq f(2x) \leq f(\varphi(x)) = g(x),$$

so, taking the supremum over x, we have $h_0(y) \leq 0$. Now take the case $y > 0$. If $x \leq -2y$, then $-x/2 \geq y$, so $\varphi(x) \geq x/2 \geq x + y$, and therefore

$$f(x+y) - g(x) \leq f(x+y) - f(\varphi(x)) \leq 0\,.$$

If $x \geq y$ then $x > 0$, so $g(x) = f(\varphi(x)) \geq f(2x) \geq f(x+y)$, so again $f(x+y) - g(x) \leq 0$. If instead $-2y \leq x \leq y$, then $f(x+y) \leq f(2y)$ and $g(x) = f(\varphi(x)) \geq f(\varphi(-2y)) \geq f(-y)$ (because $-2y < 0$ implies $\varphi(-2y) \geq -y$). Thus in that case

$$f(x+y) - g(x) \leq f(2y) - f(-y)\,.$$

Therefore
$$h_0(y) \le \max\{0, f(2y) - f(-y)\} < \infty$$
as claimed.

Note that $g \in \mathcal{I}$ and that, from the definition of h_0,

$$f(x+y) \;\le\; g(x) + h_0(y) \tag{18}$$

for all $x, y \in \mathbb{R}$. Furthermore, h_0 is nondecreasing, because, if $y' > y$, then $f(x+y') - g(x) > f(x+y) - g(x)$ for all $x \in \mathbb{R}$.

Finally, we claim that
$$h_0(y) \to -\infty$$
as $y \to -\infty$. Indeed, let $K < 0$ be arbitrary. Fix ρ so that $|x| > \rho$ implies $f(x) - g(x) < K$; such a ρ exists because of the last property required of φ. As $f(x+y) \le f(x)$ when $y \le 0$ (because f is an increasing function), it follows that

$$f(x+y) - g(x) < K \quad \text{if} \;\; y \le 0, |x| > \rho\,. \tag{19}$$

Now take any $x \in [-\rho, \rho]$. Since $x + y \le \rho + y$ and $\varphi(x) \ge \varphi(-\rho)$,

$$f(x+y) - g(x) \le f(\rho + y) - f(\varphi(-\rho))\,. \tag{20}$$

As $f(z) \to -\infty$ when $z \to -\infty$, there is some ζ so that

$$z < \zeta \;\Rightarrow\; f(z) < K + f(\varphi(-\rho))\,.$$

It follows that if $y < \zeta - \rho$ then $f(\rho + y) - f(\varphi(-\rho)) < K$. So Equations (19)-(20) imply that $f(x+y) - g(x) < K$ for all $y < \min\{0, \zeta - \rho\}$, and all $x \in \mathbb{R}$. Therefore $h_0(y) < K$ for all $y < \min\{0, \zeta - \rho\}$, proving the claim.

As $h_0 : (-\infty, \infty) \to \mathbb{R}$ is nondecreasing and has $h_0(y) \to -\infty$ when $y \to -\infty$, there is some $h \in \mathcal{I}$ so that $h_0(y) \le h(y)$ for all y. The conclusion follows then from Equation (18). ∎

Although not needed here, it is worth stating the "exponential" version of the above Lemma:

Corollary 10 *For each $\gamma \in \mathcal{K}_\infty$ there exist σ_1 and σ_2 in $\mathcal{K}_\infty$ so that*

$$\gamma(rs) \;\le\; \sigma_1(r)\sigma_2(s)$$

for all $r, s \ge 0$.

PROOF. Consider $f(x) := \ln \gamma(e^x)$. Since $\gamma \in \mathcal{K}_\infty$, $f \in \mathcal{I}$. Thus there exist g and h as in Lemma 9. We then let $\sigma_1(r) := e^{g(\ln r)}$ and $\sigma_2(r) := e^{h(\ln r)}$. ∎

Remark 11 The functions g and h in Lemma 9 can be chosen to be the same. Indeed, given g and h from the Lemma, we have that

$$f(x+y) \leq g(x) + h(y) \leq \max\{g(x), h(x)\} + \max\{g(y), h(y)\}\,,$$

so $f(x+y) \leq k(x) + k(y)$, where $k(z) := \max\{g(y), h(y)\}$ defines also a function in $\mathcal{I}$. Similarly, the functions σ_1 and σ_2 in Corollary 10 can also be chosen equal. □

Proof of Proposition 7

Let $\beta \in \mathcal{KL}$. By Lemma 8, there exist $\gamma \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}_\infty$ so that $\beta(s,t) \leq \gamma(s)\sigma\left(e^{-t}\right)$ for all $s \geq 0$ and $t \geq 0$. We apply Lemma 9 with

$$f(u) \;:=\; -\ln \sigma^{-1}(e^{-u})$$

It is clear that $f \in \mathcal{I}$. So there exist g and h in $\mathcal{I}$ so that Equation (17) holds for all reals x, y.

We define

$$\alpha_1(r) \;:=\; \exp(-g(-\ln r))$$

for $r \in (0,\infty)$, and $\alpha_1(0) := 0$. Because $g \in \mathcal{I}$, it follows that $\alpha_1 \in \mathcal{K}_\infty$. Note for future reference that

$$g(x) = -\ln \alpha_1(e^{-x}) \quad \text{for all } \; x \in \mathbb{R}\,. \tag{21}$$

Finally, we define

$$\alpha_2(r) \;:=\; \exp(h(\ln r))$$

for $r \in (0,\infty)$, and $\alpha_2(0) := 0$. Since $h \in \mathcal{I}$, it follows that $\alpha_2 \in \mathcal{K}_\infty$ as well, and we note that

$$h(y) = \ln \alpha_2(e^{y}) \quad \text{for all } \; y \in \mathbb{R}\,. \tag{22}$$

Pick now any $t \in [0,\infty)$ and any $s \in (0,\infty)$. Let

$$x := -\ln \gamma(s) - \ln \sigma(e^{-t}) \quad \text{and} \quad y := \ln \gamma(s)\,.$$

Observe that

$$-(x+y) = \ln \sigma(e^{-t})$$

so

$$f(x+y) = -\ln \sigma^{-1}(\exp(\ln \sigma(e^{-t}))) = t\,,$$

$$g(x) = -\ln \alpha_1(\exp(\ln \gamma(s) + \ln \sigma(e^{-t}))) = -\ln \alpha_1(\gamma(s)\sigma(e^{-t}))\,,$$

and

$$h(y) = \ln \alpha_2(\exp(\ln \gamma(s))) = \ln \alpha_2(\gamma(s))\,.$$

Therefore, Equation (17), applied with these x and y, gives

$$t \;\leq\; -\ln \alpha_1(\gamma(s)\sigma(e^{-t})) + \ln \alpha_2(\gamma(s))\,.$$

Taking $\exp(-\cdot)$ in both sides, we conclude that

$$e^{-t} \;\geq\; \frac{\alpha_1(\gamma(s)\sigma(e^{-t}))}{\alpha_2(\gamma(s))}\,.$$

Thus $\alpha_1(\gamma(s)\sigma(e^{-t})) \leq \alpha_2(\gamma(s))e^{-t}$, and hence

$$\gamma(s)\sigma(e^{-t}) \;\leq\; \alpha^{-1}\left(\alpha_2(\gamma(s))e^{-t}\right)\,.$$

This equation holds for $s = 0$ as well. Taking $\theta_1 := \alpha_1^{-1}$ and $\theta_2 := \alpha_2 \circ \gamma$, the theorem is proved. □

4 Remark

The proof of Theorem 3 in [6] relies upon a sharpening of the dissipation characterization obtained in [9], which gave only an estimate of the type $\nabla V(x).f(x,u) \leq -\alpha(V(x)) + \gamma_2(|u|)$, where α is of class $\mathcal{K}$, not necessarily the identity. One might conjecture, in view of this fact, that it may be possible to weaken the assumption in Equation (8) to asking merely that $\nabla V(x).f(x,u) \leq (\gamma_1(|u|) - q)\alpha(V(x)) + \gamma_2(|u|)$, where α is some function of class $\mathcal{K}$. This, however, will not be a sufficient condition: consider as an illustration the system $\dot{x} = (u^2-1)x^3$ and $V(x) = x^2$; then $\dot{V} = (u^2-1)(2V^2)$, but the system is not IISS, since for $\xi = 1$ and any input with $u(t) \equiv 2$ for $t \in [0, 1/6]$, $x(1/6)$ is undefined.

References

[1] D. Angeli, E.D. Sontag, and Y. Wang, *A characterization of integral input to state stability*, in preparation.

[2] W. Hahn, *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[3] J. Hale, *Ordinary Differential Equations, Second Edition*, Krieger, Malabar, 1980.

[4] A. Isidori, *Global almost disturbance decoupling with stability for non minimum-phase single-input single-output nonlinear systems*, Systems & Control Letters, **28** (1996), pp. 115–122.

[5] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović, *Nonlinear and Adaptive Control Design*, John Wiley & Sons, New York, 1995.

[6] L. Praly and Y. Wang, *Stabilization in spite of matched unmodelled dynamics and an equivalent definition of input-to-state stability*, Math. of Control, Signals, and Systems, **9** (1996): 1-33.

[7] E.D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. on Automatic Control, **AC-34** (1989), pp. 435–443.

[8] ——, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Springer-Verlag, New York, 1990.

[9] E.D. Sontag and Y. Wang, *On characterizations of the input-to-state stability property*, Systems & Control Letters, **24** (1995), pp. 351–359.

[10] ——, *New characterizations of the input to state stability property*, IEEE Trans. on Automatic Control, **41**(1996), pp. 1283-1294.

[11] J. Tsinias, *Sontag's "input to state stability condition" and global stabilization using state detection*, Systems & Control Letters, **20** (1993), pp. 219–226.