AN ALGEBRAIC APPROACH TO BOUNDED CONTROLLABILITY OF LINEAR SYSTEMS

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ABSTRACT

In this note we present an algebraic approach to the proof that a linear system with matrices (A,B) is null-controllable using bounded inputs iff it is null-controllable (with unbounded inputs) and all eigenvalues of A have nonpositive real parts (continuous time) or magnitude not greater than one (discrete time). We also give the analogous results for the asymptotic case. Finally, we give an interpretation of these results in the context of local nonlinear controllability.

1. INTRODUCTION

Let (A,B) be a pair of real matrices satisfying the usual Kalman reachability condition: rank(A, AB, ..., A^{n-1}B) = n = size of A. Assume that B has m columns and that the input space is now restricted to a bounded set *U* of $\mathbf{R}^m$ which contains 0 in its interior. We wish to know if the corresponding continuous and discrete time systems (with control space *U*) are null-controllable *(n.c.)*. Since these systems are *locally* n.c., it is clear that a *sufficient* condition for n.c. to hold is for A to be a (continuous or discrete) stability matrix. It is also true, but less obvious, that n.c. holds even if some eigenvalues of A are purely imaginary (magnitude=1, in discrete time); the difficulty in principle is in the control of the possible polynomially unstable modes,using bounded inputs.

This more general statement was proved for continuous time scalar (m=1) systems in [2], which also gives a proof in the (continuous) case with m>1 but with extra assumptions on A. These assumptions were dropped in [1] (see also [5]). One goal of this note is to show that this result can be obtained as a

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simple consequence of some general facts (to be developed below) in the "module theoretic" theory of linear systems together with a rather interesting lemma about polynomials. More importantly, the proof given here applies equally well to (and in fact depends of) the discrete time case. The latter is of interest in itself and also in terms of the continuous time case, since it provides a result on sampled controllability. We present also the corresponding results for asymptotic null-controllability (a.n.c.).

Finally, we shall mention an application to the problem of (local) nonlinear a.n.c. As discussed in [6,7], it is desirable to include, in the definition of a.n.c., a statement about magnitudes of controls needed to control small states. The linear results mentioned above can be translated into one such statement.

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2. MAIN RESULTS

Let $A\in\mathbf{R}^{n\times n}$, $B\in\mathbf{R}^{n\times m}$. Associate to (A,B) the continuous [resp., discrete] time system Σ with equations

(2.1) $\dot{x}(t)$ [$x(t+1)$] $= Ax(t)+Bu(t)$.

Let $\phi(t,x,u)$ denote the state of Σ at time $t>=0$ which results from $x(0)=x$ and the application of the control $u(\cdot)$ (discrete or continuous time, depending on the context; in the continuous case, take the controls to be piecewise continuous). Fix $U\subseteq\mathbf{R}^n$. An (m-input) system Σ is in *ANC(U)* (*asymptotically null-controllable using controls in U*) iff, for each state x, there is an input $u(\cdot)$ with $u(t)\in U$ for all t and such that $\phi(t,x,u)\to 0$ as $t\to\infty$. If, moreover, there is for each x an u with values in *U* such that $\phi(T,x,u)=0$ for some (finite) T, then Σ is in *NC(U)*. We say that Σ is *null-controllable (n.c.)* [resp., *asymptotically n.c. (a.n.c.)* if $\Sigma\in NC(\mathbf{R}^n)$ [resp., $\Sigma\in ANC(\mathbf{R}^n)$], and that Σ is SINC (*small-input n.c.*) [resp., SIANC (*small-input a.n.c.*)] iff Σ is in *NC(U)* [resp., *ANC(U)*] for every *U* that is a ngbd of 0 in $\mathbf{R}^m$. Finally, a continuous [resp., discrete] time Σ (or the corresponding A) is *asystable* iff each eigenvalue λ of A has $\mathrm{Re}\lambda<0$ [resp., $|\lambda|<1$] and *adequate* iff each λ satisfies $\mathrm{Re}\lambda<=0$ [$|\lambda|<=1$].

Factor the characteristic polynomial of A as $\pi=\pi_s\pi_u$, where π_u collects all the roots with $\mathrm{Re}\lambda>=0$ [$|\lambda|>=1$, in discrete time]. Let $X_u=X_u(A):=\ker\pi_u$, A_u: $X_u\to X_u$ be the restriction of A, and B_u: $\mathbf{R}^m\to X_u$ the (co-)restricted map. There results a system Σ_u (well-defined up to a choice of basis for X_u). Similarly for a system Σ_s. Recall that Σ is a.n.c. iff Σ_u is reachable. This is proved by a standard argument: If Σ is a.n.c., consider the induced system in X_u/R_u (R_u = reachable set of Σ_u). It is easy to see that this system is again a.n.c.; on the other hand, this quotient system has B=0 and an A which is not asystable, a contradiction. For the converse, note that reachability implies that Σ_u is n.c. and that, in general,

$\Sigma \in ANC(U)$ whenever $\Sigma_u \in NC(U)$, provided that $0 \in U$: This is because one may always apply first a control with values in U sending the X_u-coordinate of a given state to 0 (in finite time), and then concatenate this control with one (of infinite length) constantly = 0. Regarding n.c., recall that this is equivalent in continuous time, to reachability, and in discrete time to the image of A^n being included in the reachable set (so that it is again equivalent to reachability if A is known to be nonsingular, e.g. for Σ_u).

The main results on SINC and SIANC are as follows.

(2.2) **THEOREM**. The following statements are equivalent:
(a) Σ is SINC;
(b) $\Sigma \in NC(U)$ for some bounded ngbd of 0 U;
(c) $\Sigma \in NC(U)$ for some bounded U;
(d) Σ is n.c. and adequate.

(2.3) **THEOREM**. The following statements are equivalent:
(a) Σ is SIANC;
(b) $\Sigma \in ANC(U)$ for some bounded ngbd of 0 U;
(c) $\Sigma \in ANC(U)$ for some bounded U;
(d) Σ is a.n.c. and adequate.

Note that (d) above is equivalent to asking that Σ_u be reachable and adequate (i.e., Σ_u is SINC,) so that in particular all eigenvalues λ of A_u have real part = 0 [$|\lambda|=1$ for discrete time].

The proofs of 2.2 and 2.3 will be given below. We first consider the following feedback characterization, which is fairly obvious (using the control canonical form) in the scalar (m=1) case. Choose any norm on matrices $K \in \mathbf{R}^{m \times n}$.

(2.4) **THEOREM.** Σ is SIANC iff the following property holds: For each $\beta>0$ there is a K such that $\|K\|<\beta$ and A+BK is asystable.

PROOF (assuming 2.2-2.3). Assume that the stated property holds. By continuity (on K) of the eigenvalues of A+BK, Σ (equivalently, Σ_u) must be adequate. And the property clearly implies a.n.c. of Σ, so 2.3 applies. Conversely, assume that Σ is SIANC. Observe that if the result is proved for Σ_u then it will also be true for Σ; this is because, if $A_u+B_uK_u$ is asystable, then A+BK also is, for $K:=(0,K_u)$. We use this observation to conclude that we may assume that Σ is reachable (since Σ_u is). Consider now the smooth map

(2.5) $\Lambda : G \to \mathbf{R}^{n\times n} : (T,K) \to T(A+BK)T^{-1}$,

where $G := GL(\mathbf{R}^n)\times\mathbf{R}^{m\times n}$, seen as a product manifold. We calculate the differential of Λ at $e:=(I,0)$ (I=identity): consider for any $(S,L)\in T_eG$ the curve $(e^{tS}, I+tL)$; an application of the Baker-Campbell-Hausdorff formula gives that

(2.6) $d_e\Lambda(S,L) = [S,A]+BL$,

where [,] denotes the Lie brackett on $\mathbf{R}^{n\times n}$. It can be proved that *every* matrix can be written as $[S,L]+BL$, for fixed (A,B) and varying (S,L), if and only if (A,B) is reachable; see [3]. Thus, Λ is a submersion at e. By the implicit function theorem, Λ is open on a ngbd N of e. Pick $\beta>0$, and restrict N so that $(T,K)\in N$ implies $\|K\|<\beta$. Since $\Lambda(N)$ is an open ngbd of A, it must contain some asystable matrix C. (This last statement can be proved in a variety of ways; for instance, by a perturbation of the nonfixed coefficients of the rational canonical form of A. It is essential here, of course, that A is adequate.) Thus $C = T(A+BK)T^{-1}$, for suitable T and K such that $\|K\|<\beta$. It follows that A+BK is also asystable, as desired.

It is interesting to remark that an argument analogous to the above can be used to give a proof of the pole-shifting theorem that does not require Heymann's lemma for the reduction to the scalar case: given a reachable (A,B), the same argument shows that we may perturb A by feedback into a *cyclic* matrix; now we know that there is a u such that (A,Bu) is reachable (see [8,p.42]), and we are back in the scalar case.

We turn now to the proofs of 2.2 and 2.3. Note that (a)$\Rightarrow$(b)$\Rightarrow$(c) are trivial in both. The implication (d)$\Rightarrow$(a) in 2.3 follows from 2.2: choose U a ngbd of 0. Since Σ_u is SINC, $\Sigma_u\in$ *NC(U)*, so by a previous remark we conclude that $\Sigma\in$ *ANC(U)*, as desired. Consider now (d)$\Rightarrow$(a) in 2.2. We claim that it is enough for this to prove the result for *reachable* Σ. Indeed, assume given any n.c. adequate Σ. In continuous time, Σ is necessarily reachable. In discrete time, split A into a direct sum of a nilpotent A_o and a nonsingular A_1. The induced (A_1,B_1) is reachable, hence also SINC if the result has been proved for reachable systems. But then Σ itself is SINC, because any controlling input for the nonsingular subsystem can be concatenated by a finite sequence of zeroes which controls the nilpotent part to 0. Further, we claim that for (d)$\Rightarrow$(a) it is enough to treat the discrete time case (which is the topic of the next section). For this, assume that Σ is continuous time, reachable, and adequate. Find a sampling rate d small enough that the induced discrete time system Σ_d is itself reachable. Since $A_d = \exp(dA)$ is (discrete time) adequate, Σ_d satisfies (d) and is therefore SINC. Thus Σ is itself SINC, in fact with sampled (i.e., constant on intervals of length d) controls.

Finally, we prove that (c)⇒(d) in 2.2-2.3. We give shall only give details for the continuous time case; the discrete case in totally analogous. Since *NC(U)*$\subseteq$*ANC(U)* for all *U*, all we need to prove is that 2.3(c) implies that Σ is adequate. We proceed analogously to [2,p.92]. Assume that Σ is not adequate. Modulo GL(n), we may write the equations for Σ in such a way that the first coordinate is

(2.7) $\dot{x}_1 = \lambda x_1+\Sigma b_i u_i$,

with some λ (real)>0, or the first two coordinates as

(2.8) $\dot{x}_1 = \alpha x_1+\beta x_2+\Sigma b_i u_i$,

$\dot{x}_2 = -\beta x_1+\alpha x_2+\Sigma b_i u_i$,

with real α>0. Since the *U* in 2.3(c) is bounded, there is a large enough initial $x_1(0)$ in 2.7, or a pair $(x_1(0),x_2(0))$ with $x_1^2(0)+x_2^2(0)$ large enough in 2.8, such that the derivative $\dot{x}_1$ (or the derivative of $x_1^2+x_2^2$) is positive for any control with values in *U*. Thus there are initial vectors x(0) which cannot be controlled asymptotically to 0, a contradiction.

3. PROOF OF THE SUFFICIENCY

We establish in this section the only remaining (and the most interesting) implication of 2.2-2.3: if Σ is discrete time, reachable, and adequate, then it is SINC. The proof will proceed by first reducing to the scalar case, through the introduction of a set of ("finite memory") input transformations which acts on reachable systems. These transformations will be such that systems in the same orbit will be simultaneously SINC and such that every system becomes equivalent to a parallel connection of scalar systems. (Note that the latter requirement would be satisfied by the feedback group, but the first requirement would not, since the SINC property is not preserved under feedback.)

Specifying a system (A,B) (with m inputs) amounts (up to isomorphism) to giving a pair (X,g), where X is a finitely generated torsion **R**[z]-module and g:Λ→X is a surjective **R**[z]-linear map (here we denote $\Lambda:=\mathbf{R}[z]^m$); see for instance [4,ch.10]. We define an equivalence among reachable systems using their representation as pairs (X,g): $\Sigma_1 \sim \Sigma_2$ will mean that there exist **R**[z]-linear maps V:Λ→Λ and T:X→X such that the following diagram commutes:

(3.1)
$$\begin{array}{ccc} \Lambda & \xrightarrow{V} & \Lambda \\ g_1\downarrow & & \downarrow g_2 \\ X_1 & \xrightarrow{T} & X_2 \end{array}$$

Another way of expressing this equivalence is the following. For a given automorphism V of Λ, there is a T as here iff $V(\ker g_1) = \ker g_2$. Thus two systems are equivalent iff the kernels of their g-maps are isomorphic as *submodules* of Λ; if D_i is a (square) matrix whose columns form a basis of $\ker g_i$, this amounts to requiring that D_1 and D_2 be equivalent as polynomial matrices.

Note the following facts: (a) Σ_1 isomorphic to Σ_2 implies $\Sigma_1 \sim \Sigma_2$ (isomorphism is the same as the existence of a T so that (identity,T) is as above); (b) when m=1, $\Sigma_1 \sim \Sigma_2$ *only if* they are isomorphic (because in that case V must be a multiplication by a nonzero scalar r, so that $r^{-1}T$ gives an isomorphism); (c) $\Sigma_1 \sim \Sigma_2$ implies that A_1 is similar to A_2, and in particular that Σ_1 is adequate iff Σ_2 is; (d) for each Σ_1 there is a Σ_2 equivalent to it which is a parallel connection of scalar systems, i.e., such that the basis matrix D_2 of $\ker g_2$ is diagonal (this by existence of Smith forms, i.e. the fundamental structure theorem for f.g. modules over a p.i.d). Finally, we need the following property.

(3.2) LEMMA. Assume that $\Sigma_1 \sim \Sigma_2$. Then, Σ_1 is SINC iff Σ_2 is SINC.

PROOF. Assume that Σ_1 is SINC. Let U_2 be a ngbd of 0 in $\mathbf{R}^m$, and pick an $\alpha>0$ such that $\{\|u\|<\alpha\} \subseteq U_2$. We will prove that $\Sigma_2 \in NC(U_2)$. Let (V,T) give the equivalence $\Sigma_1 \sim \Sigma_2$, and write $V = \Sigma V_i z^i$ as a polynomial of degree r with coefficients in $\mathbf{R}^{m \times m}$. Pick any $\beta>0$ such that $r.\|V_i\|.\beta < \alpha$ for i= 1,...,r. (Use any pair of compatible norms on vectors and operators.) Take $U_1 := \{\|u\|<\beta\}$. Pick any $y \in X_2$, and let $x := T^{-1}y$, $x' := z^r x$. Since $\Sigma_1 \in NC(U_1)$, there is an $\alpha>0$ and a $w = \Sigma u_j z^j \in \Lambda$ of degree<s such that:

(3.3) $z^s x' + g_1(w) = 0$.

Applying T to both sides of this equation gives:

(3.4) $z^{r+s} y + g_2(w') = 0$,

where w'=Vw. Since degw' < r+s, (3.4) shows that w' controls y to 0. Write $w' = \Sigma u'_k z^k$. From the choice of β it follows that all $\|u'_k\|<\alpha$, so that w' has values in U_2, as desired.

It follows from the above observations that it is enough to prove the result for systems which are parallel combinations of scalar systems, since all desired properties are preserved under equivalence. But then, it is only necessary to treat the scalar case, since a sum of systems is SIANC iff each system is (just apply independent controls on each channel). Consider the following fact about polynomials:

(3.5) PROPOSITION. Let p(z) be a monic real polynomial all whose roots have magnitude <=1. Then, for each $\beta>0$, there exists a monic real polynomial $q(z) = z^k + \Sigma a_i z^i$ such that: (1) p divides q, and (2) $|a_i|<\beta$ for each i.

Assume for a moment that 2.5 is true. Let $\Sigma = (X,g)$ be a (reachable) scalar adequate system. We shall identify X with $\Lambda/(p)$, where p is the characteristic polynomial of A, and g with the canonical quotient map. Pick now an $U \subseteq \mathbf{R}$, containing a ngbd $\{|u|<\alpha\}$. Pick any state x. Let v_i, i = 0,...,n-1, be such that:

(3.6) $z^{n-1}x = \Sigma v_i z^i$ (mod p).

Let M be an upper bound on the $|v_i|$, and let $\beta := \alpha(Mn)^{-1}$. Apply the above lemma to these p, β, and let q be as there. Thus, $z^k = -\Sigma a_i z^i$ (mod p). It follows that:

(3.7) $z^{n+k-1} = -(\Sigma a_i z^i)(\Sigma v_j z^j) = \Sigma c_h z^h$ (mod p),

where the latter polynomial has degree <=n+k-2. Thus x is controllable to zero using the inputs c_h. By the choice of β, all the c_h are in U, as desired.

PROOF OF (3.5). Without loss of generality, we may assume that $\beta<1$. Note first that it is sufficient to find a polynomial q with *complex* coefficients satisfying the conclusions. This is because if p divides such a q, then it also divides q_1, where $q(z) = q_1(z)+iq_2(z)$ with both q_j real. We shall establish the result by induction on r = degree of p. Introduce for integers k>0 and complex numbers λ with $|\lambda|<1$, the polynomials:

(3.8) $p_{k,\lambda}(z) := z^k - \Sigma(\lambda^{k-i}/k)z^i$.

Note that λ is a root of $p_{k,\lambda}$, and that all its coefficients have magnitude <=1/k. If p(z) = z-λ has r=1, pick any k with $1/k<\beta$; then $q := p_{k,\lambda}$ is as desired. Assume now that $p(z) = (z-\lambda)p'(z)$, and that there is a monic polynomial q' with nonleading coefficients of magnitude $<\beta$ and divisible by p'. Pick now an integer k >degq' such that $1/k<\beta$. Let $a := \lambda^{k+1}$, and take:

(3.9) $q''(z) := p_{a,k}(z^{k+1})$.

Note that λ is a root of q'', so z-λ divides q''. It follows that p divides q:= q'q''. It only remains to see that all nonleading coefficients of q have magnitude less than β. But, by construction, these coefficients are products (not *sums of* products) of the nonzero coefficients of q' and q'', for which the magnitudes are $<\beta$.

4. A (LOCAL) NONLINEAR PROPERTY

We wish to interpret here some of the previous concepts and results in terms of local properties of nonlinear systems. Consider a continuous [resp., discrete] time system on $\mathbf{R}^n$ of the form:

(4.1) $\dot{x}(t)$ [x(t+1)] = f(x(t),u(t)),

where f is differentiable with f(0,0)=0: f(x,u) = Ax+Bu+o(x,u). Again assume controls are piecewise continuous in the continuous time case, and denote solutions by $\phi(t,x,u)$. Let Σ be the continuous [resp., discrete] time linear system associated with (A,B).

(4.2) PROPOSITION. Assume that Σ is SIANC. There exists then a ngbd V of 0 in $\mathbf{R}^n$ and a map $g:\mathbf{R}_+\to\mathbf{R}_+$, such that: (1) $g(\alpha) = o(\alpha)$ as $\alpha\to 0$, and (2) for each $x\in V$ there is a control u such that $\|u(t)\| < g(\|x\|)$ for all t and such that $\phi(t,x,u) \to 0$ as $t\to\infty$.

PROOF. Through sampling at a suitable frequency, it is again sufficient to prove the result for discrete time systems. We claim first that, for each $\beta>0$, there are an integer N and N $m\times n$-matrices K_i, i= 0,...,N-1, such that (a) all $\|K_i\|<=\beta$ and (b) $\|A^N+\Sigma A^jBK_j\| < 1/4$. (It will be convenient here to use the "L_1" norm $\|x\| = \Sigma|x_i|$ on $\mathbf{R}^n$ and $\mathbf{R}^m$, and the corresponding operator norms.) To prove this claim, consider the canonical basis vectors e_i, and find an N, and for each i, a sequence of control values u_o, ..., u_{N-1} such that $\|A^Ne_i+\Sigma A^jBu_j\| < 1/4$ and all the $\|u_j\|<\beta$. Then the matrix K_j with columns $u_j^{(1)}$, ..., $u_j^{(n)}$ has the desired properties. Consider now the (discrete) system 4.1, and take any state x. Pick $\beta>0$ and obtain the K_j as above. Apply the control $u(N-j):= K_{j-1}x$, j = 1,...,N. Then:

(4.3) $\phi(N,x,u) = (A^N+\Sigma A^jBK_j)x + o(x)$.

We can then define functions $a:\mathbf{R}_+\to\mathbf{R}_+$ and $N:\mathbf{R}_+\to\mathbf{N}$, such that the following property holds for each $\beta>0$: For each x with $\|x\|<a(\beta)$ there s a control $u(\cdot)$ such that $\|u(t)\| < \beta\|x\|$ for each t and $\|\phi(N(\beta),x,u)\| < \|x\|/2$. Define by induction $\alpha_o:= a(1)$, and $\alpha_{k+1}:= a(\alpha_k/k)$. Modify if necessary the α_i so that they monotonically decrease to 0 and $\alpha_o<1$. Let V be the ball of radius α_o centered at 0. Assume that a state x is such that $\|x\| < \alpha_k$ for some k. Then there are an N and a control $u(\cdot)$ such that $\|u(j)\| < \alpha_{k-1}\|x\|/k < \|x\|$ for all j and such that $x':= \phi(N,x,u)$ has norm $< \alpha_k/2$. Since in particular $\|x'\|<\alpha_k$, we may repeat the construction and control x' to a state x" with norm $< \alpha_k/4$, using another control with values of magnitude $< \|x\|/k$. Iterating, we obtain by concatenation an infinite length control $u(\cdot)$ with each $\|u(t)\| < \|x\|/k$ and such that $\phi(t,x,u) \to 0$ as $t\to\infty$. For a map satisfying the requirements of the proposition, take now the piecewise linear map g which is linear in between the α_i and which has values $g(\alpha_k):= \alpha_k/(k-1)$.

The converse of 4.2 is "almost" true. Specifically, the existence of V,g as there implies that the linearization Σ is adequate. In general, of course, it does not follow that Σ is a.n.c., but if the system 4.1 is itself linear (hence the same as Σ), then the conclusions imply local a.n.c., hence by linearity global a.n.c., and so the converse holds. The proof that Σ must be adequate is analogous to that of the implication (c)$\Rightarrow$(d) in 2.2-2.3. We sketch the case of continuous time systems with at least one positive real λ; the other cases are similar. Choose coordinates so that the first equation of 4.1 is:

(4.4) $\dot{x}_1 = \lambda x_1+\Sigma b_iu_i+o(x,u)$.

Pick $\alpha>0$ small enough so that $\|x\|<\alpha$ and $\|u\|<\alpha$ imply all of the following: $|o(x,u)| < \lambda\|x\|/2 + \|u\|$, $x\in V$, $g(\|x\|) < \|x\|$, and $(1+\Sigma|b_i|)g(\|x\|)/\|x\| < \lambda/2$. Pick now any nonzero vector of the type $a:= (a_1,0,...,0)'$ with $\|a\| = a_1 < \alpha$. Find u(.) as in 4.2; thus $\|u(t)\| < g(a_1) < \alpha$ for all t and $x(t):= \phi(t,a,u)$ converges to 0. Without loss of generality, we may assume that u(t) is continuous on t. Pick any t>=0 for which $x_1(t)=a_1$. The first equation gives $\dot{x}_1(t) = a_1(\lambda+c)$, where $c := [\Sigma b_i u_i(t) + o(a,u(t))]/a_1$. But:

(4.4) $|c| \le \Sigma|b_i|g(a_1)/a_1 + \lambda\|a\|/2a_1 + g(a_1)/a_1 < \lambda$.

Thus $\dot{x}_1(t)>0$ whenever $x_1(t)=a$. It follows that $x_1(\cdot)$ cannot satisfy x(0)=a and also converge to 0, a contradiction.

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