

# Further results on controllability of recurrent neural networks<sup>1</sup>

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## Abstract

This paper studies controllability properties of recurrent neural networks. The new contributions are: (1) an extension of a previous result to a slightly different model, (2) a formulation and proof of a necessary and sufficient condition, and (3) an analysis of a low-dimensional case for which the hypotheses made in previous work do not apply. © 1999 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

This paper deals with controllability properties of what are often called “recurrent neural networks”. These constitute a class of nonlinear systems which, although formally analogous to linear systems, exhibit interesting nonlinear characteristics and arise often in applications, see e.g. [3–5, 7, 9–11, 14, 15, 18]. A general model of recurrent nets (see e.g. [15]) is as follows. Assume given a Lipschitz map  $\theta: \mathbb{R} \rightarrow \mathbb{R}$ . The most typical choice is

$$\theta(x) = \tanh x: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

which is also called the “sigmoid” or “logistic” map. For each positive integer  $n$ , we let  $\theta^{(n)}$  denote the diagonal mapping

$$\theta^{(n)}: \mathbb{R}^n \rightarrow \mathbb{R}^n: \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \theta(x_1) \\ \vdots \\ \theta(x_n) \end{pmatrix}.$$

**Definition 1.1.** An  $n$ -dimensional,  $m$ -input recurrent net (with activation function  $\theta$ ) is a system with state space  $\mathbb{R}^n$  and input-value space  $\mathbb{R}^m$ , respectively, and equations of the form

$$\dot{x}(t) = \theta^{(n)}(Ax(t) + Bu(t)), \quad (1)$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ .

Since  $\theta^{(n)}$  is a globally Lipschitz map, such a system is complete (every input is admissible for every state). For general terminology about systems and control, we follow [13]. In system-theoretic terms, we may represent a net by means of a block diagram as in Fig. 1.

Observe that, if we had  $\theta =$  the identity function, we would be studying continuous-time time-invariant linear systems. However, in this paper  $\theta$  will be bounded, which excludes that case. One manner in which recurrent nets arise is when modeling situations in which rates of change of state variables are bounded or saturated (we may use  $\tanh$  to impose  $|\dot{x}_i| < 1$ ). In the field of neural networks, one thinks of the coordinates of  $x$  as the describing the time evolution of an ensemble of

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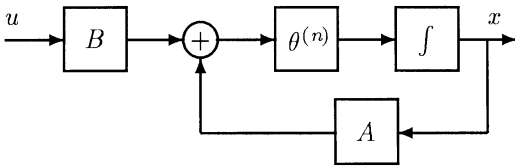


Fig. 1. Block diagram of net.

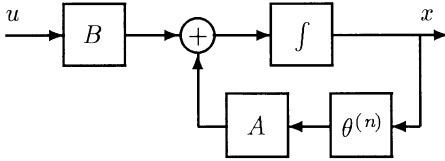


Fig. 2. Block diagram of input-affine net.

$n$  “neurons,” and the entries  $A_{ij}$  and  $B_{ij}$  as “synaptic strengths”, of the connections among neurons; in that context,  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  is called the “activation function”.

The paper [1] initiated the study of controllability properties for recurrent nets, and provided conditions for the “forward accessibility” of such systems. In [16] – see also [13] – a sufficient condition for complete controllability was obtained, and this condition is reviewed below. *One of the contributions of this paper is to show that the condition in [16] can be turned into a necessary and sufficient condition for a stronger form of complete controllability.*

There is a formal variant of the model given in the above definition, also sometimes referred to as a “recurrent neural network”, namely as follows.

**Definition 1.2.** An  $n$ -dimensional,  $m$ -input *input-affine recurrent net* (with activation function  $\theta$ ) is a system with state space  $\mathbb{R}^n$  and input-value space  $\mathbb{R}^m$ , respectively, and equations of the form

$$\dot{x}(t) = A\theta^{(n)}(x(t)) + Bu(t), \tag{2}$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ .

The block diagram representing such a system is as in Fig. 2.

As often remarked, see e.g. [2], it is sometimes possible to obtain results for input-affine recurrent networks from analogous results for (non-input-affine) nets, by means of the change of variables  $z = Ax + Bu$ . It is important to observe, however, that, under such a change of variables, one obtains equations of the form  $\dot{z} = A\theta^{(n)}(z) + Bv$ , where the new control  $v$  is the *derivative* of the original control. Thus, a solution  $x(\cdot)$  corresponding to the action of non-differentiable

(for instance, piecewise constant)  $u$  cannot be seen always as a trajectory of the new system; indeed, the function  $z$  will in general not be even absolutely continuous, so “ $z$ ” does not make sense. In addition, the feedback change of variables  $(x, u) \mapsto (Ax + Bu, v)$  can fail to be invertible because the matrix  $[A, B]$  does not have full rank (so not every state  $z$  can be written as  $Ax + Bu$  for some  $x$  and  $u$ ). In summary, obtaining results for input-affine nets by this transformation is not completely trivial, and some work is required when following this approach. In [2], it was shown how, for questions of parameter identification, this transformation can be fruitfully applied. *Another of the contributions of this paper is to follow this approach in order to provide a controllability result.* It turns out that a recent and nontrivial result about nonlinear controllability, from [6], is instrumental in the proof.

Finally, at the end of this paper we return to the model (1), and *provide a complete characterization of controllability for recurrent nets in the two-dimensional case.*

## 2. Statements of results

Given any continuous-time system  $\dot{x} = f(x, u)$ , we write  $\phi(t, x^0, u)$  for the solution  $x(t)$  corresponding to the initial state  $x(0) = x^0$  and the measurable (essentially) bounded control  $u : I \rightarrow \mathbb{R}^m$ , where  $I = [0, T]$  or  $I = [0, \infty)$ . For those systems that are considered in this paper, solutions exist globally and are unique, so this is well-defined for all  $(t, x^0, u)$ . (See [13] for generalities and basic facts about control systems.) We also write  $\phi(t, x, u)$  as  $\phi_u(t, x)$  or  $\phi_u^t(x)$ , depending on the context.

As usual, given any two states  $x^0, x^f$ , we say that  $x^0$  can be steered, or controlled, to  $x^f$  if there is some  $T \geq 0$  and some control  $u$  on  $[0, T]$  such that the solution is defined for all  $t \in [0, T]$  and  $\phi(T, x^0, u) = x^f$ . A system is *controllable* if every state  $x^0$  can be steered to every other state  $x^f$ .

For each pair of positive integers  $n$  and  $m$ , we introduce the following open and dense subset of the set of all  $n$  by  $m$  matrices:

$$\mathbf{B}_{n,m} := \{B \in \mathbb{R}^{n \times m} \mid (\forall i) \text{row}_i(B) \neq 0 \text{ and } (\forall i \neq j) \text{row}_i(B) \neq \pm \text{row}_j(B)\}$$

where  $\text{row}_i(\cdot)$  denotes the  $i$ th row of the given matrix. Observe that, in the special case  $m = 1$ , a vector

$b \in \mathbf{B}_{n,1}$  if and only if all its entries are nonzero and have different absolute values.

Finally, we let  $\Theta$  be the set of all functions  $\theta: \mathbb{R} \rightarrow \mathbb{R}$  which are Lipschitz and have the following properties:

1.  $\theta$  is an odd function, i.e.  $\theta(-r) = -\theta(r)$  for all  $r \in \mathbb{R}$ ;
2.  $\theta_\infty = \lim_{s \rightarrow +\infty} \theta(s)$  exists and is  $> 0$ ;
3.  $\theta(r) < \theta_\infty$  for all  $r \in \mathbb{R}$ ;
4. for each  $a, b \in \mathbb{R}$ ,  $b > 1$ ,

$$\lim_{s \rightarrow +\infty} \frac{\theta_\infty - \theta(a + bs)}{\theta_\infty - \theta(s)} = 0. \quad (3)$$

The activation function which appears most frequently in applications is  $\theta = \tanh$ , so the following result from [16] is worth mentioning:

**Lemma 2.1.** *The function  $\tanh \in \Theta$ .*

We recall the main result from [16]:

**Theorem 1.** *Assume that  $\theta \in \Theta$  and  $B \in \mathbf{B}_{n,m}$ . Then the system (1) is controllable.*

In order to state a necessary condition, we need the following concept, which we define for more general systems.

**Definition 2.2.** A system  $\dot{x} = f(x, u)$  is *strongly locally controllable* around a state  $x^0$  if for each neighborhood  $\mathcal{V}$  of  $x^0$  there is some neighborhood  $W$  of  $x^0$ , included in  $\mathcal{V}$ , so that, for every pair of states  $y$  and  $z$  in  $W$ ,  $y$  can be controlled to  $z$  without leaving  $\mathcal{V}$ .

This definition amounts to the requirement that any two states which are sufficiently close to  $x^0$  can be steered to one another without large excursions; see [13]. We will show, in Section 3:

**Theorem 2.** *Assume that  $\theta \in \Theta$ . Then the following two properties are equivalent, for Eq. (1):*

1.  $B \in \mathbf{B}_{n,m}$ .
2. The system (1) is strongly locally controllable around every state.

If a system with connected state space (as is the case with recurrent nets, whose state space is  $\mathbb{R}^n$ ) is strongly locally controllable around every state, then it is clearly also completely controllable. (Strong locally controllable around  $x^0$  implies that the reachable set  $\mathcal{R}(x^0)$  is both open and closed.) In fact, the proof of Theorem 1 given in [16] already establishes strong

local controllability. Thus, we only need to prove the necessity of  $B \in \mathbf{B}_{n,m}$ .

We now turn to input-affine nets. The main result, to be proved in Section 4, is:

**Theorem 3.** *Assume that  $\theta \in \Theta$ ,  $\text{rank}[A, B] = n$ , and  $B \in \mathbf{B}_{n,m}$ . Then Eq. (2) is completely controllable.*

Finally, in Section 5, we will return to the model (1), and ask now what can be said if the hypothesis that  $B \in \mathbf{B}_{n,m}$  is dropped. In general, obtaining necessary and sufficient conditions for controllability when  $B \notin \mathbf{B}_{n,m}$  appears to be a very difficult subject. We will, however, provide a complete solution for two-dimensional single-input ( $n = 2, m = 1$ ) systems.

### 3. Proof of Theorem 2

We start with a general observation.

**Lemma 3.1.** *If  $B \notin \mathbf{B}_{n,m}$  then there exists a nonzero  $p \in \mathbb{R}^n$  so that*

$$\text{sign } p^T \theta^{(n)}(Ax + Bu) = \text{sign } p^T Ax$$

for all  $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$  (with the convention  $\text{sign } 0 = 0$ ).

**Proof.** We denote, for each  $i$ ,  $b_i := \text{row}_i(B)$  and  $a_i := \text{row}_i(A)$ . There are three cases to consider: (1) some  $b_i = 0$ , (2)  $b_i - b_j = 0$  for some  $i \neq j$ , and (3)  $b_i + b_j = 0$  for some  $i \neq j$ .

In the first case, we let  $p := e_i$  ( $i$ th canonical basis vector). The conclusion follows from the equality

$$e_i^T \theta^{(n)}(Ax + Bu) = \theta(a_i x)$$

and the fact that  $\text{sign } \theta(v) = v$ . In the second case, we let  $p := e_i - e_j$ . The expression

$$\begin{aligned} (e_i - e_j)^T \theta^{(n)}(Ax + Bu) &= \theta(a_i x + b_i u) - \theta(a_j x + b_j u) \\ &= \theta(a_i x + b_i u) - \theta(a_j x + b_i u) \end{aligned}$$

is nonnegative if and only if  $a_i x \geq a_j x$  (using that  $\theta$  is monotonic), that is, when  $p^T Ax = a_i x - a_j x \geq 0$ , so also in this case we have the conclusion. Finally, if  $b_i + b_j = 0$ , we pick  $p := e_i + e_j$ . Then

$$\begin{aligned} (e_i + e_j)^T \theta^{(n)}(Ax + Bu) &= \theta(a_i x + b_i u) + \theta(a_j x + b_j u) \\ &= \theta(a_i x + b_i u) - \theta(-a_j x + b_i u) \end{aligned}$$

(since  $\theta$  is odd), which is nonnegative precisely when  $(a_i + a_j)x \geq 0$ .  $\square$

We now complete the proof of Theorem 2. Suppose that  $B \notin \mathbf{B}_{n,m}$ . Let  $p$  be as in Lemma 3.1 and let  $\mathcal{V}$  be  $\mathbb{R}^n$  if  $p^T A = 0$  and  $\{x \mid p^T A x > 0\}$  otherwise. Pick any state  $x^0 \in \mathcal{V}$ . Let  $\mathcal{R}_{\mathcal{V}}(x^0)$  be the set of states reachable from  $x^0$  without leaving  $\mathcal{V}$ . Then  $\mathcal{R}_{\mathcal{V}}(x^0)$  is included in the half-space

$$\{x \mid p^T x \geq p^T x^0\}.$$

This is because, if  $\dot{\xi} = \theta^{(n)}(A\xi + B\omega)$  and  $\xi(t) \in \mathcal{V}$  for all  $t$ , then

$$\frac{d}{dt} p^T \xi(t) = p^T \theta^{(n)}(A\xi(t) + B\omega(t)) \geq 0,$$

so the function  $\varphi(t) := p^T \xi(t)$  is nondecreasing. Thus, there cannot be any neighborhood of  $x^0$  which is reachable from  $x^0$  without leaving  $\mathcal{V}$ . This completes the proof of the theorem.  $\square$

#### 4. Proof of Theorem 3

We start by reviewing a very useful notion. We consider systems  $\dot{x} = f(x, u)$  for which the state space  $\mathcal{X}$  is an open subset of some Euclidean space  $\mathbb{R}^n$ , the input-value space  $\mathcal{U}$  is a subset of  $\mathbb{R}^m$ ,  $f$  is continuously differentiable on  $x$  and  $f$  and  $f_x$  are both continuous on  $(x, u)$ .

Given  $x^0, x^f \in \mathcal{X}$ , one says that  $x^f$  is normally reachable from  $x^0$  (with piecewise constant controls) if there are control values  $u_1, \dots, u_k \in \mathcal{U}$  and switching times  $t_1, \dots, t_k \in \mathbb{R}_{>0}$  such that

$$\phi_{u_k}^{t_k} \circ \dots \circ \phi_{u_1}^{t_1}(x^0) = x^f$$

and the map

$$(s_1, \dots, s_k) \mapsto \phi_{u_k}^{s_k} \circ \dots \circ \phi_{u_1}^{s_1}(x^0)$$

is nonsingular (i.e., its Jacobian at this point has rank  $n$ ) at  $(t_1, \dots, t_k)$ . If  $x^f$  is normally reachable from  $x^0$ , for all pairs  $x^0, x^f \in \mathcal{X}$ , we say the system is globally, or completely, normally controllable. It is easy to see that the notion of normal reachability has the following transitivity property.

**Proposition 4.1.** *If  $y$  is normally reachable from  $x$ ,  $x$  is reachable from  $z$  with piecewise constant controls*

*and  $w$  is reachable from  $y$  with piecewise constant controls, then  $w$  is normally reachable from  $z$ .*

Of course, normal controllability implies controllability. A fundamental result in control theory makes a converse statement:

**Theorem 4** (Grasse and Sussmann [6]). *Every completely controllable system is also completely normally controllable.*

The main technical fact that we need is given by the next result, which says that controllability is preserved under “backstepping” or “adding an integrator” [8]. This result is a counterpart of the well-known theorem which asserts that stabilizability is preserved under backstepping; cf. [13], (Ch. 4), but the proofs are very different. The critical step is showing that if a system is controllable then it is controllable using differentiable controls with preassigned boundary values. The technique of proof is almost identical to the one used in the much older paper [12], where it was shown that analytic systems are controllable using infinitely differentiable controls. We may now remove analyticity assumptions, thanks to the recent result that was cited in Theorem 4. We should point out, also, that the special case of the result for “systems without drift”, was previously given in [17].

**Theorem 5.** *Suppose that the input-value set  $\mathcal{U}$  is open and connected. Then,  $\dot{x} = f(x, u)$  is completely controllable if and only if the cascaded system*

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= v \end{aligned} \tag{4}$$

*with state space  $\mathcal{X} \times \mathcal{U}$  and control space  $\mathbb{R}^m$  is completely controllable.*

**Proof.** If the cascaded system is completely controllable, it is clear that the  $x$ -subsystem must be as well. We now prove the other direction. Assume that  $\dot{x} = f(x, u)$  is completely controllable; Theorem 4 then says that the system is completely normally controllable (with piecewise constant controls). Take any  $x^0, x^f \in \mathcal{X}$  and any  $y^0, y^f \in \mathcal{U}$ . We will show how to control  $(x^0, y^0)$  to  $(x^f, y^f)$ .

Pick  $\varepsilon_0 > 0$  small enough so that both  $\phi(\varepsilon_0, x^0, \underline{y}^0)$  and  $\phi(-\varepsilon_0, x^f, \underline{y}^f)$  are defined, where we wrote  $\underline{y}^0$  and  $\underline{y}^f$  to denote the constant controls taking

values  $y^0$  and  $y^f$  respectively. Let  $\tilde{x}^0 := \phi(\varepsilon_0, x^0, \underline{y}^0)$  and  $\tilde{x}^f := \phi(-\varepsilon_0, x^f, \underline{y}^f)$ . Using that the  $x$ -subsystem is completely normally controllable, in particular  $\tilde{x}^0$  is normally controllable to  $\tilde{x}^f$ . Thus, there exist  $k, u_1, \dots, u_k$ , and  $t_1, \dots, t_k \in \mathbb{R}_{>0}$ , such that

$$\phi_{u_k}^{t_k} \circ \dots \circ \phi_{u_1}^{t_1}(\tilde{x}^0) = \tilde{x}^f$$

and the map  $(s_1, \dots, s_k) \mapsto \phi_{u_k}^{s_k} \circ \dots \circ \phi_{u_1}^{s_1}(\tilde{x}^0)$  is non-singular at  $(t_1, \dots, t_k)$ . Applying the Inverse Mapping Theorem, we conclude that there is some neighborhood  $\tilde{V}$  of  $\tilde{x}^f$  and  $C^1$  functions  $\tilde{s}_1, \dots, \tilde{s}_k : \tilde{V} \rightarrow \mathbb{R}_{>0}$  such that

$$\phi_{u_k}^{\tilde{s}_k(\tilde{z})} \circ \dots \circ \phi_{u_1}^{\tilde{s}_1(\tilde{z})}(\tilde{x}^0) = \tilde{z}$$

for all  $\tilde{z} \in \tilde{V}$ . Let  $V := \phi_{\underline{y}^f}^{\varepsilon_0}(\tilde{V})$ , and define  $s_1, \dots, s_k : V \rightarrow \mathbb{R}_{>0}$  by

$$s_i(z) := \tilde{s}_i(\phi(-\varepsilon_0, \underline{y}^f, z))$$

for  $i = 1, \dots, k$ . Then,

$$\phi_{\underline{y}^f}^{\varepsilon_0} \circ \phi_{u_k}^{s_k(z)} \circ \dots \circ \phi_{u_1}^{s_1(z)} \circ \phi_{\underline{y}^0}^{\varepsilon_0}(x^0) = z$$

for all  $z \in V$ . Because  $\mathcal{U} \subseteq \mathbb{R}^m$  is open and connected, we may find a differentiable curve  $\gamma : [0, k+1] \rightarrow \mathcal{U}$  such that  $\gamma(0) = y^0$ ,  $\gamma(k+1) = y^f$  and  $\gamma(i) = u_i$  for  $i = 1, \dots, k$ . Define  $\omega : V \times \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$w(z, t) = \begin{cases} 0, & t < \varepsilon_0 \\ 1, & \varepsilon_0 \leq t < \varepsilon_0 + s_1(z) \\ \dots & \dots \\ k, & \varepsilon_0 + s_1(z) + \dots + s_{k-1}(z) \\ & \leq t < \varepsilon_0 + s_1(z) + \dots + s_k(z) \\ k+1, & \varepsilon_0 + s_1(z) + \dots + s_k(z) < t. \end{cases}$$

Then, we have

$$\phi(2\varepsilon_0 + s_1(z) + \dots + s_k(z), x^0, \gamma(\omega(z, \cdot))) = z \quad (5)$$

for all  $z \in V$ .

We now choose a sequence of functions  $\rho_l : \mathbb{R} \rightarrow \mathbb{R}$ ,  $l = 1, 2, \dots$ , so that the following properties, for each  $l$ :

- $\rho_l$  is  $C^\infty$ ,
- $\rho_l(t) \geq 0$  for all  $t \in \mathbb{R}$  and  $\rho_l(t) = 0$  for all  $|t| > 1/l$ ,
- $\int_{\mathbb{R}} \rho_l = 1$  for all  $l$ .

For instance, one could take

$$\rho_l(t) = \begin{cases} c_l e^{-\frac{1}{(l^2 t^2 - 1)^2}}, & |t| < 1/l, \\ 0, & \text{otherwise,} \end{cases}$$

where  $c_l$  is an appropriate normalizing constant.

We introduce, for each  $l$ , the function

$$\omega_l : V \times \mathbb{R} : (z, t) \mapsto \int_{\mathbb{R}} \omega(z, \tau) \rho_l(t - \tau) d\tau.$$

This is smooth in  $t$ , and the following properties hold:

1.  $\omega_l(z, t) \in [0, k+1]$  for all  $l, z \in V$  and  $t \in \mathbb{R}$ .
2. For each  $z \in V$ ,  $\omega_l(z, t) \neq \omega(z, t)$  only if  $|t - \varepsilon_0 - \sum_{i=1}^j s_i(z)| < 1/l$  for some  $j, j = 0, \dots, k$ .
3.  $\omega_l(z, 0) = 0$  and  $\omega_l(z, 2\varepsilon_0 + s_1(z) + \dots + s_k(z)) = k + 1$  for all  $z \in V$  and all  $l > 1/\varepsilon_0$ .

Note that the functions  $\gamma(\omega_l(z, \cdot))$  are valid controls, as they take values in  $\mathcal{U}$ , because of 1. Moreover, the values  $\gamma(\omega_l(z, t))$  belong to a compact subset  $K$  of  $\mathcal{U}$ , namely the image of  $\gamma$ , for all  $t$  and  $z \in V$ .

We claim that

$$h_l(z) \rightarrow \phi(2\varepsilon_0 + s_1(z) + \dots + s_k(z), x^0, \gamma(\omega(z, \cdot))) = z \quad (6)$$

as  $l \rightarrow \infty$ , uniformly on  $z$  (the last equality is true because of Eq. (5)), where we define for each  $l$  the map  $h_l : V \rightarrow \mathbb{R}^n$  as

$$h_l(z) := \phi(2\varepsilon_0 + s_1(z) + \dots + s_k(z), x^0, \gamma(\omega_l(z, \cdot))).$$

Indeed, we start by noting that we may assume, without loss of generality, that there is some  $T > 0$  such that the final times are equibounded:

$$2\varepsilon_0 + s_1(z) + \dots + s_k(z) < T \quad \text{for all } z \in \mathcal{V}$$

(take a smaller, relatively compact, neighborhood  $V$  of  $x^f$  if necessary). In general, the following continuous dependence result holds for the system  $\dot{x} = f(x, u)$ , a compact subset  $K \subseteq \mathcal{U}$ , and any fixed integer  $k$ : there is a function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ , with  $\alpha(s) \rightarrow 0$  as  $s \rightarrow 0$ , such that, whenever  $u, v$  are two inputs with values in  $K$  and of equal length  $T_0 \leq T$ , which differ in at most  $k+1$  intervals of length  $\Delta$ , then

$$\|\phi(T_0, x^0, u) - \phi(T_0, x^0, v)\| \leq \alpha(\Delta).$$

This is proved by induction, using continuity of trajectories with respect to initial conditions and to controls with uniform norm ([13], Theorem 1). We may apply this general fact, for each  $z \in V$ , and each  $l$ , to  $u = \omega_l(z, \cdot)$  and  $v = \omega(z, \cdot)$ , since property 2 above says that these differ on at most  $k+1$  intervals, each of length less than  $2/l$ . So the convergence (6) holds, since for each  $\varepsilon > 0$  we may simply pick  $l$  large enough so that  $\alpha(2/l) < \varepsilon$ , which then gives  $\|h_l(z) - z\| < \varepsilon$  for all  $z \in V$ .

Finally, we note another general fact (which appears often in control theory, notably in connection to the “Brunovsky–Lobry Lemma”): if  $V$  is a neighborhood of a point  $x^f \in \mathbb{R}^n$ , then is an  $\varepsilon > 0$  such that every continuous map  $h : V \rightarrow \mathbb{R}^n$  which is  $\varepsilon$ -close to the identity must contain a neighborhood of  $x^f$  in its image. (*Proof:* without loss of generality, we may take  $x^f = 0$ . Replace  $V$  by a subset, some closed ball  $B_\rho$  of radius  $\rho > 0$  about zero. Now take any  $\varepsilon \in (0, \rho)$ . Suppose that  $\|h(x) - x\| < \varepsilon$  for all  $x \in V$ . Then every  $p \in B_{\rho-\varepsilon}$  is in the image of  $h$ : for such a  $p$ , consider the map  $H(x) := x - h(x) + p$ ; then  $H$  maps  $B_\rho$  into itself, and hence by Brouwer’s Fixed Point Theorem there is some  $\bar{x}$  so that  $H(\bar{x}) = \bar{x}$ , which means that  $h(\bar{x}) = p$  as we wanted to prove.) Applied to the above maps  $h_i$ , which converge uniformly to the identity, we conclude, in particular that there are  $l$  and  $z_0 \in V$  such that

$$\phi(2\varepsilon_0 + s_1(z_0) + \dots + s_k(z_0), x^0, \gamma(\omega_l(z_0, \cdot))) = x^f.$$

Since both  $\gamma$  and  $\omega_l(z_0, \cdot)$  are continuously differentiable, we may write  $\gamma(\omega_l(z_0, t)) = y^0 + \int_0^t v(s) ds$  for some bounded measurable (in fact, continuous) function  $v : \mathbb{R} \rightarrow \mathbb{R}^m$ . As a control applied to the cascaded system (4),  $v$  steers  $(x^0, y^0)$  to  $(x^f, y^f)$ .

Since we had picked arbitrary  $(x^0, y^0)$  and  $(x^f, y^f)$ , this completes the proof of controllability.  $\square$

We now complete the proof of Theorem 3. From Theorems 1 and 5, we know that the cascaded system

$$\begin{aligned} \dot{x}(t) &= \theta^{(n)}(Ax(t) + By(t)), \\ \dot{y}(t) &= v \end{aligned} \tag{7}$$

is globally controllable.

Now take any two states  $z^0, z^f \in \mathbb{R}^n$  for system (2). Since  $\text{rank}[A, B] = n$ , there are pairs  $(x^0, y^0)$  and  $(x^f, y^f)$  in  $\mathbb{R}^{n+m}$  such that  $Ax^0 + By^0 = z^0$  and  $Ax^f + By^f = z^f$ . Let  $v : [0, T] \rightarrow \mathbb{R}^m$  be any control steering  $(x^0, y^0)$  to  $(x^f, y^f)$ , along a trajectory  $(x(\cdot), y(\cdot))$  of system (7). Let  $z(t) := Ax(t) + By(t)$ . This is an absolutely continuous function which satisfies  $z(0) = z^0$  and  $z(T) = z^f$ . Moreover, its derivative is  $A\dot{x} + B\dot{y} = A\theta^{(n)}(z) + Bv$ , so  $v$  is a control steering  $z^0$  to  $z^f$  for system (2).

### 5. Two-dimensional recurrent nets

From now on, we will assume, for simplicity, that in addition to  $\theta \in \Theta$ , also  $\theta$  is continuously differentiable,

and satisfies  $\theta'(s) > 0$  for all  $s \in \mathbb{R}$ ,  $\theta'$  is decreasing to zero on  $[0, \infty)$ ,  $\theta'(0) = 1$ , and  $\theta_\infty = 1$ . Note that the standard example  $\theta = \tanh$  satisfies these properties. Furthermore, in this section, all recurrent nets are two-dimensional and single-input ( $n = 2, m = 1$ ).

When  $B = (b_1, b_2)^T \notin \mathbf{B}_{n,1}$ , we may assume after a rescaling of inputs, changes of variables  $x \rightarrow -x$  or  $y \rightarrow -y$ , and/or exchanges of variables, that one of these cases holds:

1.  $B = (0, 0)^T$ ,
2.  $B = (0, 1)^T$ ,
3.  $B = (1, 1)^T$ ,

and in the first case we do not have controllability. In the remaining two cases, under a further feedback transformation of the type  $u \rightarrow ax + by + u'$ , where  $u'$  is a new control, one may transform a recurrent net, while preserving controllability properties, into one of the two canonical forms:

$$\begin{aligned} \dot{x} &= \theta(ax + by), \\ \dot{y} &= \theta(u) \end{aligned} \tag{8}$$

or

$$\begin{aligned} \dot{x} &= \theta(ax + u), \\ \dot{y} &= \theta(by + u). \end{aligned} \tag{9}$$

We now characterize controllability for each of these forms.

**Proposition 5.1.** *System (8) is controllable if and only if  $|a| \leq |b|$  and  $b \neq 0$ .*

**Proof.** First we prove the necessity part. Obviously, if  $b = 0$  the system is not controllable. So we can assume that  $|a| > |b| > 0$ . There are two cases to consider:

$a > 0$ : In this case, since  $\theta_\infty = 1$ , we may find  $A > 0$  such that  $a\theta(z) > |b|$  for all  $z \geq A$ . Then, the set  $\{(x, y) \in \mathbb{R}^2 \mid ax + by \geq A\}$  is forward invariant for system (8), which therefore cannot be controllable.

$a < 0$ : We consider the system obtained by reversing time in (8) and redefining the control as  $-u$ :

$$\begin{aligned} \dot{x} &= \theta(-ax - by), \\ \dot{y} &= \theta(u') \end{aligned} \tag{10}$$

(note that, since  $\theta$  is an odd function, we may write  $-\theta(ax + by) = \theta(-ax - by)$  and  $-\theta(u) = \theta(u')$ , where we think of  $u' := -u$  as a new control). Since (10) is not controllable, (8) is not controllable either.

We now turn to the sufficiency part. To prove controllability, it is enough to show, for any system that satisfies  $|a| \leq |b|$  and  $b \neq 0$ , that the system is null-controllable, i.e., every state can be steered to 0. Indeed, suppose that this general fact has been established. Now to steer a state back from 0 to any other state  $(x, y)$ , one may consider system (10). This also satisfies the hypotheses and hence is null-controllable. We find a control  $u$  that steers the state of this new system from  $(x, y)$  to 0, and then apply  $-u$  to the original system with the time reversed, to drive 0 to  $(x, y)$ . So we can steer any state to any other, by first passing through the origin. Moreover, since  $b \neq 0$ , the system is locally controllable around the origin, and so it suffices to prove that every state  $x$  can be steered into any arbitrarily chosen neighborhood of 0 (asymptotic null controllability).

We start with the case  $a = 0$ . Take any neighborhood  $V$  of the origin, and pick any state  $(x, y) \in \mathbb{R}^2$ . It is clear that there is some control that steers this state to some new state of the special form  $(x', 0)$ . If this state is in  $V$ , we are done. Otherwise, we have  $x' \neq 0$ . We will assume without loss of generality that  $x' > 0$ , the argument for  $x' < 0$  being analogous. We next steer our state to a state of the form  $(x'', -b)$ , along the trajectory  $(\xi, \eta)$ . Let

$$t_0 := \inf \{t > 0 \mid b\eta(t) < 0\}.$$

Then,  $\xi(t_0) = x' + \int_0^{t_0} \theta(b\eta(t)) dt > 0$  and  $\eta(t_0) = 0$ . By the definition of  $t_0$  and the continuity of  $\xi$ , we may choose  $t_1 > t_0$  such that  $\xi(t_1) > 0$ ,  $b\eta(t_1) < 0$  and  $(0, \eta(t_1)) \in V$ . Thus, we can steer our initial state to the state  $(\xi(t_1), \eta(t_1))$ , and after that apply the control  $\equiv 0$ . The system will reach the state  $(0, \eta(t_1))$  in finite time. This completes our proof for the case  $a = 0$ .

Now we assume that  $0 < |a| \leq |b|$ , and show the asymptotic null controllability. Through a change of variables

$$\tilde{x} = ax + by,$$

$$\tilde{y} = y$$

and  $v = \theta(u)$ , system (8) becomes as follows:

$$\begin{aligned} \dot{\tilde{x}} &= a\theta(\tilde{x}) + bv, \\ \dot{\tilde{y}} &= v \end{aligned} \quad (11)$$

with control value space  $(-1, 1)$ . The original system (8) is asymptotically null-controllable if and only if Eq. (11) is. Thus, we now prove that Eq. (11) is asymptotically null-controllable. If  $a < 0$ , then we

simply steer  $\tilde{y}$  to 0 and then use the identically zero control, which makes  $\tilde{x}$  go to 0 asymptotically while “freezing”  $\tilde{y}$  at zero. So we only need to deal with the case  $a > 0$ . Moreover, we may assume without loss of generality that  $b > 0$ . If this were not the case, we change variables again, making  $y' := -\tilde{y}$  and using the new control  $v' := -v$ , which brings the system into the above form with  $b$  replaced by  $-b$ . In summary, we have  $0 < a \leq b$  from now on.

We start by defining the following set:

$$\Omega := \{(x, y) \mid x > 0, y > 0, x + ay\theta(x) < by\}.$$

We claim that every initial state  $(\tilde{x}^0, \tilde{y}^0) \in \Omega$  can be steered to the origin asymptotically. To see this, we first apply the feedback control

$$v = \frac{a\tilde{y}^0}{\tilde{x}^0 - b\tilde{y}^0} \theta(\tilde{x})$$

to obtain the closed-loop system:

$$\begin{aligned} \dot{\tilde{x}} &= \frac{a\tilde{x}^0}{\tilde{x}^0 - b\tilde{y}^0} \theta(\tilde{x}), \\ \dot{\tilde{y}} &= \frac{a\tilde{y}^0}{\tilde{x}^0 - b\tilde{y}^0} \theta(\tilde{x}). \end{aligned} \quad (12)$$

Let  $(\xi, \eta)$  be the trajectory of Eq. (12) that corresponds to the initial state  $\xi(0) = \tilde{x}^0$ ,  $\eta(0) = \tilde{y}^0$ . Since  $\tilde{x}^0 > 0$  and  $\tilde{x}^0 - b\tilde{y}^0 < -a\tilde{y}^0\theta(\tilde{x}^0) < 0$ , it follows that  $\xi(t) \rightarrow 0$  monotonically as  $t \rightarrow \infty$ . Consider the function  $\alpha(t) := \tilde{y}^0\xi(t) - \tilde{x}^0\eta(t)$ . Then, from the form of the equations, it follows that  $\dot{\alpha} \equiv 0$  and  $\alpha(0) = 0$ . Thus  $\alpha \equiv 0$ , so  $\eta(t) = (\tilde{y}^0/\tilde{x}^0)\xi(t) \rightarrow 0$  as well.

By definition of  $\Omega$ ,  $0 < a\tilde{y}^0\theta(\tilde{x}^0) < b\tilde{y}^0 - \tilde{x}^0$ . Since  $\xi$  is decreasing, we have that  $\theta(\xi(t)) \leq \theta(\tilde{x}^0)$  for all  $t \geq 0$ . So

$$\left| \frac{a\tilde{y}^0}{\tilde{x}^0 - b\tilde{y}^0} \theta(\xi(t)) \right| < 1$$

for all  $t \geq 0$ , so the control  $v$  takes values in  $(-1, 1)$ , as required. This proves the claim. Moreover, the same claim is also true for  $-\Omega$ , because of the symmetry of the system (that is, if we change variables  $x' := -\tilde{x}$ ,  $y' := -\tilde{y}$ , and  $v' := -v$ , the system equations are the same but  $\Omega$  gets transformed into  $-\Omega$ ).

Now suppose given an arbitrary state  $(\tilde{x}^0, \tilde{y}^0) \in \mathbb{R}^2$ . We first steer it to some state of the form  $(0, \tilde{y}_1)$ . This can always be done with controls bounded by one, because  $a \leq b$ . If  $\tilde{y}_1 = 0$ , we are done. Otherwise, if  $\tilde{y}_1 > 0$ , we steer it to some  $(\tilde{x}_2, \tilde{y}_2) \in \Omega$ . If  $\tilde{y}_1 < 0$ , we steer it to some  $(\tilde{x}_2, \tilde{y}_2) \in -\Omega$ . These motions can be accomplished by the application of a short duration

control, positive or negative respectively. By the claim proved above, we may then steer  $(\tilde{x}_2, \tilde{y}_2)$  to the origin asymptotically.  $\square$

**Proposition 5.2.** *System (9) is completely controllable if and only if  $a \neq b$ .*

**Proof.** We first show the necessity statement. Suppose that  $a = b$ , and consider the initial state  $(0, 0)$ . Pick any control  $u$ , and let  $\xi$  be the solution of  $\dot{x} = \theta(ax + u)$  with  $\xi(0) = 0$ . By uniqueness of solutions,  $(\xi, \xi)$  is the trajectory for the entire system with the initial state  $(0, 0)$ . Thus any state  $(x, y)$  reachable from  $(0, 0)$  must have  $x = y$ , and the system is not controllable.

Now we turn to proving sufficiency when  $a \neq b$ . Arguing exactly as before, it suffices to prove asymptotic null-controllability, because the condition  $a \neq b$  is again satisfied for the time reversed equations ( $a \rightarrow -a$  and  $b \rightarrow -b$ ), and the linearization at the origin is controllable due to the condition  $a \neq b$ . We may further assume (exchange equations if necessary) that  $a > b$ . There are two cases to consider.

$b \geq 0$ : Here  $a > b \geq 0$ . Let

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid 0 < x < y, ax > by\}.$$

**Claim 1.** *Pick  $(x^0, y^0) \in \Omega$ . Then, there is a  $C^1$  function*

$$k : (0, \infty) \rightarrow \mathbb{R}$$

which has the following property:

$$y^0 \theta(sax^0 + k(s)) - x^0 \theta(sby^0 + k(s)) = 0 \quad \text{for all } s \in (0, \infty). \quad (13)$$

Indeed, we consider the following function:

$$f(s, u) := y^0 \theta(sax^0 + u) - x^0 \theta(sby^0 + u).$$

For each fixed  $s$ ,  $f(s, -\infty)f(s, \infty) = -(x^0 - y^0)^2 < 0$ . So there exists some  $u = k(s)$  such that  $f(s, k(s)) = 0$ . Moreover, for each  $u \in \mathbb{R}$  for which  $f(s, u) = 0$ , necessarily

$$\frac{\partial f}{\partial u}(s, u) > 0. \quad (14)$$

To verify this, take any  $u$  so that  $f(s, u) = 0$ . Since  $ax^0 > by^0$ , also

$$\theta(sax^0 + u) > \theta(sby^0 + u)$$

and therefore

$$x^0 \theta(sax^0 + u) > x^0 \theta(sby^0 + u) = y^0 \theta(sax^0 + u),$$

where the last equality follows from Eq. (13). This implies that  $(x^0 - y^0)\theta(sax^0 + u) > 0$ , which, since  $x^0 - y^0 < 0$ , means that

$$sax^0 + u < 0. \quad (15)$$

We know, then, that  $sby^0 + u < sax^0 + u < 0$ . The derivative  $\theta'$  is increasing on  $(-\infty, 0]$  (since  $\theta$  is odd and we assumed that  $\theta'$  is decreasing on  $[0, \infty)$ ), so

$$0 < \theta'(sby^0 + u) \leq \theta'(sax^0 + u).$$

Multiplying this inequality by  $x^0 < y^0$  gives (14). We conclude that  $u = k(s)$  is the unique solution of the equation  $f(s, u) = 0$  and, by the Implicit Mapping Theorem, that  $k$  is  $C^1$ . So Claim 1 is proved.

**Claim 2.** *Any  $(x^0, y^0) \in \Omega$  can be controlled to the origin asymptotically.*

To prove this, we apply the feedback control  $u = k(x/x^0)$ . Let  $\xi$  solve  $\dot{\xi} = \theta(a\xi + k(\xi/x^0))$  with  $\xi(0) = x^0$ . Notice that  $u$  is well-defined (the argument of  $k$  is positive) at least for small  $t > 0$ , since  $\xi(0)/x^0 = 1$ . We let, for all  $t$  so that  $\xi(t) > 0$ :

$$s(t) := \frac{\xi(t)}{x^0}.$$

Thus,  $a\xi(t) = s(t)ax^0$  for all such  $t$ . Now we let  $\eta(t) := y^0 s(t)$ . It follows that

$$\dot{\eta} = \frac{y^0}{x^0} \dot{\xi} = \frac{y^0}{x^0} \theta(a\xi + k(s)) = \frac{y^0}{x^0} \theta(sax^0 + k(s))$$

which in turn equals  $\theta(sby^0 + k(s))$  by Eq. (13). So, substituting  $y^0 s(t) = \eta(t)$ , we have that  $\dot{\eta} = \theta(b\eta + k(s))$ . Thus  $(\xi, \eta)$  is the trajectory with  $(\xi(0), \eta(0)) = (x^0, y^0)$ , as long as  $\xi(t)$  remains positive. Moreover, Eq. (15) insures that  $a\xi + k(s) < 0$ , so  $\dot{\xi} < 0$  whenever  $\xi > 0$ . This implies that  $\xi(t) \rightarrow 0$  as  $t$  increases (it could be the case that  $\xi$  becomes zero in finite time, or  $\xi(t) \rightarrow 0$  as  $t \rightarrow \infty$ ), and, so also  $\eta(t) = y^0 \xi(t)/x^0$  approaches zero. This proves Claim 2.

Observe that the same claim holds true for  $-\Omega$ , because of the symmetry of the system (that is, if we change variables  $x' := -x$ ,  $y' := -y$ , and  $u' := -u$ , the system equations are the same but  $\Omega$  gets transformed into  $-\Omega$ ).

Finally, take an arbitrary  $(x^0, y^0) \in \mathbb{R}^2$ . Suppose that this point is not already in  $\Omega \cup (-\Omega) \cup \{(0, 0)\}$ . We may first steer this point to a state of the form  $(0, y_1)$ . (This can obviously be done, for instance by first using

$u = -ax - x$  to bring  $x$  close to zero, and then using local controllability of the first equation). If  $y_1 = 0$ , then we are done.

Suppose  $y_1 > 0$ . If we apply  $u = 1$  thereafter,  $\zeta(t) \rightarrow \infty$  and  $\eta(t) \rightarrow \infty$  along the ensuing trajectory (because the right-hand side in each of the equations in Eq. (9) is bounded below by  $\theta(1) > 0$ ). We evaluate the limit of  $\zeta/\eta$  using L'Hôpital's rule. We have that  $\dot{\zeta} = \theta(a\zeta + 1) \rightarrow 1$  and either  $\eta \rightarrow 1$  (if  $b > 0$ ) or  $\eta \rightarrow \theta(1)$  (if  $b = 0$ ), so in either case the limit in question is  $\geq 1$ . Since  $a > b$ ,  $a\zeta - b\eta$  eventually becomes positive. Let  $t_0$  be infimum of the  $t \geq 0$  so that  $a\zeta(t) - b\eta(t) > 0$ . For  $t \leq t_0$ ,

$$\dot{\zeta}(t) = \theta(a\zeta(t) + 1) \leq \theta(b\eta(t) + 1) = \dot{\zeta}(t)$$

so  $\eta - \zeta$  increases while  $t \leq t_0$ . Thus  $\eta(t_0) - \zeta(t_0) \geq y_1 > 0$ , and so for some small  $t > t_0$  we have that  $\eta(t) > \zeta(t)$  and  $a\zeta(t) > b\eta(t)$ , so  $(\zeta, \eta)$  enters  $\Omega$ . If  $y_1 < 0$ , we may apply  $u = -1$  and conclude that we eventually enter  $-\Omega$ . In any case, we can then steer the state of the system to zero asymptotically. This completes consideration of the case  $b \geq 0$ .

$b < 0$ : In this case we may first steer  $x$  to 0 and subsequently apply the identically zero control. The  $y$  coordinate will go to 0 asymptotically, while  $x$  is "frozen" at zero.

This completes our discussion of the two cases.  $\square$

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