IEEE Trans. Neural Networks **3** *(1992): 981-990.*

FEEDBACK STABILIZATION USING TWO-HIDDEN-LAYER NETS

Eduardo D. Sontag
SYCON - Rutgers Center for Systems and Control
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903

ABSTRACT

This paper compares the representational capabilities of one hidden layer and two hidden layer nets consisting of feedforward interconnections of linear threshold units.

It is remarked that for certain problems two hidden layers are required, contrary to what might be in principle expected from the known approximation theorems. The differences are not based on numerical accuracy or number of units needed, nor on capabilities for feature extraction, but rather on a much more basic classification into "direct" and "inverse" problems. The former correspond to the approximation of continuous functions, while the latter are concerned with approximating one-sided inverses of continuous functions —and are often encountered in the context of inverse kinematics determination or in control questions.

A general result is given showing that nonlinear control systems can be stabilized using two hidden layers, but not in general using just one.

Key words : Neural nets, nonlinear control systems, feedback

1 Introduction

This paper concerns itself with the global stabilization of nonlinear systems

$$x(t+1) \;=\; P(x(t), u(t)) \tag{1}$$

by means of state feedback laws $u(t) = K(x(t))$ which can be implemented using neural networks. Such control laws have attracted some interest lately (see e.g. [7] and references there). Our objective here is not to provide a practical stabilization technique, but rather to explore the capabilities and the ultimate limitations of alternative network architectures. We do so by showing that, contrary to what might be expected from the well-known representation theorems [4], [3], [6], single hidden layer nets are *not* sufficient for stabilization, but two hidden layer nets are enough —assuming that threshold processors are used.

The basic reason underlying the lack of sufficiency of one hidden layer is that, often, control laws for nonlinear systems require the use of discontinuous mappings, and these cannot be well-approximated as superpositions of maps which are constant on halfspaces.

In fact, the same phenomenon appears in a more general class of nonlinear questions, not necessarily in control theory, questions that deal with *inverse* or *indirect* problems. In these, one is interested in obtaining a one-sided inverse to a continuous map. For instance, inverse kinematics calculations in robotics are of this type. Other authors, most notably [2] and [1], had previously noted the need for two hidden layers; while they stated their results mostly in terms of numerical accuracy and numbers of neurons, the underlying reasons also had to do with limitations of superpositions. This difference in capabilities was also implicit —but expressed in the language of piecewise linear maps— in the algebraic reference [8].

The remarks in this paper suggest that one could roughly classify learning problems into "direct" and "indirect" ones, the former being more suitable for solution by one hidden layer nets, and the latter by two hidden layer nets. Of course, a particular inverse or indirect problem may well be solvable using one hidden layer nets; certainly linear problems are like that. But our rough classification might be still helpful in dealing with the difficult issue of selection of architectures.

Mathematically, the main results are quite simple, and they are to be expected in view of the older work by the author which dealt with piecewise linear sets and systems. The only difficulties are in generalizing the arguments in [9] to deal with a slightly more restrictive class of feedback laws than in [9], and in proving the negative result. The exposition here is self-contained, however, and no use is made of the results in [9] and [8]. Moreover, we organized the paper in such a manner that readers not familiar with the control application will still be able to read the sections on direct and indirect problems independently of the rest.

1.1 Summary of Results on Representability

We will deal with functions that can be computed by nets consisting of feedforward interconnections, via additive links, of processors ("neurons") each of which has a scalar response θ. In our positive results we take this processing element to be the standard "hardlimiter" function from the neural net and perceptron literature: $\theta = \mathcal{H}$, where $\mathcal{H}(x) = 0$ for $x \leq 0$ and $\mathcal{H}(x) = 1$ for $x > 0$. In negative results, more general functions θ can be used. The ouput is not passed through a final neuron, as done in some studies of feedforward nets, as this would limit the range of values that can be computed.

It is by now well-known —see e.g. [4], [3], [6]— that functions computable by nets with a single hidden layer can approximate continuous functions, uniformly on compacts, under only weak assumptions on θ. Consider now the following *inversion* problem: *Given a continuous function $f : \mathbb{R}^m \to \mathbb{R}^p$, a compact subset $C \subseteq \mathbb{R}^p$ included in the image of f, and an $\varepsilon > 0$, find a function $\phi : \mathbb{R}^p \to \mathbb{R}^m$ so that $\|f(\phi(x)) - x\| < \varepsilon$ for all $x \in C$.* It is trivial to see that in general discontinuous functions ϕ are needed. We show later that nets with just one hidden layer are not enough to guarantee the solution of all such problems, but nets with two hidden layers are. The basic obstruction is due, in essence, to the impossibility of approximating by single-hidden-layer nets the characteristic function of any bounded polytope, while for some (non one-to-one) f the only possible one-sided inverses ϕ must be close to such a characteristic function. On the other hand, it is fairly trivial to get these approximations with two hidden layers.

1.2 Summary of Control Results

We assume that system (1) is so that states $x(t)$ evolve in $\mathbb{R}^n$, controls $u(t)$ take values in $\mathbb{R}^m$, and that $P : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuous and has $P(0,0) = 0$.

The system (1) is *asymptotically controllable* if for each state x_0 there is some infinite control sequence $u(0), u(1), \ldots$ such that the corresponding solution with $x(0) = x_0$ satisfies that $x(t) \to 0$ as $t \to \infty$. This condition is obviously the weakest possible one if any type of controller is to stabilize the system; see [12], in particular Chapter 4, for a discussion of such issues.

The main objective is to find a map (feedback law)

$$K \,:\, \mathbb{R}^n \to \mathbb{R}^m \,,$$

computable by a net, which stabilizes any given compact subset of the state space $C \subseteq \mathbb{R}^n$ to $x = 0$, that is, so that the closed-loop system

$$x(t+1) \;=\; P(x(t), K(x(t))) \tag{2}$$

(denoted also $x^+ = P(x, K(x))$) is asymptotically stable and contains C in the domain of attraction. (In general, for different C, a different K may be needed; this is due to the limitations imposed by having only a finite number of simple processing elements in the net.) As we are interested in global behavior, we make the simplifying assumption that the system can be *locally* stabilized with linear feedback, i.e. there is some matrix F so that the closed loop system with right-hand side $P(x(t), Fx(t))$ is locally asymptotically stable.

We will show that asymptotic controllability is then not only necessary but also sufficient in order to guarantee the existence of a two-hidden-layer net that stabilizes any given compact. On the other hand, we will construct an example of a system which satisfies all the assumptions —in fact, it is so that $F = 0$ locally stabilizes and so that every state can be driven in two time steps to the origin— but for which every *one*-layer net results in some nontrivial periodic orbit.

The discussion is entirely in terms of discrete-time systems (1). However, just as in [9], one may immediately apply all results to continuous-time systems

$$\dot{x} \;=\; f(x, u) \tag{3}$$

through the use of sample-and-hold control. Thus, given an asymptotic controllable system (3) which satisfies the first-order stability condition, and given any compact subset C of the state space, there is a sampling period $\delta > 0$ and a two-hidden-layer net K so that the controls $u(t) =$ constant value $K(x(k\delta))$ on each sampling interval $t \in [k\delta, (k+1)\delta)$ stabilize states in C. See for instance [13] or [12], Section 4.8, for more on the topic of nonlinear stabilizability for continuous-time systems.

Also, only the full state feedback problem is treated in detail, but [9] shows how to deal with partial observations in the analogous case of piecewise linear feedback (in fact, the main results in that reference are for the partially observed case).

2 Definitions and Results

In this section we give the basic definitions, discuss elementary properties, and provide precise statements of results. Proofs are deferred to Section 3, which deals with properties of certain sets of functions, including those associated to nets, and Section 4, which develops the material on stabilization.

2.1 Feedforward Nets

We will find it more convenient not to define a "net" but rather a "function computed by a net," because different sets of net parameters (weights, thresholds) may give rise to the same behavior —for instance, permuting the neurons and all incoming and outgoing weights results in the same map. The functions so defined will correspond to the nets discussed in the Introduction.

A function $\theta : \mathbb{R} \to \mathbb{R}$ is assumed given. In neural net practice, one often takes θ to be the *standard sigmoid* $\theta(x) = \sigma(x) = \frac{1}{1+e^{-x}}$ or equivalently, up to translations and change of coordinates, the hyperbolic tangent $\theta(x) = \tanh(x)$. Another usual choice is the *hardlimiter*, *Heaviside*, or *threshold* function

$$\theta(x) = \mathcal{H}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

which can be approximated well by $\sigma(\gamma x)$ when the "gain" γ is large. The main results given will be for $\theta = \mathcal{H}$.

Definition 2.1 A function $f : \mathbb{R}^p \to \mathbb{R}$ is *computable by a strict zero-hidden-layer net* if it is an affine function, that is, there exist a vector $v \in \mathbb{R}^p$ and a scalar $\tau \in \mathbb{R}$ such that $f(u) = v.u + \tau$, where the dot indicates inner product. For any integer $d \geq 1$, the function $f : \mathbb{R}^p \to \mathbb{R}$ is *computable by a strict d-hidden-layer net (with processsors of type θ)* if there exist an integer l, constants $w_1, \ldots, w_l \in \mathbb{R}$, and functions $f_1, \ldots, f_l$ so that

$$f(u) = \sum_{i=1}^{l} w_i \theta(f_i(u)) \tag{4}$$

and each f_i is computable by a strict $(d-1)$-hidden-layer net. □

In other words, the functions computable by nets with no hidden layers are those in the span of the coordinate functions and the constants, and those computable by d layers constitute the span of the functions $\theta(f(x))$, for f computable with one less layer. Note that constant terms (or "biases" in neural net terminology) can always be included in the sum in (4), as one could take one of the f_i's to be constant. A d-hidden-layer net is sometimes called a "$(d+2)$-layer net" if one counts the inputs and outputs as a layer. We prefer the hidden-layer terminology, as less ambiguous.

In particular, a function f is computable by a strict one-hidden-layer net if there are real numbers $w_1, \ldots, w_l, \tau_1, \ldots, \tau_l$ and vectors $v_1, \ldots, v_l \in \mathbb{R}^p$ such that, for all $u \in \mathbb{R}^p$,

$$f(u) \ = \ \sum_{i=1}^{l} w_i \, \theta(v_i.u + \tau_i) \ . \tag{5}$$

Most results mentioned here will deal with $d = 1$ or $d = 2$. For fixed θ, and under mild assumptions on θ, nets with one hidden layer can be used to approximate arbitrary continuous functions uniformly on compacts sets; see for instance [3], [6]. For other problems, as discussed below, two hidden layers are needed.

Definition 2.2 A *function computable by a strict net with possible direct input to output connections (and d hidden layers)* is by definition a function $g : \mathbb{R}^p \to \mathbb{R}$ of the form $Fu + f(u)$, where F is linear and f is computable by a strict d-hidden-layer net as above. □

For multivariable maps $f : \mathbb{R}^p \to \mathbb{R}^m$, "computable by a d-hidden-layer net" means by definition that each coordinate function $f_i : \mathbb{R}^p \to \mathbb{R}$, $i = 1, \ldots, p$ is so computable, and similarly when direct connections are allowed.

Remark 2.3 We use the terminology "strict" to differentiate from the case in which one would also allow in the sum (4) terms of the form $w_i f_i(u)$, where f_i is computable with $d-1$ layers. In graph-theoretic terms, such more general functions are computable by nets in which forward connections are allowed between arbitrary intermediate nodes (not necessarily in adjacent layers). With the possible exception of direct connections from inputs to outputs, however, we will not need such "nonstrict" nets. The positive results will hold already for strict nets, while the negative result, for $d = 1$, will show that certain problems cannot be solved by one-layer nets with possible i/o conections, which in that case ($d = 1$) are the same as nonstrict nets. For simplicity, from now on we drop the word "strict." □

2.2 Certain Properties of Classes of Functions

To explain the different approximation capabilities of one- and two-hidden layer nets, we first consider, in general, the following properties on classes of functions.

Suppose given, for each positive integer p, an $\mathbb{R}$-linear space of functions $\mathcal{F}_p$ from $\mathbb{R}^p$ into $\mathbb{R}$, so that for each $f \in \mathcal{F}_p$, each constant $c \in \mathbb{R}$, and every $k = 1, \ldots, p$, the function

$$g(u_1, \ldots, u_{p-1}) \;:=\; f(u_1, \ldots, u_{k-1}, c, u_k, \ldots, u_{p-1})$$

obtained by setting the kth coordinate to c belongs to $\mathcal{F}_{p-1}$.

For each positive integers p and m, we denote $\mathcal{F}_p^m := (\mathcal{F}_p)^m$, thought of as a linear space of maps $\mathbb{R}^p \to \mathbb{R}^m$. Thus, by definition, if $f = (f_1, \ldots, f_m)' : \mathbb{R}^p \to \mathbb{R}^m$ is any map, then $f \in \mathcal{F}_p^m$ if and only if each coordinate function f_i is in $\mathcal{F}_p$. We call any $\mathcal{F} = \{\mathcal{F}_p^m\}_{p,m}$ obtained in this fashion a *compatible class of functions*. Consider the following possible properties of such an $\mathcal{F}$:

(INV) For any m, p, any continuous function $f : \mathbb{R}^m \to \mathbb{R}^p$, any compact subset $C \subseteq \mathbb{R}^p$ included in the image of f, and any $\varepsilon > 0$, there exists some $\phi \in \mathcal{F}_p^m$ so that $\|f(\phi(x)) - x\| < \varepsilon$ for all $x \in C$.

(SEC) For any open subset $\mathcal{U} \subseteq \mathbb{R}^p \times \mathbb{R}^m$ and every compact subset $C \subseteq \mathbb{R}^p$ included in the projection $\pi_1(\mathcal{U})$ of $\mathcal{U}$ on the first p coordinates, there exists some $\phi \in \mathcal{F}_p^m$ so that $(x, \phi(x)) \in \mathcal{U}$ for all $x \in C$.

(SEC^0) For any open subset $\mathcal{U} \subseteq \mathbb{R}^p \times \mathbb{R}^m$ and every compact subsets $C \subseteq \pi_1(\mathcal{U})$ and C_0 so that $C_0 \times \{0\} \subseteq \mathcal{U}$ there exists some $\phi \in \mathcal{F}_p^m$ so that $(x, \phi(x)) \in \mathcal{U}$ for all $x \in C$ and also $\phi(x) = 0$ for all $x \in C_0$.

The first of these corresponds to approximations of (one-sided) inverses of continuous maps, the second to finding sections of projections, and the last to finding sections of such projections which are guaranteed to vanish in a prescribed compact. It turns out that the last property is sufficient for solving stabilization problems for nonlinear control systems, while the first is necessary if such problems are to be solved.

Clearly (SEC^0) implies (SEC). It is also true that (SEC) implies (INV): indeed, assume given any continuous $f : \mathbb{R}^m \to \mathbb{R}^p$, $\varepsilon > 0$, and $C \subseteq f(\mathbb{R}^p)$ as in the statement of (SEC), and let $\mathcal{U}$ be defined as the subset of $\mathbb{R}^p \times \mathbb{R}^m$ consisting of all pairs (x, y) such that $\|f(y) - x\| < \varepsilon$. This is open, by continuity of f, and a section of π_1 provides an ε-approximation to the inverse of f.

2.3 Results for Nets

From Lemma 3.6 and Proposition 3.5 (see Section 3 below), we will derive the following fact:

Proposition 2.4 Let $\mathcal{F} = \{\mathcal{F}_p^m\}_{p,m}$ be the set of maps computable by two-hidden-layer nets with processsors of type $\mathcal{H}$. Then, $\mathcal{F}$ satisfies (SEC^0) (and hence also (SEC) and (INV)).

The proof will be based on the identification of maps computable by such two-hidden-layer nets with maps that are piecewise constant on each element of a finite polyhedral partition, and the proof that the latter type of maps form what we will call a "complete" compatible class of functions, therefore satisfying (SEC^0).

On the other hand, we will have the following:

Proposition 2.5 The set of functions computable by one-hidden-layer nets with $\theta = \mathcal{H}$, even with possible direct input to output connections, does not satisfy (INV) (nor, therefore, (SEC) or (SEC^0)).

The same negative result holds with any continuous θ such as the standard sigmoid. The proof of Proposition 2.5 is based on a more general argument that shows that solving (INV) implies being able to find a certain type of approximation to a characteristic function of a bounded polyhedron, and these approximations cannot be formed out of "ridge" functions, those otained as linear combinations of scalar functions of linear combinations.

2.4 Results for Feedback

We say that a subset $C \subseteq \mathbb{R}^n$ is *asymptotically stable for the closed-loop system* (2) if (2) is locally asymptotically stable about $x = 0$ and C is included in the domain of attraction.

Let $\mathcal{F} = \{\mathcal{F}_p^m\}_{p,m}$ be a compatible class of functions. The system (1) is *$\mathcal{F}$-stabilizable on compacts* if for each compact subset $C \subseteq \mathbb{R}^n$ there exists some $K \in \mathcal{F}_n^m$ so that C is asymptotically stable for the closed-loop system (2).

The two main technical results on stabilization, proved in Section 4, are as follows.

Theorem 1 *Assume that (1) is an asymptotically controllable system so that the origin $x = 0$ is locally asymptotically stable for the zero-input equation*

$$x(t+1) \;=\; P(x(t), 0) \;.$$

Let $\mathcal{F} = \{\mathcal{F}_p^m\}_{p,m}$ be a class of functions satisfying (SEC^0). Then (1) is $\mathcal{F}$-stabilizable on compacts.

If $\mathcal{F} = \{\mathcal{F}_p^m\}_{p,m}$ is a compatible class of functions, we denote $\mathcal{F} + \mathcal{L} = \{\mathcal{F}_p^m + \mathcal{L}\}_{p,m}$ the new compatible class of functions obtained by taking as $\mathcal{F}_p^m + \mathcal{L}$ the set of all the maps of the form $f + L$, with $f \in \mathcal{F}_p^m$ and $L : \mathbb{R}^p \to \mathbb{R}^m$ a linear map.

Theorem 2 *Assume that (1) is an asymptotically controllable system, and that P is differentiable about $x = 0, u = 0$. Let $P(x,u) = Ax + Bu + o(x,u)$. Assume further that the pair (A, B) is stabilizable in the linear systems sense:*

$$\text{rank}\,[zI - A, B] = n \quad \text{for all } z \in \mathbb{C}, |z| \geq 1 \;.$$

Let $\mathcal{F} = \{\mathcal{F}_p^m\}_{p,m}$ be a compatible class of functions satisfying (SEC^0). Then (1) is $\mathcal{F}+\mathcal{L}$-stabilizable on compacts.

Because of Proposition 2.4, from these follow the main positive results for nets:

Corollary 2.6 If (1) is asymptotically controllable and $x = 0$ is locally asymptotically stable for the zero-input equation $x^+ = P(x, 0)$, then (1) is stabilizable on compacts using two-hidden-layer nets with processors of type $\mathcal{H}$. □

Corollary 2.7 If (1) is asymptotically controllable and its linearization $x^+ = Ax+Bu$ at the origin is stabilizable, then (1) is stabilizable on compacts using two-hidden-layer nets with processors of type $\mathcal{H}$ and possible direct input to output connections. □

While these Corollaries could also be proved directly, it is far more interesting to see them as consequences of the possibility of constructing sections of maps. In particular, it is then not hard to see that the stabilization property is robust under small perturbations in the feedback law.

In Section 4 we also prove that the conclusions of these Corollaries cannot hold for single-hidden-layer nets, as well as many other sets of functions. This follows from:

Theorem 3 *Assume that $\mathcal{F}$ is a compatible class of functions which does not satisfy property (SEC). Then there exists a system (1) which:*

- *is asymptotically controllable, and*
- *is so that the origin is locally asymptotically stable for the zero-input dynamics $x^+ = P(x, 0)$*

but is not $\mathcal{F}$-stabilizable on compacts.

From Proposition 2.5 we are then able to conclude:

Proposition 2.8 There exists a system (1) which is asymptotically controllable, and is so that the origin is locally asymptotically stable for the zero-input dynamics, but which is not stabilizable on compacts using nets with one hidden layer, $\theta = \mathcal{H}$, and possible direct input to output connections.

The rest of the paper will develop the technical details and provide proofs.

3 The Property (SEC^0)

One way of generating classes of functions satisfying property (SEC^0) is through certain types of piecewise constant functions.

Definition 3.1 Let p be a positive integer. A class of subsets $\mathcal{B}$ of $\mathbb{R}^p$ will be said to be a *Boolean basis* if $\mathcal{B}$ is a Boolean algebra ($\mathcal{B}$ is closed under finite intersections and complements) and it contains a basis of open sets (every open subset of $\mathbb{R}^p$ is a union of open sets belonging to $\mathcal{B}$). □

Definition 3.2 Let $\mathcal{F} = \{\mathcal{F}_p^m\}_{p,m}$ be a compatible class of functions. The class $\mathcal{F}$ will be said to be *complete* if for each p there exists a Boolean basis $\mathcal{B}_p$ such that $\mathcal{F}_p$ contains the characteristic functions of all the elements of $\mathcal{B}_p$. □

Note that if $v = (v_1, \ldots, v_m)'$ is any element in $\mathbb{R}^m$ and χ is the characteristic function of any set $W \in \mathcal{B}_p$, then χv, seen as the map $\mathbb{R}^p \to \mathbb{R}^m : x \mapsto \chi(x)v$, is in $\mathcal{F}_p^m$, because each of its coordinates $x \mapsto v_i\chi(x)$ belongs to $\mathcal{F}_p$, which is closed under scalar multiplications. More generally, if $v_1, \ldots, v_k$ are elements in $\mathbb{R}^m$, and $\chi_1, \ldots, \chi_k$ are characteristic functions of disjoint sets $W_i \in \mathcal{B}_p$, the map

$$\sum_{i=1}^{k} v_i\, \chi_i(\cdot)\ ,$$

which takes the constant value v_i on W_i, is in $\mathcal{F}_p^m$.

As an illustration of the above concepts, the class of all those subsets of $\mathbb{R}^p$ which can be written as a finite union of intersections of closed and open sets forms a Boolean basis in our sense, and the same is true for the Boolean algebra generated by all the open spheres —related to the "radial basis functions" used in some neural network applications. Another, more relevant, example, is as follows.

Consider, for each fixed p, the open halfspaces in $\mathbb{R}^p$, i.e. the sets defined by inequalities of the type $v.u > \tau$, for some $\tau \in \mathbb{R}$ and $v \in \mathbb{R}^p$. Now take the Boolean algebra generated by all such halfspaces. This defines a class of subsets $\mathcal{B}_p$ each of which is a finite union of intersections of finitely many open and closed subspaces ($v.u \geq \tau$). Since every open cube is an intersection of open halfspaces, and cubes form a basis for the topology of $\mathbb{R}^p$, $\mathcal{B}_p$ is a Boolean basis.

As each closed halfspace can be written as the union of a hyperplane and an open halfspace, every element of $\mathcal{B}_p$ can also be written as a finite disjoint union of sets of the form

$$H \bigcap P \tag{6}$$

where H is an affine manifold (possibly the whole space) and P is an open polyhedron, that is, a set defined by finitely many affine inequalities of the type $v.u > \tau$. Sets of the form (6) are called *relatively open* polyhedra. As in [8] and [9], we define:

Definition 3.3 Elements of $\mathcal{B}_p$ are called *piecewise linear sets*. The linear span of the characteristic functions of such piecewise linear sets is the set of (polyhedrally) *piecewise constant* functions from $\mathbb{R}^p$ into $\mathbb{R}$. More generally, a map $f : \mathbb{R}^p \to \mathbb{R}^m$ is said to be piecewise constant if each coordinate function f_i is. □

It follows from the preceeding discussion that the piecewise constant functions are also spanned by the characteristic functions of the relatively open polyhedra. The set of piecewise constant maps is complete, because each $\mathcal{B}_p$ is a Boolean algebra by definition.

Remark 3.4 In [8] and [9], one also defines more generally *piecewise linear* maps, as those whose graphs are piecewise linear sets, or equivalently, maps that are affine (rather than constant) on each element of a finite polyhedral partition. For such maps one may develop a fairly elegant algebraic theory, and various computational complexity issues have been studied too (see [10]). Their study is conveniently carried out by introducing the first-order logical theory of real numbers with addition, and studying elimination of quantifier issues for it. Of course, piecewise linear maps also constitute a Boolean complete set. It is easy to see that in order to represent general piecewise linear maps one will need richer structures than feedforward nets. Essentially, what are needed are pairs of nets, one for partitioning the state space and the other for implementing an affine function in each; the two nets interact multiplicatively. Such pairs of nets may be more useful in practice –in particular, they are better suited for modeling gain-scheduling approaches to control; see [9]. On the other hand, the subset of piecewise constant maps, and the maps obtained by adding to them a fixed linear map, are enough for establishing a general existence result, and hence we restrict atention to them in this paper. □

The main property of complete sets of functions that we need is the following trivial observation:

Proposition 3.5 If $\mathcal{F}$ is complete, then it satisfies property (SEC^0).

Proof. Let $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{C}_0$ be as in the statement of the property, and let $\mathcal{B}_p$ be as in the definition of completeness. Consider the open set

$$V \;\; := \;\; \{x \in \mathbb{R}^m \,|\, (x, 0) \in \mathcal{U}\} \; .$$

For each $x \in \mathcal{C}_0$, we pick a neighborhood $\mathcal{O}_x$ of x contained in V. Since $\mathcal{B}_p$ is a Boolean basis, we may take $\mathcal{O}_x \in \mathcal{B}_p$ for all such x. We write $u_x := 0$ for each $x \in C_0$.

Now consider any $x \in \mathcal{C}\backslash\mathcal{C}_0$. As $\mathcal{C} \subseteq \pi_1(\mathcal{U})$, there is some $u_x \in \mathbb{R}^p$ so that $(x, u_x) \in \mathcal{U}$, and thus we may pick some neighborhood $\mathcal{O}_x \in \mathcal{B}_p$ of x with the property that $(z, u_x) \in \mathcal{U}$ for all $z \in \mathcal{O}_x$. Moreover, as $\mathcal{C}_0$ is closed, we may take $\mathcal{O}_x$ to be disjoint from $\mathcal{C}_0$.

The sets $\mathcal{O}_x$ cover the compact $\mathcal{C}$; choose a finite subcover, say corresponding to points $x_1, \ldots, x_k$, and write $\mathcal{O}_i$ instead of $\mathcal{O}_{x_i}$ and u_i instead of u_{x_i}. Without loss of generality, we assume that $x_1, \ldots, x_l$ are in $\mathcal{C}_0$ and $x_{l+1}, \ldots, x_k$ are in $\mathcal{C}\backslash\mathcal{C}_0$. Note that by construction, none of $\mathcal{O}_{l+1}, \ldots, \mathcal{O}_k$ intersect $\mathcal{C}_0$, so the union of $\mathcal{O}_1, \ldots, \mathcal{O}_l$ must cover $\mathcal{C}_0$, and that $u_i = 0$ for $i = 1, \ldots, l$. Define $\mathcal{W}_1 := \mathcal{O}_1$ and for each $i = 1, \ldots, k-1$:

$$\mathcal{W}_{i+1} \;:=\; \mathcal{O}_{i+1} \backslash \left(\bigcup_{j \le i} \mathcal{O}_j \right)$$

so that the $\mathcal{W}_j$'s are disjoint and still cover $\mathcal{C}$. Since $\mathcal{B}_p$ is a Boolean algebra, each $\mathcal{W}_j$ belongs again to it, and thus the linear combination of characteristic functions $\phi := \sum_{i=1}^{k} u_i\, \chi_i(\cdot)$ is in $\mathcal{F}_p^m$. This combination ϕ satisfies $\phi(x) = 0$ for all $x \in C_0$ because C_0 is included in the union of $\mathcal{W}_1, \ldots, \mathcal{W}_l$, and also $(x, \phi(x)) \in \mathcal{U}$ for all $x \in C$ by construction. ∎

The next remark relates Definitions 2.1 and 3.3.

Lemma 3.6 A function $f : \mathbb{R}^N \to \mathbb{R}$ is piecewise constant if and only if it is computable by a two-hidden-layer net with processsors of type $\mathcal{H}$.

Proof. Let f be piecewise constant. By definition, f is a linear combination of characteristic functions of open polyhedra (6). Thus in order to show that f is computable by a two-hidden-layer net it is sufficient to prove that such characteristic functions are. Let H be the set of solutions of $\mu_i.u = c_i$, $i = 1, \ldots, k$, and let P be defined by the inequalities $\nu_i.u > d_i$, $i = 1, \ldots, l$. Then

$$\mathcal{H}\left(-\sum_{i=1}^{k} \left[\mathcal{H}(\mu_i.u - c_i) + \mathcal{H}(-\mu_i.u + c_i) \right] - \sum_{i=1}^{l} \left[1 - \mathcal{H}(\nu_i.u - d_i) \right] + \frac{1}{2} \right)$$

is the characteristic function of $H \bigcap P$.

Conversely, assume that f is computable by a d-hidden-layer net, for any $d > 0$. We prove by induction on d that f is piecewise constant. As f is a linear combination of terms of the form $\mathcal{H}(g(x))$, with g computable with $d-1$ hidden layers, it is sufficient to show the result for one such term. If $d = 1$ then $f(x) = \mathcal{H}(v.u + \tau)$ is the characteristic function of the half-space $v.u + \tau > 0$. Now let $d \ge 2$, so by induction we may assume that g is constant on each of the relatively open polyhedra $P_1, \ldots, P_k$. It follows that f is constant on each of the same polyhedra. ∎

From Lemma 3.6 and Proposition 3.5 applied to the (polyhedrally) piecewise constant functions, we have the desired conclusion, Proposition 2.4.

3.1 A Necessary Condition

We show here that property (INV) —and therefore also (SEC) and (SEC^0)— does not hold for certain classes of functions $\mathcal{F}$, including those computed by single-layer nets.

Lemma 3.7 Assume that $\mathcal{F}$ satisfies (INV). Then, there exists some $\psi \in \mathcal{F}_2^1$ such that

- $\psi(x) \in (-1,0) \bigcup (2,4)$ for all x with $\|x\| < 3/2$.
- $\psi(x) > 2$ for $\|x\| < 1/2$.
- $-1 < \psi(x) < 0$ for $5/4 < \|x\| < 3/2$.

Proof. Let S be the open unit ball in $\mathbb{R}^3$ centered at $(0,0,3)'$, that is the set where $x_1^2+x_2^2+(x_3-3)^2 < 1$, and let T be the solid torus in $\mathbb{R}^3$ obtaining by rotating about the x_3-axis the disk in the x_2, x_3-plane with $x_1 = 0$ and $(x_2 - 5/4)^2 + (x_3 + 1/2)^2 < 1/4$. Observe that S projects along the x_3 axis onto the unit disk

$$D = \{x \in R^2 \mid \|x\| < 1\}$$

and that T projects onto the annulus

$$A = \{x \in R^2 \mid 3/4 < \|x\| < 7/4\} .$$

Let C_0 be the closed disk in $\mathbb{R}^2$ of radius $3/2$ centered at zero, which is included in $A \bigcup D$.

As $S \bigcup T$ is open, there exists some continuous map $\rho : \mathbb{R}^3 \to \mathbb{R}$ so that $\rho(x) \in (0, 1/4)$ if $x \in S \bigcup T$, and which is identically $= 1/4$ outside $S \bigcup T$. (For instance, one may take $\rho = (4 + 4d^2)^{-1}$, where $d(x)$ is the distance to the complement of $S \bigcup T$. One may even take an infinitely differentiable function d whose zero set is this complement; see Exercise 2.2.1 in [5].)

Finally, let $m = p = 3$ in property (INV), applied with $\varepsilon = 1/4$,

$$f(x_1, x_2, x_3) := (x_1, x_2, \rho(x_1, x_2, x_3)) ,$$

and the set $C := C_0 \times \{0\}$. Thus there is some $\phi = (\phi_1, \phi_2, \phi_3)' \in \mathcal{F}_3^3$ so that

$$\|f(\phi(x)) - x\| < 1/4$$

whenever $x \in C$. From this inequality, it holds then that $\rho(\phi(x)) < 1/4$ for such x, so necessarily

$$\phi(x) \in S \bigcup T \tag{7}$$

for all $x \in C$. Let $\psi(x_1, x_2) := \phi_3(x_1, x_2, 0)$. Then, (7) implies that $\psi(x_1, x_2) \in (2, 4)$ when $\phi(x_1, x_2, 0) \in S$, and $\psi(x_1, x_2) \in (-1, 0)$ when $\phi(x_1, x_2, 0) \in T$, so always $\psi(x_1, x_2) \in (-1, 0) \bigcup (2, 4)$ for $(x_1, x_2) \in C_0$.

It also follows that

$$\|(\phi_1(x), \phi_2(x)) - (x_1, x_2)\| < 1/4 \tag{8}$$

for all $x = (x_1, x_2, x_3) \in C$. In the particular case in which $\|(x_1, x_2)\| < 1/4$, (8) implies that $\|(\phi_1(x), \phi_2(x))\| < 1/2$, which together with (7) implies that $\phi(x) \in S$, and therefore that $\psi(x_1, x_2) > 2$. Similarly, if $\|(x_1, x_2)\| > 5/4$ then (8) implies that $\|(\phi_1(x), \phi_2(x))\| > 1$, and this together with (7) gives that $\phi(x) \in T$, and hence that $-1 < \psi(x_1, x_2) < 0$. ∎

We next prove that single-layer nets cannot satisfy the above properties. In order to do so, it is convenient to prove something a bit more general. Let $\mathcal{Q}$ be any class of functions from $\mathbb{R}^2$ into $\mathbb{R}$ which satisfies the following three properties, where $\mathcal{S}_q := \{u \mid q(x) = 0\}$:

1. Each $q \in \mathcal{Q}$ is continuous, and if q is not constant then it satisfies the following openness condition: If $u \in \mathcal{S}_q$ for some $u \in \mathbb{R}^2$, and if $\{\varepsilon_n\}$ is any sequence converging to zero, then there is some sequence $u_k \to u$ and a subsequence $\{\varepsilon_{n_k}\}$ such that $q(u_k) = \varepsilon_{n_k}$ for all k. (Note that if $\nabla q(u) \neq 0$ for each u for which $q(u) = 0$, then the openness condition is satisfied.)

2. If q and $\widetilde{q}$ are any two nonconstant elements in $\mathcal{Q}$ then either $\mathcal{S}_q \bigcap \mathcal{S}_{\widetilde{q}}$ is finite (possibly empty) or there is some $\lambda \in \mathbb{R}$ such that $\widetilde{q}(u) = \lambda q(u)$ for all u (and in particular $\mathcal{S}_q = \mathcal{S}_{\widetilde{q}}$).

3. Each set $\mathcal{S}_q$ is connected and unbounded.

For instance, the set $\mathcal{Q}$ consisting of all affine functions $q(u) = v.u + \tau$ satisfies the above properties.

If $\mathcal{Q}$ is as above, we will say that $f : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{Q}$-*superposition* if it can be written in the form

$$f(u) \;=\; \sum_{i=1}^{k} \alpha_i(q_i(u)) \;+\; g(u) \tag{9}$$

where the functions q_i are in $\mathcal{Q}$, k is some positive integer, each function $\alpha_i : \mathbb{R} \to \mathbb{R}$ is continuous except possibly at $x = 0$, and g is continuous.

As an example, any $f : \mathbb{R}^2 \to \mathbb{R}$ computable by a single-hidden layer net with processors of type $\mathcal{H}$ is such a superposition, with $\mathcal{Q}$ being the set of affine functions. When possible direct input to output connections are allowed, the function is still a superposition: one may include the linear term Fu, either by taking an extra α_i equal to the identity, or taking $g(u) = Fu$. Note also that functions computable by nets with any number of hidden units but θ continuous are also superpositions (just use the "g" term).

Together with Lemma 3.7, the following implies Proposition 2.5.

Proposition 3.8 Let $\mathcal{Q}$ be as above, and let $f : \mathbb{R}^2 \to \mathbb{R}$ and $R > 0$ be so that:

(a) $f(u) \in (-1, 0) \bigcup (2, +\infty)$ for all $\|u\| < R$.

(b) $f(u) > 2$ on some disk $\|u\| < \varepsilon$.

(c) $f(u) \in (-1, 0)$ on some annulus $R - \varepsilon < \|u\| < R$.

Then, f cannot be a $\mathcal{Q}$-superposition.

Proof. Assume that f would be a superposition, as in (9). The functions q_i may be taken to be all nonconstant; otherwise a constant term can be added to g. Let $\mathcal{S}_i := \mathcal{S}_{q_i}$, $i = 1, \ldots, k$ (it may be the case that $\mathcal{S}_i = \mathcal{S}_j$ for some $i \neq j$). The only possible discontinuities of f are on the $\mathcal{S}_i$'s.

Let F be the set of points that are in the intersection of two or more distinct sets of the type $\mathcal{S}_i$. By the second property of $\mathcal{Q}$, F is known to be finite. We claim now that there exists some point u^1 with $\|u^1\| < R$ which is not in F and is such that there exist two sequences

$$y_n^1 \to u^1 \;\text{ and }\; z_n^1 \to u^1 \tag{10}$$

with $f(y_n^1) > 2$ and $f(z_n^1) < 0$ for all n.

Indeed, consider for each u^0 of norm R the function $s(\mu) := \mu u^0$, $\mu \in (0, 1)$. For small μ, $s(\mu) > 2$, by part (b) in the statement, and for large μ part (c) implies that $s(\mu) < 0$. On the other hand, $s(\mu)$ is always either > 2 or < 0, by part (a). Let μ^0 be the supremum of the values μ for which $s(\mu) > 2$. Then, $u^1 = \mu^0 u^0$ is so that two sequences as above exist. As a different point is obtained for each different u^0, and we only need to avoid the finite set F, one can take $u^1 \notin F$, as claimed.

Note that f is discontinuous at u^1, so $u^1 \in \mathcal{S}_i$ for at least one i. After reordering terms if necessary, assume that

$$u^1 \;\in\; \mathcal{S} = \mathcal{S}_1 = \ldots = \mathcal{S}_l$$

and $u^1 \notin \mathcal{S}_j$ for $j = l+1, \ldots, k$. As the sets S_j are closed (each q_i is continuous), there is a neighborhood of u^1 disjoint from all such S_j's. We write

$$f(u) \;=\; f_1(u) + f_2(u)$$

where $f_1(u) := \sum_{i=1}^{l} \alpha_i(q_i(u))$. Then, f_2 is continuous at $u = u^1$, and therefore the last term in

$$\begin{aligned} |f_1(y_n^1) - f_1(z_n^1)| \;&\geq\; |f(y_n^1) - f(z_n^1)| - |f_2(y_n^1) - f_2(z_n^1)| \\ &>\; 2 - |f_2(y_n^1) - f_2(z_n^1)| \end{aligned}$$

tends to zero, which implies that $|f_1(y_n^1) - f_1(z_n^1)| > 3/2$ for all large n.

The set $\mathcal{S}$ is connected and unbounded, and it contains some point in the interior of the disk $\|u\| < R$. Therefore it contains points of the form $\|u\| = R - \varepsilon$, for every ε small. Together with the fact that F is finite, this means that there must exist some $u^2 \in \mathcal{S}$ such that $f(u) \in (-1, 0)$ for all u near u^2 and so that f_2 is continuous at u^2.

By the third property of $\mathcal{Q}$, there is a fixed $q \in \mathcal{Q}$ and real numbers λ_i so that $q_i = \lambda_i q$ for all $i = 1, \ldots, l$. Now we use the openness property for the function q: after choosing if necessary a subsequence of $\{y_n^1\}$, as $q(u^2) = 0$ and $q(y_n^1) \to 0$, there exists some sequence $y_n^2 \to u^2$ so that $q(y_n^2) = q(y_n^1)$ for all n. Using now that also $q(z_n^1) \to 0$, and picking yet another subsequence if necessary, there also exists some sequence $z_n^2 \to u^2$ so that $q(z_n^2) = q(z_n^1)$. Thus also $\lambda_i q(y_n^2) = \lambda_i q(y_n^1)$ and $\lambda_i q(z_n^2) = \lambda_i q(z_n^1)$ for all n, so $f_1(y_n^1) = f_1(y_n^2)$ and $f_1(z_n^1) = f_1(z_n^2)$ for all n and each $i = 1, \ldots, l$. We conclude that

$$\begin{aligned} |f(y_n^2) - f(z_n^2)| \;&\geq\; |f_1(y_n^1) - f_1(z_n^1)| - |f_2(y_n^2) - f_2(z_n^2)| \\ &>\; 3/2 - |f_2(y_n^1) - f_2(z_n^1)| \,. \end{aligned}$$

Since the last term tends to zero, by continuity of f_2 at u^2, it follows that $|f(y_n^2) - f(z_n^2)| > 1$ for all large n, contradicting the fact that $f(u) \in (-1, 0)$ for all u near u^2. ∎

4 Stabilization

Given any system (1) and an input sequence $\omega = (u_1, \ldots, u_k) \in (\mathbb{R}^m)^k$, we use the notation $P(x, \omega)$ to denote the state reached after applying ω, that is,

$$P(x, \omega) \;:=\; P(P(\ldots(P(P(x, u_1), u_2), \ldots), u_{k-1}), u_k) \,.$$

The notation includes the empty sequence ω, for which $P(x, \omega) = x$. For any subset $S \subseteq \mathbb{R}^n$, we denote by

$$\mathcal{C}^1(S) := \{x \in \mathbb{R}^n \mid P(x, u) \in S \text{ for some } u \in \mathbb{R}^m\} \tag{11}$$

the set of states which can be controlled to S in one step.

The notation $\|x\|$ is used for Euclidean norm in the state space $\mathbb{R}^n$ or in control-value space $\mathbb{R}^m$, $\mathbb{B}(x, \varepsilon)$ denotes the open ball of radius ε centered at x, and $\overline{\mathbb{B}}(x, \varepsilon)$ is the closure of $\mathbb{B}(x, \varepsilon)$. More generally, $\mathbb{B}(S, \varepsilon)$ is the open ε-neighborhood of a set S, that is, the set of points x so that $\|x - s\| < \varepsilon$ for some $s \in S$.

We start with a simple consequence of local stability.

Lemma 4.1 Assume given a control system (1) for which the origin $x = 0$ is a locally asymptotically stable state for the zero-input equation

$$x(t+1) \;=\; Q(x(t)) \,, \tag{12}$$

where $Q(x) := P(x, 0)$. Then, there exist:

- a compact set A_0 included in the domain of attraction of the origin in (12) and invariant under Q,
- bounded open sets B_0 and L_0 so that

$$A_0 \subseteq L_0 \subseteq \operatorname{clos} L_0 \subseteq B_0 \ ,$$

  and
- a real number $\varepsilon > 0$ and an integer $s > 0$,

such that the following two properties hold:

(i) If $x \in B_0$ and $u \in \mathbb{R}^m$ has norm $\|u\| < \varepsilon$ then $P(x, u) \in L_0$.

(ii) If $x \in B_0$ and $(u_1, \ldots, u_s) \in (\mathbb{R}^m)^s$ is a sequence that satisfies $\|u_i\| < \varepsilon$ for all $i = 1, \ldots, s$, then there is some $i \in \{1, \ldots, s\}$ so that $P(x, (u_1, \ldots, u_i)) \in A_0$.

Proof. Pick any bounded open set V which contains 0 and whose closure F is contained in the domain of attraction of 0 in (12). By asymptotic stability, there is then some integer s so that $Q^s(F) \subseteq V$. (This is a well-known consequence of stability, and can be proved first locally as in [12], Lemma 4.8.10, and then following by a standard compactness argument.) Let

$$A_0 \ := \ F \bigcup Q(F) \bigcup \ldots \bigcup Q^{s-1}(F) \ .$$

Note that A_0 is invariant under Q, and it is compact because each $Q^i(F)$ is. Moreover, as $Q(Q^{s-1}(F)) \subseteq V$ by the choice of s, it is also true that for each $x^0 \in A_0$ there is some $i \leq s$ so that $Q^i(x^0) \in V$. In particular, this implies that A_0 is in domain of attraction of the origin, because V is in the domain of attraction.

We now claim that there are positive real numbers ε_i, $i = 0, \ldots, s-1$ and bounded open sets $N_0, \ldots, N_s$ and $M_0, \ldots, M_s$ with the properties that:

$$Q^i(F) \ \subseteq \ M_i \ \subseteq \ \operatorname{clos} M_i \ \subseteq \ N_i \tag{13}$$

and

$$P\left(N_i \times \mathbb{B}(0, \varepsilon_i)\right) \ \subseteq \ M_{i+1} \tag{14}$$

for each $i = 0, \ldots, s-1$. Let $M_s = N_s := V$, and define the N_i, M_i, ε_i recursively for decreasing $i = s-1, \ldots, 0$ as follows. Assume that N_{i+1}, M_{i+1} have been defined, and $i \geq 0$. Then, since

$$P\left(Q^i(F) \times \{0\}\right) \ = \ Q^{i+1}(F) \ \subseteq \ M_{i+1}$$

it follows by continuity of P and openness of M_{i+1} that there is a bounded open set N_i and an $\varepsilon_i > 0$ so that (14) holds and $Q^i(F) \subseteq N_i$. As $Q^i(F)$ is a compact set and N_i is open, there is some open set M_i so that (13) holds. This completes the recursive construction.

Let $\varepsilon > 0$ be the smallest of the ε_i's, and denote

$$B_0 \ := \ N_0 \bigcup N_1 \bigcup \ldots \bigcup N_{s-1}$$

and

$$L_0 \ := \ M_0 \bigcup M_1 \bigcup \ldots \bigcup M_{s-1} \ .$$

These are bounded sets and, because of (13), they satisfy $A_0 \subseteq L_0 \subseteq \operatorname{clos} L_0 \subseteq B_0$. Note also that

$$M_s = V \subseteq F \subseteq M_0 \subseteq L_0 \,. \tag{15}$$

Let $x \in B_0$ and $\|u\| < \varepsilon$. By definition of B_0, $x \in N_i$ for some $i = 0, \ldots, s-1$. Then (14) implies that $P(x,u) \in M_{i+1} \subseteq L_0$, the last inclusion by definition of L_0 and by (15) when $i = s-1$. Thus conclusion (i) holds.

Pick now any $x \in B_0$ and any control sequence $(u_1, \ldots, u_s)$ with $\|u_i\| < \varepsilon$ for all i. Assume that $x \in N_j$, where $j \in \{0, \ldots, s-1\}$. Then applying repeatedly (14) there results that $P(x,(u_1, \ldots, u_{s-j})) \in M_s \subseteq F \subseteq A_0$, and this proves (ii). ∎

Lemma 4.2 Assume that the compact $C \subseteq \mathbb{R}^n$ is in the domain of null-asycontrollability for the system (1), and let A_0 be a compact neighborhood of the origin. Then, there exists an integer $r \geq 1$ and a sequence of compact sets

$$A_0, A_1, \ldots, A_r$$

so that $C \subseteq A_0 \bigcup \ldots \bigcup A_r$ and $A_i \subseteq \mathcal{C}^1(A_{i-1})$ for each $i = 1, \ldots, r$.

Proof. Pick any $x \in C$. By definition of null-asycontrollability, there must exist some input sequence ω so that $P(x,\omega) \in \operatorname{int} A_0$, and hence so that also some neighborhood V_x of x is controlled into $\operatorname{int} A_0$ by ω. Covering C by such V_x's, and using compactness, we conclude that there is some integer $r \geq 1$ and some *finite* subset $U \subseteq \mathbb{R}^m$ so that every element of C can be controlled to $\operatorname{int} A_0$, and hence also into A_0, using inputs with values in U and of length at most r. Without loss of generality, we may assume that $0 \in U$. We let

$$A_l \;:=\; \{x \in C \,|\, P(x,(u_1, \ldots, u_l)) \in A_0 \text{ for some } (u_1, \ldots, u_l) \in U^l\}$$

for each $l = 1, \ldots, r$. Note that C is covered by the A_l's, by choice of r and U. Moreover, if $l \geq 1$ then, for each $x \in A_l$ and each $(u_1, \ldots, u_l)$ as in the definition of A_l, it holds that $P(x, u_1) \in A_{l-1}$.

It only remains to prove that each A_l is compact. For this, note that

$$C_l \;:=\; \{(x, u_1, \ldots, u_l) \,|\, x \in C, (u_1, \ldots, u_l) \in U^l, P(x,(u_1, \ldots, u_l)) \in A_0\}$$

is compact for each l, since C, A_0, U are all compact and P is continuous, and A_l is the projection of C_l on the x coordinates. ∎

Lemma 4.3 Assume that (1) is a given system, and $A_0, \ldots, A_r$ is a sequence of nonempty compact sets so that $A_i \subseteq \mathcal{C}^1(A_{i-1})$ for each $i = 1, \ldots, r$. Let B_0 and L_0 be bounded open sets with $A_0 \subseteq L_0 \subseteq \operatorname{clos} L_0 \subseteq B_0$. Then, there exist two sequences of bounded open sets $B_1, \ldots, B_r$ and $L_1, \ldots, L_r$ so that the following properties hold for each $i, j \in \{0, \ldots, r\}$:

1. $A_i \;\subseteq\; L_i \;\subseteq\; \operatorname{clos} L_i \;\subseteq\; B_0 \bigcup \ldots \bigcup B_i$.
2. $L_i \bigcap B_j \;=\; \emptyset \quad \text{if } \; i < j$.
3. $B_j \;\subseteq\; \mathcal{C}^1(L_i) \quad \text{if } \; j = i + 1$.

Proof. We assume that $B_0, \ldots, B_l$ and $L_0, \ldots, L_l$ have been already obtained, so that the desired properties hold for all $i, j \in \{0, \ldots, l\}$. (Note that when $l = 0$, property 1 holds by hypothesis, and 2 and 3 are vacuous.) We need to construct B_{l+1} and L_{l+1} so that the following are verified:

(a) $A_{l+1} \subseteq L_{l+1} \subseteq \operatorname{clos} L_{l+1} \subseteq B_0 \bigcup \ldots \bigcup B_{l+1}$.

(b) $L_i \bigcap B_{l+1} = \emptyset$ for each $i = 0, \ldots, l$.

(c) $B_{l+1} \subseteq \mathcal{C}^1(L_l)$.

Let

$$G := (B_0 \bigcup \ldots \bigcup B_l)^{\mathrm{c}}$$

(superscript denotes complement), a closed set, and introduce the compact set

$$F := \operatorname{clos} \left(L_0 \bigcup \ldots \bigcup L_l \right) .$$

Note that by property 1, F is contained in the union of $B_0, \ldots, B_l$, so $F \bigcap G = \emptyset$. Thus there exists some $\delta > 0$ so that

$$d(F, G) > \delta , \tag{16}$$

where d denotes distance between sets, $d(D, E) = \inf\{\|x - y\|, x \in D, y \in E\}$, $d = +\infty$ if either of D or E is empty. Now pick any x in the compact set

$$E := A_{l+1} \bigcap G .$$

Since $x \in A_{l+1}$ and by hypothesis $A_{l+1} \subseteq \mathcal{C}^1(A_l)$, there is some $u_x \in \mathbb{R}^m$ so that $P(x, u_x) \in A_l \subseteq L_l$. Therefore by continuity of $P(\cdot, u_x)$ there is some $\varepsilon(x) > 0$ so that $P(z, u_x) \in L_l$ whenever $\|x - z\| < \varepsilon(x)$, and we may take $\varepsilon(x) < \delta$. Take the open set

$$B_{l+1} := \bigcup_{x \in E} \mathbb{B}(x, \varepsilon(x))$$

and observe that

$$E \subseteq B_{l+1} \subseteq \mathbb{B}(E, \delta) \tag{17}$$

so in particular B_{l+1} is bounded. By construction (inputs u_x above), property (c) holds. Furthermore,

$$A_{l+1} \subseteq \left(B_0 \bigcup \ldots \bigcup B_l \right) \bigcup (A_{l+1} \bigcap G) \subseteq \left(B_0 \bigcup \ldots \bigcup B_l \right) \bigcup B_{l+1} , \tag{18}$$

the last inclusion by (17). Since the complement of $B_0 \bigcup \ldots \bigcup B_l \bigcup B_{l+1}$ is closed and disjoint from the compact A_{l+1} (by (18)), there is some open set L_{l+1} which contains A_{l+1} and so that (a) holds.

We still need to establish (b). For this, it is sufficient to show that B_{l+1} does not intersect F. Assume otherwise that would be some $x \in B_{l+1} \bigcap F$. Hence $x \in \mathbb{B}(E, \delta)$ (by (17)), so there is some $y \in E \subseteq G$ with $\|x - y\| < \delta$, but together with $x \in F$ this would contradict the inequality (16). ∎

Proof of Theorem 1.

First apply Lemma 4.1, to obtain A_0, B_0, ε, s as there. Now let C be any compact set in $\mathbb{R}^n$ which is to be stabilized, and apply Lemma 4.2 to this C and the A_0 just obtained. Let r and $A_1, \ldots, A_r$ be as in this second Lemma. Next apply Lemma 4.3, with these data, to obtain $B_1, \ldots, B_r$ as well as $L_1, \ldots, L_r$. Observe that

$$C \subseteq A_0 \bigcup \ldots \bigcup A_r \subseteq B_0 \bigcup \ldots \bigcup B_r$$

by property 1 in Lemma 4.3. We now define a set $\mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^m$, to be used when applying the definition of property (SEC^0), as follows:

$$\mathcal{U} := \mathcal{U}_0 \bigcup \ldots \bigcup \mathcal{U}_r ,$$

where

$$\mathcal{U}_0 \;:=\; \{(x,u)\,|\,x \in B_0, \|u\| < \varepsilon\}$$

and

$$\mathcal{U}_i \;:=\; \{(x,u)\,|\,x \in B_i, u \in \mathbb{R}^m, P(x,u) \in L_{i-1}\}$$

for all $i = 1,\ldots,r$. By property 3 in Lemma 4.3, B_i is included in the projection $\pi_1(\mathcal{U}_i)$ for each $i = 1,\ldots,r$, and the same is true obviously for $i = 0$. Note that each $\mathcal{U}_i$, and hence also $\mathcal{U}$, is open, because the sets B_i and L_i are open.

We will take C_0 in the definition of property (SEC^0) to be equal to A_0. Note that then $C_0 \subseteq B_0$, which means that $C_0 \times \{0\} \subseteq \mathcal{U}_0 \subseteq \mathcal{U}$, as needed in applying (SEC^0) with this C_0. As C is included in the union of the B_i's, it is a subset of $\pi_1(\mathcal{U})$. So we can apply the property, to obtain the desired feedback $K \in \mathcal{F}_p^m$.

We now prove that this feedback law provides stability. For each $x \in \mathbb{R}^n$ define

$$\mu(x) \;:=\; \max\{i\,|\,x \in B_i\}$$

with $\mu(x) := +\infty$ if $x \notin B_0 \bigcup \ldots \bigcup B_r$. Note that $\mu(x) \le r$ for all $x \in C$.

Claim: Pick any $x \in B_0 \bigcup \ldots \bigcup B_r$, and let $u := K(x)$ and $y := P(x,u)$. Then:

$$\mu(y) \;\le\; \max\{\mu(x) - 1, 0\}\;.$$

Indeed, let $k := \mu(x)$, so x is in none of the B_j, $j > k$, and thus necessarily $(x,u) \in \mathcal{U}_0 \bigcup \ldots \bigcup \mathcal{U}_k$. Let $(x,u) \in \mathcal{U}_j$, where $j \le k$.

If $j = 0$ then $x \in B_0$ and $\|u\| < \varepsilon$, so conclusion (i) in Lemma 4.1 implies that $y \in L_t$, where $t = 0$. If $j > 0$ then the definition of $\mathcal{U}_j$ implies that $y \in L_t$, where $t = j-1 \le k-1$. In either case, property 2 in Lemma 4.3 implies that $y \notin B_h$ for all $h > t$, so $\mu(y) \le t$. This proves the claim.

Pick now any initial state x^0 in C and consider the trajectory $x^{i+1} := P(x^i, K(x^i)), i = 1,2,\ldots$. From the above claim, we conclude that the sequence $\mu(x^i)$ becomes identically zero after at most r steps. At that point x^i is in L_0, so after at most s more steps it enters A_0, by conclusion (ii) in Lemma 4.1, after which, since $K(x) \equiv 0$ on $C_0 = A_0$, the dynamics is that of $x^+ = P(x,0)$, which is asymptotically stable by hypothesis. This proves that C is in the domain of attraction. Local asymptotic stability also follows from the fact that $K(x) \equiv 0$ about $x = 0$. This completes the proof of Theorem 1. ∎

Proof of Theorem 2.

This is an immediate consequence of Theorem 1: it is only necessary to first apply a linear stabilizing feedback $u = Fx$, and then apply the previous result to the new system

$$x^+ \;=\; P(x, Fx + u)$$

which is now locally asymptotically stable for $u = 0$. There results a feedback $K \in \mathcal{F}_n^m$ stabilizing this new system, which is the equivalent to saying that $K + F$ stabilizes the original system. ∎

4.1 One Hidden Layer is Not Enough

Theorem 3 is an immediate consequence of the following more precise result:

Proposition 4.4 Let $\mathcal{F}$ be a compatible class of functions that does not satisfy property (SEC). Then, there exists a system (1), for which the origin is locally asymptotically stable for the zero-input dynamics, and for which every state can be controlled to zero in at most two steps (hence asymptotically controllable), and there is a compact subset C of the state space, so that the following happens:

For every feedback law $K \in \mathcal{F}_n^m$, the closed-loop system (2) has a nontrivial periodic orbit that intersects C.

Proof. Let $\mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^q$ be open and let $C \subseteq \mathbb{R}^n$ be a compact subset included in the projection $\pi_1(\mathcal{U})$ of $\mathcal{U}$ on the first n coordinates, for which there exists no $\phi \in \mathcal{F}_n^q$ so that $(x, \phi(x)) \in \mathcal{U}$ for all $x \in C$.

We first claim that we can assume $0 \notin C$. This is because one may always consider the compact set $\widetilde{C} := \{(1,x) \,|\, x \in C\} \subseteq \mathbb{R}^{n+1}$ and the open set $\widetilde{\mathcal{U}} := \mathbb{R} \times \mathcal{U} \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^q$, for which $\widetilde{C}$ is included in the projection of $\widetilde{\mathcal{U}}$ in the first $n+1$ coordinates. If there would be a $\psi \in \mathcal{F}_{n+1}^q$ so that $((1,x), \psi(1,x)) \in \widetilde{\mathcal{U}}$ for all $x \in C$ (property (SEC) for $\widetilde{C}$ and $\widetilde{\mathcal{U}}$) then $(x, \phi(x)) \in \mathcal{U}$ for all $x \in C$, where $\phi(\cdot) := \psi(1, \cdot) \in \mathcal{F}_n^q$, contradicting the above. Thus we assume from now on that $0 \notin C$.

Let $\lambda > 0$ be a real number chosen in such a manner that the set

$$D := \frac{1}{\lambda} C = \left\{ \frac{1}{\lambda} x \,\middle|\, x \in C \right\}$$

does not intersect C. Such a λ exists because $0 \notin C$ and C is compact. Consider the following two disjoint closed subsets of $\mathbb{R}^n \times \mathbb{R}^q$:

$$F_1 := \{(x,u) \,|\, x \in C \text{ and } (x,u) \notin \mathcal{U}\}$$

and

$$F_2 := \{(x,u) \,|\, x \in D\} \,.$$

Now let γ and ψ be continuous functions $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ chosen as follows (if desired, they can also picked infinitely differentiable, by Exercise 2.2.1 in [5]):

$$\psi(x,u) = \begin{cases} 1 & \text{if } (x,u) \in F_1 \\ 0 & \text{if } (x,u) \in F_2 \\ \in (0,1) & \text{otherwise} \end{cases}$$

and $\gamma(x,u) = 0$ on $F_1 \bigcup F_2$ and > 0 otherwise. Finally, let

$$\alpha(x, u_1, u_2) := \gamma(x, u_1) u_2 + \lambda + \left(\frac{1}{\lambda} - \lambda \right) \psi(x, u_1)$$

for any $x \in \mathbb{R}^n, u_1 \in \mathbb{R}^q, u_2 \in \mathbb{R}$. Observe that, for all x, u_1, u_2:

$$\alpha(x, u_1, u_2) = 1/\lambda \quad \text{if } (x, u_1) \in F_1 \,, \tag{19}$$

$$\alpha(x, u_1, u_2) = \lambda \quad \text{if } x \in D \tag{20}$$

and

$$\alpha(x, u_1, u_2) = a(x, u_1) u_2 + b(x, u_1), \; a(x, u_1) \neq 0, \quad \text{if } (x, u_1) \notin F_1 \bigcup F_2 \,. \tag{21}$$

As $(0,0) \notin F_1 \bigcup F_2$ (because $x = 0$ is in neither C nor D), we may pick, because of (21), some u_2^0 so that $\alpha(0, 0, u_2^0) = 0$.

Consider now the system with input space $\mathbb{R}^m = \mathbb{R}^{q+1}$, state space $\mathbb{R}^n$, and equations

$$x^+ = P(x, (u_1, u_2)) := \alpha(x, u_1, u_2 + u_2^0)\, x \,.$$

Note that $P(0, (0,0)) = 0$, as needed for the definition of system. Moreover, the Jacobian of P with respect to x, evaluated at $x = 0, u_1 = 0, u_2 = 0$ is zero, so the linearization of this system at the origin has asymptotically stable dynamics $x^+ = 0$. It follows that the origin is locally asymptotically stable for the zero-input dynamics.

We claim that every state can be controlled to zero in at most two steps. Take any state $x \in \mathbb{R}^n$. If $x \notin C \bigcup D$ then for any $u_1 \in \mathbb{R}^q$ one may find, by (21), an u_2 so that $\alpha(x, u_1, u_2 + u_2^0) = 0$, and any such pair (u_1, u_2) drives the state to zero in one step. If $x \in C$ then by the assumption $C \subseteq \pi_1(\mathcal{U})$ there is some u_1 so that $(x, u_1) \in \mathcal{U}$. It follows that $(x, u_1) \notin F_1 \bigcup F_2$, so again it is possible to control to 0 in one step. Finally, assume that $x \in D$. Pick any u_1, u_2. Then (20) gives that $z := P(x, (u_1, u_2)) = \lambda x \in C$. Thus in one more step z can be controlled to zero, as wanted.

Take any $K \in \mathcal{F}_n^m$, and write $K(\cdot) = (\phi(\cdot), \rho(\cdot))$, with $\phi \in \mathcal{F}_n^q$. Then there must exist some $x \in C$ so that $(x, \phi(x)) \notin \mathcal{U}$ (by the choice of C,$\mathcal{U}$ contradicting property (SEC)). Consider this x, and take $u_1 := \phi(x), u_2 := \rho(x)$. As $(x, u_1) \notin \mathcal{U}$, it follows that $(x, u_1) \in F_1$, so by (19) it follows that

$$P(x, K(x)) \; = \; \frac{1}{\lambda} x \; .$$

Now $z := (1/\lambda)x \in D$, so as in the previous paragraph it follows that $P(z, K(z)) = \lambda z = x$. In conclusion, the closed-loop system (2) has a periodic orbit $x, z, x, z, \ldots$ with $x \neq z$ and $x \in C$. This completes the proof of the Proposition, and hence also of Theorem 3. ∎

References

[1] Blum, E.K., and L. Kwan Li, "Approximation theory and feed-forward networks," *Neural Networks* **4**(1991): 511-516.

[2] Chester, D., "Why two hidden layers and better than one," *Proc. Int. Joint Conf. on Neural Networks, Washington, DC, Jan. 1990*, IEEE Publications, 1990, p. I.265-268.

[3] Cybenko, G., "Approximation by superpositions of a sigmoidal function," *Math. Control, Signals, and Systems* **2**(1989): 303-314.

[4] Funahashi, K., "On the approximate realization of continuous mappings by neural networks," *Neural Networks* **2**(1989): 183-192.

[5] Hirsch, M.W., *Differential Topology*, Springer-Verlag, NY, 1976

[6] Hornik, K.M., M. Stinchcombe, and H. White, "Multilayer feedforward networks are universal approximators," *Neural Networks* **2**, 1989, p. 359-366.

[7] Narendra, K.S., and K. Parthasarathy, "Identification and control of dynamical systems using neural networks," *IEEE Trans. Neural Nets* **1**(1990): 4-27.

[8] Sontag, E.D., "Remarks on piecewise-linear algebra," *Pacific J.Math.*, **98**(1982): 183-201.

[9] Sontag, E.D., "Nonlinear regulation: The piecewise linear approach," *IEEE Trans. Autom. Control* **AC-26**(1981): 346-358.

[10] Sontag, E.D., "Real addition and the polynomial hierarchy," *Inform. Proc. Letters* **20**(1985): 115-120.

[11] Sontag, E.D., "Feedforward nets for interpolation and classification," *J. Comp. Syst. Sci.*, to appear.

[12] Sontag, E.D., *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, Springer, New York, 1990.

[13] Sontag, E.D., "Feedback stabilization of nonlinear systems," in *Robust Control of Linear Systems and Nonlinear Control* (M.A. Kaashoek, J.H. van Schuppen, and A.C.M. Ran, eds,.) Birkhäuser, Cambridge, MA, 1990, pp. 61-81.