SIAM J. CONTROL OPTIM.
Vol. 43, No. 1, pp. 256–276

SEPARATION PRINCIPLES FOR INPUT-OUTPUT AND INTEGRAL-INPUT-TO-STATE STABILITY*

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Abstract. We present new characterizations of input-output-to-state stability. This is a notion of detectability formulated in the ISS (input-to-state stability) framework. Equivalent properties are presented in terms of asymptotic estimates of the state trajectories based on the magnitudes of the external input and output signals. These results provide a set of *separation principles* for input-output-to-state stability—characterizations of the property as conjunctions of weaker stability notions. When applied to the notion of integral ISS, these characterizations yield analogous results.

Key words. nonlinear systems, input-to-output stability, detectability, Lyapunov method

AMS subject classifications. 93D20, 93D05, 93D09

DOI. 10.1137/S0363012902419047

1. Introduction.

Detectability is a central notion in control theory. It plays a major role both in static state-feedback design (LaSalle's invariance principle and Jurdjevic–Quinn control) as well as in stabilization by means of dynamic output feedback or observer design. Several possibilities are available when formulating such a notion in the context of nonlinear control. According to the specific problem under consideration, they capture some or most of the useful features of its linear counterpart. One approach that has proved to be especially powerful for systems subject to exogenous disturbances is to define zero-detectability in terms of estimates involving (possibly nonlinear) gains with respect to input and output norms. This leads to the so-called input-output-to-state stability (IOSS) property. Such a notion not only allows one to extend LaSalle-type stability results to the case of nonautonomous systems [2], but it also provides a machinery, fully compatible with the formalism of the input-to-state stability (ISS) property [8, 9, 13, 14, 15, 17, 18, 19, 23, 24, 25, 30, 31], that helps one understand relevant issues such as minimum-phase behavior or certainty equivalence [20, 11].

Although general nonlinear systems may often exhibit an overwhelming variety of behaviors, it turns out that many of the "reasonable" formulations of the detectability property (meaning at least compatible with the linear notion of detectability) are equivalent to each other. In this paper, we discuss characterizations of IOSS in terms of the asymptotic behavior of system solutions. This leads to several useful decompositions of the IOSS property in terms of weaker notions. These *separation principles* are in direct analogy to those previously provided for ISS [28] and input-to-output sta-

*Received by the editors November 28, 2002; accepted for publication (in revised form) October 30, 2003; published electronically June 15, 2004.
http://www.siam.org/journals/sicon/43-1/41904.html

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bility [12]. These results all generalize from differential equations the fact that global asymptotic stability can be characterized by the combination of (neutral) stability and attractivity.

As an application of our results, we will also discuss several ways of reformulating the notion of integral input-to-state stability (iISS) (cf. [5, 26]) in terms of asymptotic gains. This is accomplished by treating the iISS property as the IOSS property for suitable auxiliary systems. Similar results were also obtained for the so-called derivative-ISS property (DISS) in the recent work [7].

As mentioned, the main results are (far from obvious) generalizations to systems with outputs of the analogous separation principles which appeared in [28] dealing with the ISS notion. Actually, the ISS case is a special case of IOSS when the output map is identically zero. However, it takes much more effort to handle the general case of nonzero output maps. It can be seen from later sections that IOSS amounts to the requirement of convergence to 0 (or to balls whose radii are proportional to the norms of the input signals) *only* for those trajectories which evolve in a certain set constrained by the output signals. In the ISS case, by comparison, the output map is identically zero, so the constraints become trivial, and the restricted set becomes the whole state space.

2. Basic definitions. Consider systems in the following general form:

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)), \tag{1}$$

where, for each $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{U}$, a subset of $\mathbb{R}^m$. We assume that the maps $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h: \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz continuous, with $f(0,0) = 0$ and $h(0) = 0$. The symbol $|\cdot|$ denotes the usual Euclidean norms. The open ball in $\mathbb{R}^l$ centered at the origin with radius r will be denoted by $\mathcal{B}_l(r)$.

By an *input* we mean a measurable and locally essentially bounded function $u : \mathcal{I} \to \mathbb{U}$, where $\mathcal{I}$ is a subinterval of $\mathbb{R}$ which contains the origin. Whenever the domain $\mathcal{I}$ of an input u is not specified, it will be understood that $\mathcal{I} = \mathbb{R}_{\geq 0}$. Given a system with input-value set $\mathbb{U}$, we will also often consider the same system with inputs restricted to some subset $\mathcal{O} \subseteq \mathbb{U}$. We use $\mathcal{M}_{\mathcal{O}}$ to denote the set of all such inputs.

Given any input u and any $\xi \in \mathbb{R}^n$, the unique maximal solution of the initial value problem $\dot{x} = f(x,u)$, $x(0) = \xi$ (defined on some maximal open subinterval of $\mathcal{I}$) is denoted by $x(\cdot, \xi, u)$. When $\mathcal{I} = \mathbb{R}_{\geq 0}$, this maximal subinterval has the form $[0, T_{\xi,u})$. The system is said to be *forward complete* if for every initial state ξ and for every input u defined on $\mathbb{R}_{\geq 0}$, $T_{\xi,u} = +\infty$. The corresponding output is denoted by $y(\cdot, \xi, u)$, that is, $y(t, \xi, u) = h(x(t, \xi, u))$ on the domain of definition of the solution.

The L_∞-norm (possibly infinite) of a function v defined on $\mathcal{I}$ is denoted by $\|v\|$, i.e.,

$$\|v\| = (\text{ess}) \sup\{|v(t)|, t \in \mathcal{I}\}.$$

In particular, for a maximal trajectory $x(\cdot, \xi, u)$ and the corresponding output function $y(\cdot, \xi, u)$ of (1) defined on $[0, T_{\xi,u})$, $\|u\|$, $\|x\|$, and $\|y\|$ denote the L_∞-norm of $u(\cdot)$, $x(\cdot, \xi, u)$, and $y(\cdot, \xi, u)$, respectively, on $[0, T_{\xi,u})$. We will make a slight abuse of notation and use sup and lim sup to mean the essential supremum where appropriate. For a function v defined on an interval $\mathcal{I}$, if $\mathcal{I}_1 \subseteq \mathcal{I}$, we use $v_{\mathcal{I}_1}$ to denote the restriction of v to $\mathcal{I}_1$, i.e., $v_{\mathcal{I}_1}(t) = v(t)$ if $t \in \mathcal{I}_1$, and $v_{\mathcal{I}_1}(t) = 0$ otherwise. Notice the following fact: for a measurable function v defined on some interval $[0, T)$ for some $T \leq \infty$,

$$\limsup_{t \to T} |v(t)| = \lim_{t \to T} \|v_{[t,T)}\|. \tag{2}$$

We use standard terminology (cf. [10]): $\mathcal{N}$ is the class of continuous, increasing functions from $[0,\infty)$ to $[0,\infty)$; $\mathcal{K}$ is the set of $\mathcal{N}$ functions γ that are strictly increasing and satisfy $\gamma(0)=0$; $\mathcal{K}_\infty$ is the set of $\mathcal{K}$ functions that are unbounded; $\mathcal{L}$ is the set of functions $[0,+\infty)\to[0,+\infty)$ which are continuous, decreasing, and converge to 0 as their argument tends to $+\infty$; $\mathcal{KL}$ is the class of functions $[0,\infty)^2\to[0,\infty)$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second one. A positive definite function $\gamma:[0,\infty)\to[0,\infty)$ is one such that $\gamma(0)=0$ and $\gamma(s)>0$ for all $s>0$.

The following notions were introduced in [22] (see also [4]) and [16, 29], respectively.

DEFINITION 2.1. *The system* (1) *satisfies the* unboundedness observability *(UO) property if, for each state ξ and control u such that $T_{\xi,u}<\infty$, it holds that* $\limsup_{t\to T_{\xi,u}}|y(t,\xi,u)|=+\infty$*, that is, for each state ξ and control u,*

$$T_{\xi,u}<\infty \quad\Rightarrow\quad \|y\|=+\infty.$$

DEFINITION 2.2. *The system* (1) *is* input-output-to-state stable *(IOSS) if there exist some $\beta\in\mathcal{KL}$, $\gamma_u\in\mathcal{K}$, and $\gamma_y\in\mathcal{K}$ such that*

$$|x(t,\xi,u)|\le\beta(|\xi|,t)+\gamma_u(\|u\|)+\gamma_y\left(\|y_{[0,t]}\|\right) \tag{3}$$

for all $t\in[0,T_{\xi,u})$, all $\xi\in\mathbb{R}^n$, and all $u(\cdot)$. □

Clearly, the IOSS property implies the UO property. The following local version of it will also be used in the proof of our main result.

DEFINITION 2.3. *The system* (1) *is* locally IOSS *if there exist $\delta>0$ and functions $\beta\in\mathcal{KL}$, $\gamma_u,\gamma_y\in\mathcal{K}$ so that for any $\xi\in\mathbb{R}^n$, any $u(\cdot)$,*

$$\max\{|\xi|,\|u\|,\|y\|\}\le\delta \;\Rightarrow\; |x(t,\xi,u)|\le\beta(|\xi|,t)+\gamma_u(\|u\|)+\gamma_y(\|y_{[0,t]}\|) \tag{4}$$

for all $t\in[0,T_{\xi,u})$. □

Below we discuss several other properties for systems as in (1) regarding estimates of the state variables on the basis of external information provided by past input and output signals.

2.1. A catalog of properties.

DEFINITION 2.4. *Consider system* (1)*. We say that*

- *the* input-output limit property *(IO-LIM) holds if for some $\gamma_u,\gamma_y\in\mathcal{K}$,*

$$\inf_{t\in[0,T_{\xi,u})}|x(t,\xi,u)|\le\max\{\gamma_u(\|u\|),\gamma_y(\|y\|)\}\quad\forall\,\xi\in\mathbb{R}^n,\ \forall\,u(\cdot); \tag{5}$$

- *the* input-output asymptotic gain property *(IO-AG) holds if solutions are ultimately bounded by some nonlinear gain function of $\|u\|$ and $\|y\|$, that is, for some $\gamma_u,\gamma_y\in\mathcal{K}$,*

$$\limsup_{t\to T_{\xi,u}}|x(t,\xi,u)|\le\max\{\gamma_u(\|u\|),\gamma_y(\|y\|)\}\quad\forall\,\xi\in\mathbb{R}^n,\ \forall\,u(\cdot); \tag{6}$$

- *the* input-output-to-state boundedness property *(IO-BND) holds if for some $\sigma_0,\sigma_u,\sigma_y\in\mathcal{N}$, it holds that*

$$|x(t,\xi,u)|\le\max\{\sigma_0(|\xi|),\sigma_u(\|u\|),\sigma_y(\|y_{[0,t]}\|)\} \tag{7}$$

for all $\xi\in\mathbb{R}^n$, all $u(\cdot)$, and all $t\in[0,T_{\xi,u})$;

- *the* input-output global stability property *(IO-GS) holds if the functions* $\sigma_0, \sigma_u, \sigma_y$ *in* (7) *can be taken to be of class* $\mathcal{K}$.

Thinking of these detectability properties as "stability modulo inputs and outputs," we can identify IOSS with asymptotic stability, IO-GS with (neutral) stability, and IO-AG with attractivity. In this context it seems perfectly natural that IOSS should be equivalent to the combination of IO-GS and IO-AG, and indeed that is one of the decompositions which appears in our main result. Related results follow by considering other "basic" stability-like notions, such as IO-LIM.

It is not hard to see that each of the IO-AG, the IO-GS, and the IO-BND properties implies the UO condition. This follows from the fact that, for any ξ, u, if $T_{\xi,u} < \infty$, then $|x(t,\xi,u)| \to \infty$ as $t \to T_{\xi,u}$. That the IO-LIM property also implies the UO condition follows from the next remark.

Remark 2.5. The IO-LIM property can be defined equivalently by replacing the "inf" in (5) by "lim inf", that is,

$$\liminf_{t\to T_{\xi,u}} |x(t,\xi,u)| \le \max\{\gamma_u(\|u\|), \gamma_y(\|y\|)\}. \tag{8}$$

It is straightforward that (8) implies (5). To see that (5) implies (8), take any $T \in [0, T_{\xi,u})$. Applying (5) to $x(t,\xi_T,u_T)$ with $\xi_T = x(T,\xi,u)$ and $u_T(t) = u(t+T)$, one gets

$$\inf_{t\in[T,T_{\xi,u})} |x(t,\xi,u)| \le \max\{\gamma_u(\|u_T\|), \gamma_y(\|y_T\|)\} \le \max\{\gamma_u(\|u\|), \gamma_y(\|y\|)\}, \tag{9}$$

where $y_T = h(x(\cdot,\xi_T,u_T))$. Since T can be arbitrary, one obtains (8). □

Another characterization of IO-LIM is as follows. This statement is proved (along with some related characterizations of the IO-LIM property) in Appendix A.

LEMMA 2.6. *System* (1) *satisfies the IO-LIM property if and only if there exist* γ_u, $\gamma_y \in \mathcal{K}$ *such that*

$$\inf_{t\in[0,T_{\xi,u})} \Big\{ |x(t,\xi,u)| - \max\{\gamma_u(\|u_{[0,t]}\|), \gamma_y(\|y_{[0,t]}\|)\} \Big\} \le 0 \qquad \forall\, \xi \in \mathbb{R}^n,\ \forall\, u(\cdot). \tag{10}$$

The following implication will be needed.

LEMMA 2.7. *If* (1) *satisfies the IO-LIM property, then it satisfies the IO-BND property.*

Proof. Consider a system as in (1). For each subset $\mathcal{O}$ of the input space $\mathbb{U}$, each subset C of $\mathbb{R}^n$, and each $\mathcal{Y} \subset \mathbb{R}^p$ we denote

$$\mathcal{R}_{\mathcal{O}/\mathcal{Y}}(C) := \Big\{ x(t,\xi,u) :\ \xi \in C, u \in \mathcal{M}_{\mathcal{O}}, t \in [0, T_{\xi,u}) \\ \text{and } h(x(\lambda,\xi,u)) \in \mathcal{Y}\ \ \forall \lambda \in [0,t] \Big\}.$$

Then $\mathcal{R}_{\mathcal{O}/\mathcal{Y}}(C)$ is the reachable set with initial conditions in C, controls in $\mathcal{O}$, and subject to an output constraint.

Suppose a system as in (1) satisfies the IO-LIM property and, consequently, the UO property as well. By Lemma 2.6 one sees that (10) holds for some $\gamma_u, \gamma_y \in \mathcal{K}$. Pick an arbitrary $s > 0$. We let

$$\Omega = \mathcal{B}_n(2s),\ C = \mathrm{cl}(\Omega),\ K = \mathrm{cl}\left(\mathcal{B}_n\left(\frac{3s}{2}\right)\right),\ \mathcal{Y} = \mathrm{cl}\left(\mathcal{B}_p\left(\gamma_y^{-1}\frac{(s)}{2}\right)\right)$$

and

$$\mathcal{Y}_o = \mathcal{B}_p(\gamma_y^{-1}(s)),\ \mathcal{O} = \mathrm{cl}(\mathcal{B}_m(\gamma_u^{-1}(s))).$$

Finally we define $\sigma_0(s) := \sup\{|p| :\ p \in \mathcal{R}_{\mathcal{O}/\mathcal{Y}}(C)\}$.

Claim. $\sigma_0(s) < \infty$.

Proof. For any $\xi \in \mathbb{R}^n$ and any $u \in \mathcal{M}_{\mathcal{O}}$, by (10), there exists $\tau \in [0, T_{\xi,u})$ so that

$$|x(\tau, \xi, u)| \le \frac{3}{2} \max\left\{s, \gamma_y(\|y_{[0,\tau]}\|)\right\}.$$

Considering separately the cases $\|y_{[0,\tau]}\| \le \gamma_y^{-1}(s)$ and $\|y_{[0,\tau]}\| > \gamma_y^{-1}(s)$, we obtain that either

$$|x(\tau, \xi, u)| \le \frac{3s}{2}$$

or there is some $t \in [0, \tau]$ so that

$$|h(x(t, \xi, u))| > \gamma_y^{-1}(s).$$

In other words, there exists $\tau \in [0, T_{\xi,u})$ such that either $x(\tau, \xi, u) \in K$ or for some $t \in [0, \tau]$, $h(x(t, \xi, u)) \notin \mathcal{Y}_o$. We then apply Lemma B.1 to conclude that $\mathcal{R}_{\mathcal{O}/\mathcal{Y}}(C)$ is bounded, and thus $\sigma_0(s) < \infty$.

Let $\sigma_u(r) = \sigma_0(\gamma_u(r))$ and $\sigma_y(r) = \sigma_0(\gamma_y(r))$. Now pick $\xi \in \mathbb{R}^n$, an input u, and $t \in [0, T_{\xi,u})$. Let $s = \max\{|\xi|, \gamma_u(\|u\|), \gamma_y(\|y_{[0,t]}\|)\}$. If $s = 0$, then $|x(t, 0, 0)| = 0$. If $s > 0$, by definition of σ_0 we have

$$|x(t, \xi, u)| \le \sigma_0(s) \le \max\{\sigma_0(|\xi|), \sigma_u(\|u\|), \sigma_y(\|y_{[0,t]}\|)\}. \tag{11}$$

This completes the proof of IO-LIM ⇒ IO-BND. □

Remark 2.8. Note that in the above proof, if the function γ_u can be chosen to be the zero function, that is, if (10) can be strengthened to

$$\inf_{t \in [0, T_{\xi,u})} \left[|x(t, \xi, u)| - \gamma_y(\|y_{[0,t]}\|)\right] \le 0,$$

then the function σ_u in (11) can be chosen to be the zero function. □

We next comment on some straightforward characterizations of the IO-AG property.

Remark 2.9. It is immediate from the definition that the IO-AG property (6) is equivalent to the UO property in combination with the following:

$$\limsup_{t \to \infty} |x(t, \xi, u)| \le \max\left\{\gamma_u(\|u\|),\ \gamma_y(\|y\|)\right\} \tag{12}$$

for all ξ, u for which $T_{\xi,u} = \infty$. □

Combining this remark with (2), the following can be easily shown (using an argument as in Remark 2.5).

LEMMA 2.10. *A system as in* (1) *satisfies the IO-AG property if and only if it is UO and, for some* $\gamma_u, \gamma_y \in \mathcal{K}$*, the following holds for all* ξ *and* u *for which* $T_{\xi,u} = \infty$:

$$\limsup_{t \to \infty} |x(t, \xi, u)| \le \max\left\{\gamma_u\left(\limsup_{t \to \infty} |u(t)|\right),\ \gamma_y\left(\limsup_{t \to \infty} |y(t)|\right)\right\}. \tag{13}$$

2.2. IOSS and OSS properties. Consider a system

$$\dot{x} = f(x), \;\; y = h(x) \tag{14}$$

without input. This can be considered as a system as in (1) with $\mathbb{U}$ consisting of a single point. We use $x(\cdot, \xi)$ to denote the solution of (14) with the initial state ξ defined on a maximal interval $[0, T_\xi)$, and we let $y(t, \xi) = h(x(t, \xi))$.

DEFINITION 2.11. *We say that the system* (14) *is*

- locally stable modulo outputs *(O-LS) if for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all ξ and all $t \in [0, T_\xi)$ it holds that*

$$\max\{|\xi|, \|y_{[0,t]}\|\} \leq \delta_\varepsilon \;\Rightarrow\; |x(t, \xi)| \leq \varepsilon; \tag{15}$$

- output-to-state stable *(OSS) (see* [29]*) if there exist some $\beta \in \mathcal{KL}$ and some $\gamma \in \mathcal{K}$ such that, for any trajectory $x(\cdot)$ of the system, it holds that*

$$|x(t, \xi)| \leq \beta(|\xi|, t) + \gamma(\|y_{[0,t]}\|) \qquad \forall\, t \in [0, T_\xi). \tag{16}$$

For a system as in (1), we say that the system is *zero-input OSS* (zero-OSS) or *zero-input O-LS* (zero-O-LS) if the zero input system $\dot{x} = f(x, 0), \; y = h(x)$ is OSS or O-LS, respectively. The following technical lemma will be needed in the proof of our main result.

LEMMA 2.12. *The zero-OSS property implies the local IOSS property.*

Proof. Suppose the system (1) is zero-OSS. Then, by Theorem 3 in [29] there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions α_1, α_2, α, and ρ such that $\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)$ and

$$\frac{\partial V(\xi)}{\partial \xi} f(\xi, 0) \leq -2\alpha(|\xi|) + \rho(|h(\xi)|) \qquad \forall\, \xi \in \mathbb{R}^n. \tag{17}$$

Let $\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined as

$$\sigma(s, r) = sr + \max_{|\xi| \leq s, |\mu| \leq r} \left\{ \frac{\partial V(\xi)}{\partial \xi} \left[f(\xi, \mu) - f(\xi, 0) \right] \right\}. \tag{18}$$

Note that $\sigma(s, 0) = 0$ for all $s \geq 0$. Since V is smooth and $\frac{\partial V}{\partial \xi}(0) = 0$, we also have $\sigma(0, r) = 0$ for all $r \geq 0$. Furthermore, $\sigma(\cdot, r) \in \mathcal{K}$ for each $r \geq 0$ and $\sigma(s, \cdot) \in \mathcal{K}$ for each $s \geq 0$. By Corollary IV.5 in [5], there exist σ_1, $\sigma_2 \in \mathcal{K}_\infty$ so that $\sigma(s, r) \leq \sigma_1(s)\sigma_2(r)$. Hence, combining (17) and (18), we get

$$\begin{aligned}
\frac{\partial V(\xi)}{\partial \xi} f(\xi, \mu) &= \frac{\partial V(\xi)}{\partial \xi} f(\xi, 0) + \frac{\partial V(\xi)}{\partial \xi} \left[f(\xi, \mu) - f(\xi, 0) \right] \\
&\leq -2\alpha(|\xi|) + \rho(|h(\xi)|) + \sigma(|\xi|, |\mu|) \\
&\leq -2\alpha(|\xi|) + \rho(|h(\xi)|) + \sigma_1(|\xi|)\sigma_2(|\mu|).
\end{aligned}$$

Therefore

$$\alpha^{-1}(\sigma_2(|\mu|) + \rho(|h(\xi)|)) \leq |\xi| \leq \sigma_1^{-1}(1) \quad \Rightarrow \quad \frac{\partial V(\xi)}{\partial \xi} f(\xi, \mu) \leq -\alpha(|\xi|),$$

from which it follows that for some $\mathcal{K}_\infty$-functions χ_1, χ_2, and $\tilde{\alpha}$ and some $c > 0$, it holds that

$$\max\{\chi_1(|\mu|), \chi_2(|h(\xi)|)\} \leq V(\xi) \leq c \quad \Rightarrow \quad \frac{\partial V(\xi)}{\partial \xi} f(\xi, \mu) \leq -\tilde{\alpha}(V(\xi)).$$

Pick any initial state ξ and input u. Denote the trajectory $x(t,\xi,u)$ by $x(t)$, and the output $y(t,\xi,u)$ by $y(t)$. Pick any $T \in [0, T_{\xi,\mu})$, and let

$$v^* = \max\{\chi_1(\|u\|), \chi_2(\|y_{[0,T]}\|)\}.$$

For almost all $t \in [0,T]$, we have

$$v^* \le V(x(t)) \le c \;\Rightarrow\; \frac{d}{dt}V(x(t)) \le -\tilde{\alpha}(V(x(t))).$$

By Lemma 13 in [29], one sees that there exists some $\beta_0 \in \mathcal{KL}$ which depends only on $\tilde{\alpha}$ so that

$$\max_{t\in[0,T]} V(x(t)) \le c \quad \Rightarrow\; V(x(t)) \le \max\{\beta_0(V(x(0)),t), v^*\} \quad \forall\, t \in [0,T];$$

that is, if $V(x(t)) \le c$ for all $t \in [0,T]$,

$$V(x(t)) \le \max\{\beta_0(V(x(0)),t), \chi_1(\|u\|),\ \chi_2(\|y_{[0,T]}\|)\} \quad \forall\, t \in [0,T]. \tag{19}$$

Let $\hat{\beta}_0(r) = \beta_0(r,0)$. Without loss of generality, we assume that $\hat{\beta}_0(r) > r$, and thus, $\hat{\beta}_0 \in \mathcal{K}_\infty$. Let

$$\delta = \min\left\{\chi_1^{-1}\left(\frac{c}{2}\right),\ \chi_2^{-1}\left(\frac{c}{2}\right),\ \alpha_2^{-1}\left(\hat{\beta}_0^{-1}\left(\frac{c}{2}\right)\right)\right\}.$$

Claim. If $\max\{|x(0)|,\ \|u\|,\ \|y\|\} \le \delta$, then $V(x(t)) \le c$ for all $t \in [0, T_{\xi,u})$.

Suppose the claim fails. This means

$$\max\{\tilde{\beta}_0(\alpha_2(|x(0)|)),\ \chi_1(\|u\|),\ \chi_1(\|y\|)\} \le \frac{c}{2},$$

but for some $t \in [0, T_{\xi,u})$, $V(x(t)) \ge c$. Let

$$t_0 = \inf\left\{t \ge 0 : V(x(t)) \ge \frac{c}{2}\right\}.$$

Then $t_0 < T_{\xi,u}$. By the continuity property of $x(\cdot)$, there is some $0 < \varepsilon < T_{\xi,u} - t_0$ such that on $[0, t_0+\varepsilon)$, $V(x(t)) < c$. By (19), we have

$$V(x(t)) \le \max\{\hat{\beta}_0(V(x(0))),\ \chi_1(\|u\|),\ \chi_2(\|y\|)\} \le \frac{c}{2}$$

for all $t \in [0, t_0+\varepsilon]$. This contradicts the definition of t_0.

Finally, applying (19) together with the proved claim, one sees that if

$$\max\{|x(0)|, \|u\|, \|y\|\} < \delta,$$

then

$$|x(t)| \le \max\{\beta(|x(0)|, t),\ \gamma_1(\|u\|),\ \gamma_2(\|y_{[0,t]}\|)\}\ \forall\, t \in [0, T_{\xi,u}),$$

where $\beta(s,r) = \alpha_1^{-1}(\beta_0(\alpha_2(s),r)), \gamma_i(r) = \alpha_1^{-1}(\chi_i(r))$ for $i = 1,2$. $\square$

3. Equivalent characterizations of IOSS. The following is our main result in the context of IOSS.

THEOREM 1. *Consider a system as in* (1) *with* $\mathbb{U} = \mathbb{R}^m$. *The following properties are equivalent:*

1. *(IOSS),*
2. *(IO-AG)* & *(IO-GS),*
3. *(IO-AG)* & *(zero-OSS),*
4. *(IO-AG)* & *(local IOSS),*
5. *(IO-AG)* & *(zero-O-LS),*
6. *(IO-LIM)* & *(IO-GS),*
7. *(IO-LIM)* & *(zero-OSS),*
8. *(IO-LIM)* & *(local IOSS),*
9. *(IO-LIM)* & *(zero-O-LS).*

Among the properties listed in Theorem 1, it is easy to see that the IOSS property implies every other one, and the zero-O-LS property is implied by any one of the local IOSS, the zero-OSS, and the IO-GS properties. Thus, to prove Theorem 1, it is enough to show the following implication:

$$\text{(IO-LIM) \& (zero-O-LS)} \Rightarrow \text{(IOSS)}.$$

We will proceed by the following technical lemmas.

LEMMA 3.1. *(IO-LIM)* & *(zero-O-LS)* $\Rightarrow$ *(zero-OSS).*

LEMMA 3.2. *(IO-LIM)* & *(local IOSS)* $\Rightarrow$ *(IO-LIM)* & *(IO-GS).*

LEMMA 3.3. *(IO-LIM)* & *(IO-GS)* $\Rightarrow$ *(IO-AG)* & *(IO-GS).*

Observe that Lemmas 3.1–3.3 together with Lemma 2.12 provide the following chain:

$$\text{(IO-LIM) \& (zero-O-LS)} \Rightarrow \text{(IO-LIM) \& (zero-OSS)} \Rightarrow \text{(IO-LIM) \& (local IOSS)}$$

$$\Rightarrow \text{(IO-LIM) \& (IO-GS)} \Rightarrow \text{(IO-AG) \& (IO-GS)}.$$

To complete the proof of Theorem 1, we will need the following result.

PROPOSITION 3.4. *(IO-AG)* & *(IO-GS)* $\Rightarrow$ *(IOSS).*

The proofs of the lemmas and the proposition will be given in section 6.

Remark 3.5. Theorem 1 is a satisfying theoretical result in that it unifies a number of properties and provides a generalization of the separation principle for asymptotically stable differential equations. Moreover, the theorem is a useful tool for recognizing IOSS systems. The definition of IOSS rarely lends itself to direct verification. More often, this property is shown using the Lyapunov characterization provided in [16]. In cases where construction of an appropriate Lyapunov function proves difficult, Theorem 1 provides a number of alternative conditions which may be tested.

As an example, consider the following family of systems without inputs, with state $(x, z) \in \mathbb{R}^n \times \mathbb{R}$:

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{z} &= |x|, \\ y &= z. \end{aligned} \tag{20}$$

Suppose that the system is forward complete. Provided that the function f is locally Lipschitz, this system satisfies the IO-LIM property, which can be shown as follows. For each initial condition ξ, consider two cases. If $\int_0^\infty |x(s,\xi)|\,ds = \infty$, then $\lim_{t\to\infty} y(t,\xi) = \infty$, and so the IO-LIM bound (8) holds trivially for any $\gamma_y \in \mathcal{K}_\infty$. Otherwise, from the fact that $\int_0^\infty |x(s,\xi)|\,ds < \infty$, we have $\liminf_{t\to\infty} |x(t,\xi)| = 0 \le \|y\|$. Thus, the IO-LIM estimate (8) holds in both cases.

Hence system (20) is known to satisfy the IOSS property (OSS in this case), provided that it satisfies one of the stability properties as in Theorem 1. For instance, if the system $\dot{x} = f(x)$ is locally stable, then the system (20) is (zero)-O-LS (since the z component of the state is trivially bounded by the output), and so the system enjoys the OSS property.

On the other hand, the Lyapunov approach would not be well suited to exploring the IOSS property in this situation, since little is assumed about the dynamics. □

4. On iISS. In this section, we indicate how the equivalences shown in Theorem 1 can be used to derive asymptotic characterizations of the iISS property.

DEFINITION 4.1 (see [26]). *A system as in* (1) *is* integral input-to-state stable *(iISS) if there exist functions $\beta \in \mathcal{KL}$, $\sigma \in \mathcal{K}$, and $\gamma \in \mathcal{K}$ such that, for all $\xi \in \mathbb{R}^n$ and all u, the solution $x(t,\xi,u)$ is defined for all $t \ge 0$, and*

$$|x(t,\xi,u)| \le \beta(|\xi|\,,t) + \gamma\left(\int_0^t \sigma(|u(s)|)\,ds\right) \tag{21}$$

for all $t \ge 0$.

To make use of our main result, we reformulate the iISS property in terms of the IOSS property.

LEMMA 4.2. *System* (1) *is integral input-to-state stable with an estimate as in* (21) *if and only if the augmented system*

$$\dot{x} = f(x,u), \quad \dot{e} = \sigma(|u|), \quad y = e \tag{22}$$

is IOSS.

Proof. Let the augmented system be IOSS; then, for $e(0) = 0$ we have

$$\begin{aligned} |x(t,\xi,u)| &\le |x(t,\xi,u)| + |e(t)| \le \beta(|\xi|,t) + \gamma_1(\|u\|) + \gamma_2(\|y_{[0,t]}\|) \\ &= \beta(|\xi|,t) + \gamma_1(\|u\|) + \gamma_2\left(\int_0^t \sigma(|u(s)|)\,ds\right) \end{aligned} \tag{23}$$

for all $t \in [0, T_{\xi,u})$. By causality, (23) can be rewritten as

$$|x(t,\xi,u)| \le \beta(|\xi|,t) + \gamma_1(\|u_{[0,t]}\|) + \gamma_2\left(\int_0^t \sigma(|u(s)|)\,ds\right). \tag{24}$$

Since on any finite interval, the integral term in (24) is finite, it follows that $T_{\xi,u} = \infty$. In turn, this implies that (24) holds on $[0,\infty)$. This estimate implies iISS for (1), by virtue of Theorem 1 in [6].

To see the converse, clearly, $|e(t)| = |y(t)| \le \|y_{[0,t]}\|$. Also observe that the iISS property implies that the augmented system is forward complete. Thus, it is enough to show that a suitable estimate holds on $[0,\infty)$ for the x component of the state. By (21), we have

$$\begin{aligned} |x(t,\xi,u)| &\le \beta(|\xi|,t) + \gamma\left(\int_0^t \sigma(|u(s)|)\,ds\right) \\ &= \beta(|\xi|,t) + \gamma(e(t) - e(0)) \\ &\le \beta(|\xi|,t) + \gamma(|y(t)| + |y(0)|) \le \beta(|\xi|,t) + \gamma(2\|y_{[0,t]}\|) \end{aligned} \tag{25}$$

for all $t \ge 0$. This completes the proof. □

4.1. Lyapunov characterizations of iISS. This IOSS formulation of the iISS property allows us to exploit known results on IOSS to develop new characterizations of iISS. For instance, the following new Lyapunov characterization for iISS follows directly from the Lyapunov characterization for IOSS presented in [16]. We prove the next two results under the assumption that the gain σ in (21) is locally Lipschitz. Remark 4.7 indicates how this can always be achieved through a simple modification.

THEOREM 2. *System* (1) *is iISS if and only if there exist functions* $\alpha_1, \alpha_2, \alpha$, γ_1,γ_2, σ *of class* $\mathcal{K}_\infty$ *and a smooth function* $V : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$, *with*

$$\alpha_1(|\xi| + |\eta|) \leq V(\xi, \eta) \leq \alpha_2(|\xi| + |\eta|) \qquad \forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}, \tag{26}$$

such that the following dissipation inequality is satisfied:

$$\frac{\partial V(\xi,\eta)}{\partial \xi} f(\xi, \mu) + \frac{\partial V(\xi,\eta)}{\partial \eta} \sigma(|\mu|) \leq -\alpha(|\xi| + |\eta|) + \gamma_1(|\eta|) + \gamma_2(|\mu|) \tag{27}$$

for all $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}$, $\mu \in \mathbb{U}$. □

Remark 4.3. It is easy to see that estimates (26) and (27) imply

$$\alpha_1(|\xi|) \leq V(\xi, \eta) \leq \alpha_2(|\xi| + |\eta|) \qquad \forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}, \tag{28}$$

and for all $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}$, $\mu \in \mathbb{U}$,

$$\frac{\partial V(\xi,\eta)}{\partial \xi} f(\xi, \mu) + \frac{\partial V(\xi,\eta)}{\partial \eta} \sigma(|\mu|) \leq -\alpha(|\xi|) + \gamma_1(|\eta|) + \gamma_2(|\mu|). \tag{29}$$

On the other hand, suppose that for a given V, equations (28) and (29) hold for some $\alpha_1, \alpha_2, \alpha$, γ_1, γ_2, σ of class $\mathcal{K}_\infty$; then one can again show that the following type of estimate holds for the x-component of (22):

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \rho_1(\|e\|_{[0,t]}) + \rho_2(\|u\|) \qquad \forall\, t \geq 0,$$

where ρ_1, $\rho_2 \in \mathcal{K}$. Combining this with the fact that $|e(t)| \leq \|e\|_{[0,t]}$, one sees that the augmented system (22) is IOSS. Hence, the corresponding system as in (1) is iISS.

Thus, a system as in (1) is iISS if and only if there exists some smooth function $V : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ for which (28) and (29) hold for some $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha, \gamma_1, \gamma_2$ and σ.

In [5], an equivalent Lyapunov characterization for iISS was formulated as in the following: for some $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$ and some *continuous positive definite* function α

$$\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|) \qquad \forall \xi \in \mathbb{R}^n, \tag{30}$$

$$\frac{\partial V(\xi)}{\partial \xi} f(\xi, \mu) \leq -\alpha(|\xi|) + \gamma(|\mu|). \tag{31}$$

The significance of the new Lyapunov characterization by (28)–(29) (or (26)–(27)) is that in (27) or (29) one can require the function α be of class $\mathcal{K}_\infty$. This may lead to some interesting applications in feedback design. For instance, if V is an iISS-Lyapunov function defined by (30)–(31), then, given a $\mathcal{K}_\infty$-function, $\rho \circ V$ may fail to be an iISS-Lyapunov function. However, if V is an iISS-Lyapunov function satisfying (28)–(29), then, for any $\rho \in \mathcal{K}_\infty$, $\rho \circ V$ is again an iISS-Lyapunov function satisfying the same type of estimates as in (28)–(29) (cf. [27, 2] regarding changing "supply rates"). □

4.2. Asymptotic characterizations of iSS.

DEFINITION 4.4. *A system as in* (1) *satisfies the* bounded energy weakly converging state *(BEWCS) property if for some* σ *of class* $\mathcal{K}_\infty$ *the following holds:*

$$\int_0^{+\infty} \sigma(|u(s)|)\,ds < +\infty \Rightarrow \liminf_{t\to+\infty} |x(t,\xi,u)| = 0. \tag{32}$$

To be more precise, (32) means that for any ξ and any u for which

$$\int_0^{\infty} \sigma(|u(s)|)\,ds < \infty,$$

it holds that $T_{\xi,u} = \infty$, and $\liminf_{t\to+\infty} |x(t,\xi,u)| = 0$.

DEFINITION 4.5. *A system as in* (1) *satisfies the* bounded energy frequently bounded state *(BEFBS) property if for some* σ *of class* $\mathcal{K}_\infty$ *the following holds:*

$$\int_0^{+\infty} \sigma(|u(s)|)\,ds < +\infty \;\Rightarrow\; \liminf_{t\to+\infty} |x(t,\xi,u)| < +\infty. \tag{33}$$

To be more precise, (33) *means that for any* ξ *and any* u *for which* $\int_0^\infty \sigma(|u(s)|)\,ds < \infty$*, it holds that* $T_{\xi,u} = \infty$ *and* $\liminf_{t\to+\infty} |x(t,\xi,u)| < \infty$.

Remark 4.6. Note that, given any input function u, any $T < \infty$, and any $\mathcal{K}_\infty$-function σ, one has $\int_0^T \sigma(|u(s)|) < \infty$. Hence, together with the causality of the trajectories, the BEFBS property implies the forward completeness property. That is, if a system is BEFBS, then the system is forward complete. Since the BEWCS property implies the BEFBS property, the BEWCS property also implies the forward completeness property. □

We say that a system as in (1) is *zero-GAS* if the corresponding zero-input system $\dot{x} = f(x,0)$ is globally asymptotically stable, and is *zero-LS* if the zero-input system is locally (neutrally) stable.

THEOREM 3. *The following properties are equivalent for system* (1) *with* $\mathbb{U} = \mathbb{R}^m$:

1. *iISS,*
2. *BEWCS and zero-LS,*
3. *BEFBS and zero-GAS.*

Proof. Implication 1 ⇒ 3 follows immediately from the definition of iISS. We show next 3 ⇒ 2 and 2 ⇒ 1.

[3 ⇒ 2]. Assume that the system (1) is zero-GAS and satisfies (33) for some $\sigma \in \mathcal{K}_\infty$. By the Lyapunov characterization of zero-GAS in [5, Lemma IV.10], there exists a smooth Lyapunov function $U(\xi)$ such that for some $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$,

$$\tilde{\alpha}_1(|\xi|) \le U(\xi) \le \tilde{\alpha}_2(|\xi|) \qquad \forall \xi \in \mathbb{R}^n,$$

and for some $\tilde{\alpha}, \gamma, \sigma_0 \in \mathcal{K}_\infty$, it holds that

$$\frac{\partial U(\xi)}{\partial \xi} f(\xi,\mu) \le -\tilde{\alpha}(|\xi|) + \gamma(|\xi|)\sigma_0(|\mu|) \qquad \forall \xi \in \mathbb{R}^n, \mu \in \mathbb{U}. \tag{34}$$

By Proposition II.5 in [5], there exists some smooth $\mathcal{K}$-function χ such that for the function V defined by $V = \chi \circ U$ it holds that

$$\frac{\partial V(\xi)}{\partial \xi} f(\xi,\mu) \le -\rho(|\xi|) + \sigma_1(|\mu|) \qquad \forall \xi \in \mathbb{R}^n, \mu \in \mathbb{U}, \tag{35}$$

for some $\mathcal{K}_\infty$ function σ_1 and some positive definite function ρ. Let $\tilde\sigma = \max\{\sigma_1, \sigma\}$, where σ is as in (33). Pick $\xi \in \mathbb{R}^n$ and u with $\int_0^\infty \tilde\sigma(|u(s)|)\,ds < +\infty$. By the BEFBS assumption,

$$m := \liminf_{t\to+\infty} |x(t,\xi,u)| < +\infty.$$

We want to show $m = 0$. For the sake of contradiction, assume $m > 0$. For any $r > 0$, let

$$w(r) := \max_{|\xi|\le r} V(\xi).$$

Claim. $w(3m) - w(2m) > 0$.

Proof. Suppose $w(3m) - w(2m) = 0$. Then there exists some ξ_0 with $|\xi_0| \le 2m$ at which V takes the maximum value $w(3m)$. Since ξ_0 is an interior point of the open ball centered at 0 with radius $3m$, it follows that $\frac{\partial}{\partial\xi}V(\xi_0) = 0$. This contradicts (35) applied with $\mu = 0$.

We let T be such that

$$\int_T^{+\infty} \tilde\sigma(|u(s)|)\,ds < w(3m) - w(2m).$$

By the definition of m, there exists $\tau \ge T$ such that $|x(\tau,\xi,u)| < 2m$. By virtue of (35), for all $t \ge \tau$

$$\begin{aligned} V(x(t,\xi,u)) - V(x(\tau,\xi,u)) &\le \int_\tau^t \sigma_1(|u(s)|)\,ds \\ &< \int_T^{+\infty} \tilde\sigma(|u(s)|)\,ds < w(3m) - w(2m). \end{aligned} \tag{36}$$

Hence $V(x(t,\xi,u)) < w(3m)$ for all $t \ge \tau$. This implies that

$$U(x(t,\xi,u)) \le \chi^{-1}(w(3m)) \quad \forall\, t \ge \tau$$

(note that $w(3m)$ is in the range of the $\mathcal{K}$-function χ). Hence, $x(t,\xi,u)$ stays bounded on $[\tau,\infty)$, and consequently, $x(t,\xi,u)$ is bounded on $[0,\infty)$. Let $M > 0$ be such that $|x(t,\xi,u)| < M$ for all t. By (34), one sees that

$$\frac{d}{dt}U(x(t)) \le -\tilde\alpha(|x(t)|) + \gamma(M)\sigma_1(|u(t)|),$$

where $x(t)$ denotes the considered trajectory $x(t,\xi,u)$. This is enough to conclude that for some $\beta \in \mathcal{KL}$ and some $\alpha \in \mathcal{K}_\infty$ it holds that

$$\alpha(|x(t,\xi,u)|) \le \beta(|\xi|,t) + \int_0^t \tilde\sigma(|u(s)|)\,ds,$$

and therefore, as shown in [26], $|x(t,\xi,u)| \to 0$. This implies $m = 0$, which is clearly a contradiction. ∎

[2 ⇒ 1]. Suppose that (1) satisfies the BEWCS property with an estimate as in (32). Consider the auxiliary system (22), where σ is the energy supply function as in (32). Note that this system is forward complete (cf. Remark 4.6). By (32), one sees that for any $\gamma_y \in \mathcal{K}_\infty$ it holds that

$$\liminf_{t\to\infty} |x(t,\xi,u)| \le \gamma_y(\|y\|).$$

Therefore, for any choice of γ_u and $\gamma_y \in \mathcal{K}_\infty$, the following asymptotic property is true:

$$\liminf_{t\to+\infty} |x(t,\xi,u)| \le \max\{\gamma_u(\|u\|), \gamma_y(\|y\|)\}. \tag{37}$$

Since $|e(t)| = |y(t)| \le \|y\|$ for all $t \ge 0$, system (22) satisfies the IO-LIM property. Also, it follows from the zero-LS and the BEWCS properties that the system (1) is zero-GAS. Hence, the corresponding augmented system (22) is zero-OSS. By applying the main result in section 3, we find that (22) is IOSS, and therefore, by virtue of Lemma 4.2, system (1) is iISS. □

Remark 4.7. Notice that, in the construction of the augmented system (22), the only requirement on the function σ is that it must be class $\mathcal{K}_\infty$. In order to apply the results for IOSS systems, though, the local Lipschitz condition of the dynamics is needed. This issue can be dealt with as follows. First of all, one may always choose a function $\hat\sigma \in \mathcal{K}_\infty$ that majorizes σ so that $\hat\sigma^{-1}$ is locally Lipschitz. Then the iISS, the BEWCS, or the BEFBS estimates as in (21), (32), or (33), respectively, are not violated by replacing σ by $\hat\sigma$. Consider the change of input variables by $u \doteq \frac{w}{|w|}\hat\sigma^{-1}(|w|)$. Clearly $\hat\sigma(|u|) = |w|$, and therefore, iISS, BEWCS, or BEFBS of (1) with $\hat\sigma$ as the input energy supply function is easily seen to be equivalent to iISS, BEWCS, or BEFBS, respectively, for

$$\dot x = f(x, w\sigma^{-1}(|w|)/|w|) \tag{38}$$

with the new supply function $(s) = |s|$. The augmented system corresponding to (38) will therefore be

$$\dot x = f(x, w\sigma^{-1}(|w|)/|w|), \quad \dot e = |w|, \quad y = e. \tag{39}$$

The proofs of the previously derived results, when applied to system (39) and (38), provide proofs of the results in Theorem 3 for the original system (1). Therefore the Lipschitz condition on σ can be dropped. □

5. Uniform detectability. In this section we develop some machinery which will be needed to prove Theorem 1. We will derive a separation principle for the property we call *uniform OSS*. Here we consider systems with inputs (unlike in the definition of OSS), but we think of those inputs not as additive disturbances but as multiplicative time-varying uncertainties. We restrict the possible values of these inputs by considering systems as in (1) with $u \in \mathcal{M}_\mathcal{O}$ for some $\mathcal{O} \subset \mathbb{U}$. Throughout this section, we assume that $\mathcal{O}$ is compact.

DEFINITION 5.1. *For a system as in* (1), *the* global detectability property *holds if the following hold:*

- *there exist* $\bar\sigma_1$, $\bar\sigma_2 \in \mathcal{K}$ *so that*

$$|x(t,\xi,u)| \le \max\{\bar\sigma_1(|\xi|), \bar\sigma_2(\|y_{[0,t]}\|)\} \quad \forall\, \xi \in \mathbb{R}^n,\ \forall\, u,\ \forall\, t \in [0, T_{\xi,u}); \tag{40}$$

- *there exists* $\gamma \in \mathcal{K}_\infty$ *so that*

$$\limsup_{t\to T_{\xi,u}} |x(t,\xi,u)| \le \gamma(\|y\|) \tag{41}$$

for all $\xi \in \mathbb{R}^n$, *all* $u \in \mathcal{M}_\mathcal{O}$. □

Note that the last condition (41) is just the IO-AG condition as in (6) with $\gamma_u = 0$. It can be seen that if a system is globally detectable, then the system satisfies the UO property.

DEFINITION 5.2. *We say that the* uniform output-to-state stability property *holds for* (1) *with* $u \in \mathcal{M}_{\mathcal{O}}$ *if for some* $\tilde{\gamma} \in \mathcal{K}_\infty$ *and some* $\beta \in \mathcal{KL}$ *the system satisfies*

$$|x(t, \xi, u)| \le \max\{\beta(|\xi|, t), \tilde{\gamma}(\|y_{[0,t]}\|)\} \tag{42}$$

for all $\xi \in \mathbb{R}^n$*, all* $u \in \mathcal{M}_{\mathcal{O}}$*, and all* $t \in [0, T_{\xi,u})$. □

Clearly, the uniform output-to-state stability property implies the global detectability property. The main result of this section says that the converse is true as well when $\mathcal{O}$ is compact.

THEOREM 4. *Consider a system as in* (1) *with* $\mathcal{O}$ *compact. The system is* globally detectable *if and only if it is* uniformly output-to-state stable.

To prove Theorem 4, we will need the following result.

LEMMA 5.3. *Consider system* (1) *with* $u \in \mathcal{M}_{\mathcal{O}}$ *for some compact set* $\mathcal{O}$*. Assume that the system* (1) *satisfies the global detectability property. Then there exists some* $\gamma_1 \in \mathcal{K}$ *such that for all* $\varepsilon > 0$ *and all* $r > 0$ *there exists* $T_{\varepsilon,r}$ *so that for any* $\xi \in \mathbb{R}^n$ *with* $|\xi| \le r$ *and for any* $u \in \mathcal{M}_{\mathcal{O}}$*, if* $T_{\varepsilon,r} < T_{\xi,u}$*, then*

$$|x(t, \xi, u)| \le \max\left\{\varepsilon,\ \gamma_1\left(\|y_{[0,t]}\|\right)\right\}$$

for all $t \in [T_{\varepsilon,r}\,,\ T_{\xi,u})$.

Proof. Suppose a system satisfies the global detectability property for $u \in \mathcal{M}_{\mathcal{O}}$ as in (40) and (41). Let $\rho(r) := \max\{\bar{\sigma}_2(r), \gamma(r)\}$. It can be seen that, with $z = x$, $w = y$, and $\hat{z}(t) = \max\{|z(t)| - \rho(|w(t)|),\ 0\}$, the system is globally error-detectable as defined in [3] (see also Appendix B of this paper). Hence, by Lemma B.6, the system is uniformly globally error-detectable. Combining this with Remark B.5, we have proved Lemma 5.3. □

Modifying $T_{\varepsilon,r}$ if necessary, we may restate Lemma 5.3 as in the following.

COROLLARY 5.4. *Consider system* (1) *with* $u \in \mathcal{M}_{\mathcal{O}}$ *for some compact set* $\mathcal{O}$*. Assume that the system* (1) *satisfies the global detectability property. Then there exists a continuous map* $T : \mathbb{R}_{>0} \times \mathbb{R}_{\ge 0}$ *with the properties that for each* $r > 0$*,* $T(\cdot, r)$ *is decreasing, and for each* $\varepsilon > 0$*,* $T(\varepsilon, \cdot)$ *is increasing, so that for any* $\varepsilon > 0$*,* $|\xi| \le r$*, and any* $u \in \mathcal{M}_{\mathcal{O}}$*, the following holds:*

$$T(\varepsilon, r) < T_{\xi,u} \ \Rightarrow\ \Big\{|x(t, \xi, u)| \le \max\left\{\varepsilon,\ \gamma_1\left(\|y_{[0,t]}\|\right)\right\} \quad \forall\, t \in [T(\varepsilon, r)\,,\ T_{\xi,u})\Big\}. \tag{43}$$

The following lemma on $\mathcal{KL}$ functions will also be needed. This fact is proved in [21] and is stated as Lemma 4.1 in [1]. (That reference requires φ to take nonnegative values, but this can always be assumed without loss of generality, simply replacing φ by $\max\{\varphi, 0\}$.)

PROPOSITION 5.5. *If a function* $\varphi : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ *satisfies*

- *for all* $r > 0$*,* $\varepsilon > 0$*, there exists some* $T = T(\varepsilon, r) > 0$ *so that* $\varphi(s, t) < \varepsilon$ *for all* $s \le r$ *and* $t \ge T$*;*
- *for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *so that* $\varphi(s, t) \le \varepsilon$ *for all* $s \le \delta$ *and all* $t \ge 0$*,*

then there exists some $\beta \in \mathcal{KL}$ *so that* $\varphi(s, t) \le \beta(s, t)$ *for all* $s \ge 0$*,* $t \ge 0$. □

We are now ready to prove Theorem 4.

Proof of Theorem 4. Suppose that the system (1) with $u \in \mathcal{M}_{\mathcal{O}}$ is globally detectable. Then estimate (40) holds for some $\bar{\sigma}_1, \bar{\sigma}_2 \in \mathcal{K}$, and there exist some

$\gamma_1 \in \mathcal{K}$ and some $T(\cdot,\cdot)$ as in Corollary 5.4 such that (43) holds. Without loss of generality, we assume that $\gamma_1 = \bar{\sigma}_2$. For each $s \geq 0$, $t \geq 0$, set

$$\varphi(s,t) := \sup\left\{|x(t,\xi,u)| - \gamma_1(\|y_{[0,t]}\|) : \ |\xi| \leq s,\ u \in \mathcal{M}_{\mathcal{O}},\ t < T_{\xi,u}\right\}$$

(with the convention that $\sup\emptyset = -\infty$).

It follows from definitions of $T(\cdot,\cdot)$ and φ that for any $r > 0$ and any $\varepsilon > 0$, $\varphi(s,t) < \varepsilon$ for all $s \leq r$ and $t \geq T(\varepsilon, r)$. By (40), $\varphi(s,t) \leq \bar{\sigma}_1(|\xi|)$ for all $s \geq 0$ and $t \geq 0$. Hence, φ satisfies both conditions as in Proposition 5.5, and thus, there exists some $\beta \in \mathcal{KL}$ so that $\varphi(s,t) \leq \beta(s,t)$ for all $s \geq 0$, $t \geq 0$. Combining this with the definition of φ, we get

$$|x(t,\xi,u)| \leq \beta(|\xi|, t) + \gamma_1(\|y_{[0,t]}\|)$$

for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{M}_{\mathcal{O}}$, and all $t < T_{\xi,u}$. □

6. Proofs of the technical lemmas. To prove Lemmas 3.1–3.3, we need the following result.

DEFINITION 6.1. *A system as in* (1) *satisfies the* input-output local stability property *(IO-LS) if there exist $\delta > 0$ and $\mathcal{K}$ functions $\alpha_0, \alpha_u, \alpha_y \in \mathcal{K}$ so that for all ξ, all $u(\cdot)$, and all $t \in [0, T_{\xi,u})$ it holds that*

$$\max\{|\xi|, \|u\|, \|y_{[0,t]}\|\} \leq \delta \Rightarrow |x(t,\xi,u)| \leq \max\{\alpha_0(|\xi|), \alpha_u(\|u\|), \alpha_y(\|y_{[0,t]}\|)\}. \tag{44}$$

LEMMA 6.2. *(IO-LS) & (IO-BND) ⇔ (IO-GS).*

Proof. The implication (IO-GS) ⇒ (IO-LS) & (IO-BND) is obvious; therefore, we only show the converse. Let δ, α_0, α_u, α_y, σ_0, σ_u, σ_y be given as in (44) and (7). Pick a constant $c > 0$ and three class-$\mathcal{K}$ functions $\beta_\star$ such that for each $\star \in \{0, u, y\}$, $\sigma_\star(s) \leq \beta_\star(s) + c$ for all $s > 0$. Pick class-$\mathcal{K}$ functions $\gamma_\star$ so that for each $\star \in \{0, u, y\}$ it holds that

$$\gamma_\star(s) \geq \begin{cases} \max\{\alpha_\star(s), 3\beta_\star(s)\} & \forall\, 0 \leq s \leq \delta, \\ 3[\beta_\star(s) + 3c] & \forall s \geq \delta. \end{cases} \tag{45}$$

Consider any ξ and $u(\cdot)$. Then (IO-GS) holds with gains γ_0, γ_u, and γ_y. Indeed, if $|\xi| \leq \delta$, $\|u\| \leq \delta$, and $\|y_{[0,t]}\| \leq \delta$, this follows by (IO-LS). If instead $|\xi| > \delta$, the definition of (IO-BND) implies

$$\begin{aligned} |x(t,\xi,u)| &\leq \sigma_0(|\xi|) + \sigma_u(\|u\|) + \sigma_y(\|y_{[0,t]}\|) \\ &\leq \beta_0(|\xi|) + \beta_u(\|u\|) + \beta_y(\|y_{[0,t]}\|) + 3c \\ &\leq [\beta_0(|\xi|) + 3c] + [\gamma_u(\|u\|) + \gamma_y(\|y_{[0,t]}\|)]/3 \\ &\leq (1/3)[\gamma_0(|\xi|) + \gamma_u(\|u\|) + \gamma_y(\|y_{[0,t]}\|)] \\ &\leq \max\{\gamma_0(|\xi|), \gamma_u(\|u\|), \gamma_y(\|y_{[0,t]}\|)\}. \end{aligned} \tag{46}$$

The case $\|u\| > \delta$ can be treated in a similar way. Finally, we are left to deal with the case $\delta < \|y_{[0,t]}\| < +\infty$. (The case $\|y\| = +\infty$ is trivial.) By (UO), $x(t,\xi,u)$ is well defined, and therefore the argument as in (46) can be repeated. □

Remark 6.3. Observe that in the proof of Lemma 6.2, if the functions α_u and σ_u as in (44) and (7) can be chosen zero, and if the $\|u\|$ term is not presented in the $\max\{\cdots\} \leq \delta$ phrase in (44), then the function γ_u in (46) can also be chosen zero. That is, one has the following.

If for some $\alpha_0,\ \alpha_y \in \mathcal{K}$, $\sigma_0,\ \sigma_y \in \mathcal{N}$ it holds that

$$\max\{|\xi|, \|y_{[0,t]}\|\} \le \delta \Rightarrow |x(t,\xi,u)| \le \max\{\alpha_0(|\xi|),\ \alpha_y(\|y_{[0,t]}\|)\}, \tag{47}$$

and

$$|x(t,\xi,u)| \le \max\{\sigma_0(|\xi|),\ \sigma_y(\|y_{[0,t]}\|)\} \qquad \forall\,\xi \in \mathbb{R}^n,\ \forall\,u,\ \forall\,t \in [0,T_{\xi,u}), \tag{48}$$

then for some $\tilde{\sigma}_0, \tilde{\sigma}_y \in \mathcal{K}$ it holds that

$$|x(t,\xi,u)| \le \max\{\tilde{\sigma}_0(|\xi|),\ \tilde{\sigma}_y(\|y_{[0,t]}\|)\} \qquad \forall\,\xi \in \mathbb{R}^n,\ \forall\,u,\ \forall\,t \in [0,T_{\xi,u}). \tag{49}$$

6.1. Proof of Lemma 3.3. Let $\gamma_u, \gamma_y, \sigma_0, \sigma_u, \sigma_y \in \mathcal{K}$ be as in (5) and (7). Pick any ξ, u and any $\varepsilon > 0$. By (5), there is some $T \in [0, T_{\xi,u})$ such that

$$|x(T,\xi,u)| \le \max\{\gamma_u(\|u\|),\ \gamma_y(\|y\|)\} + \varepsilon.$$

Applying (7) to the initial state $\xi_T := x(T,\xi,u)$ and the control $u_T(t) := u(t+T)$ (whose corresponding output function is $y_T(t,\xi_T,u_T) = y(t+T,\xi,u)$), we conclude that

$$\begin{aligned}\sup_{t\ge T}|x(t,\xi,u)| = \sup_{0\le t<T_{\xi_T,u_T}} |x(t,\xi_T,u_T)| &\le \max\{\sigma_0(|\xi_T|), \sigma_u(\|u_T\|), \sigma_y(\|y\|)\}\\ &\le \max\left\{\sigma_0(\gamma_u(\|u\|)+\varepsilon), \sigma_0(\gamma_y(\|y\|)+\varepsilon), \sigma_u(\|u\|), \sigma_y(\|y\|)\right\}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\limsup_{t\to T_{\xi,u}} |x(t,\xi,u)| \le \max\{\hat{\gamma}_u(\|u\|),\ \hat{\gamma}_y(\|y\|)\}, \tag{50}$$

where $\hat{\gamma}_u(s) = \max\{\sigma_0(\gamma_u(s)), \sigma_u(s)\}$, $\hat{\gamma}_y(s) = \max\{\sigma_0(\gamma_y(s)), \sigma_y(s)\}$. ∎

Remark 6.4. We remark that if γ_u and σ_u as in (5) and (7) can be chosen to be the zero function, then the gain function $\hat{\gamma}_u$ in (50) can also be chosen to be the zero function. ∎

6.2. Proof of Lemma 3.1. Consider the zero-input system

$$\dot{x} = f(x,0), \quad y = h(x). \tag{51}$$

Denote the trajectory corresponding to an initial state ξ by $x(t,\xi)$. This trajectory is defined on a maximal interval $[0, T_\xi)$. Applying Lemma 2.7 and Remark 2.8 to the zero-input system, we know that there exists some $\sigma_0,\ \sigma_y \in \mathcal{N}$ such that for every trajectory of the system the following holds:

$$|x(t,\xi)| \le \max\{\sigma_0(|\xi|),\ \sigma_y(\|y_{[0,t]}\|)\} \qquad \forall\,t \in [0,T_\xi). \tag{52}$$

By Remark 6.3, one sees that, together with the zero-O-LS property (as defined in (15)), (52) implies that the system is locally detectable, that is, for some $\tilde{\sigma}_0, \tilde{\sigma}_y \in \mathcal{K}$, (49) holds.

Applying Remark 6.4 to (51) with the IO-LIM estimate (5) with $\gamma_u = 0$ and (49), one sees that the following holds for every trajectory $x(t,\xi)$ of (51):

$$\limsup_{t\to T_\xi} |x(t,\xi)| \le \gamma\left(\|y_{[0,T_\xi)}\|\right). \tag{53}$$

The combination of (52) and (53) means that the system (51) is globally detectable. By Theorem 4, the system (51) is OSS. ∎

6.3. Proof of Lemma 3.2. Clearly (local-IOSS) $\Rightarrow$ (IO-LS). Moreover, by Lemma 2.7, (IO-LIM) $\Rightarrow$ (IO-BND). Combining this with Lemma 6.2, we get

$$\text{(local IOSS) \& (IO-LIM)} \;\Rightarrow \text{(IO-LS) \& (IO-BND)} \Leftrightarrow \text{(IO-GS)}, \tag{54}$$

which completes the proof. □

6.4. Proof of Proposition 3.4. In order to complete the proof, the following result will be needed.

LEMMA 6.5. *Suppose that system* (1) *satisfies the IO-AG and IO-GS properties. Then there exists some locally Lipschitz* $\varphi \in \mathcal{K}_\infty$ *such that the system*

$$\dot{x}(t) = f(x(t), d(t)\varphi(|x(t)|)), \qquad y(t) = h(x(t)), \tag{55}$$

where d *denotes the disturbance functions taking values in the closed unit ball of* $\mathbb{R}^m$*, is globally detectable.*

Proof. Let σ_0, σ_u, $\sigma_y \in \mathcal{K}$ be such that the IO-GS estimate (7) holds, and let γ_u, $\gamma_y \in \mathcal{K}$ be such that the IO-AG estimate (6) holds for the system. Without loss of generality, we may assume that $\sigma_u = \gamma_u$ and $\sigma_y = \gamma_y$. Pick any locally Lipschitz $\mathcal{K}_\infty$-function φ such that $\sigma_u(\varphi(s)) \leq s/2$ for all $s \geq 0$. Below we show that with the function φ, the corresponding system (55) is globally detectable. For any $\xi \in \mathbb{R}^n$ and any d, we let $x_\varphi(t, \xi, d)$ denote the corresponding solution for (55), $y_\varphi = h(x_\varphi)$, and let $[0, \tau_{\xi,d})$ denote the maximal interval of the solution. Observe that for any ξ, d it holds that

$$x_\varphi(t, \xi, d) = x(t, \xi, u_d) \qquad \forall\, 0 \leq t < \tau_{\xi,d},$$

where $u_d(t) = d(t)\varphi(|x_\varphi(t, \xi, d)|)$. Consequently,

$$\begin{aligned} |x_\varphi(t, \xi, d)| &\leq \max\left\{\sigma_0(|\xi|),\ \sigma_u\left(\varphi\left(\|x_\varphi(\cdot, \xi, d)_{[0,t)}\|\right)\right),\ \sigma_y(\|(y_\varphi)_{[0,t)}\|)\right\} \\ &\leq \max\left\{\sigma_0(|\xi|),\ \frac{\|x_\varphi(\cdot, \xi, d)_{[0,t)}\|}{2},\ \sigma_y(\|(y_\varphi)_{[0,t)}\|)\right\} \qquad \forall\, 0 \leq t < \tau_{\xi,d}\,. \end{aligned}$$

Thus, for any $0 \leq t < \tau_{\xi,d}$,

$$\|x_\varphi(\cdot, \xi, d)_{[0,t)}\| \leq \max\left\{\sigma_0(|\xi|),\ \frac{\|x_\varphi(\cdot, \xi, d)_{[0,t)}\|}{2},\ \sigma_y(\|(y_\varphi)_{[0,t)}\|)\right\}.$$

Consequently, for any $0 \leq t < \tau_{\xi,d}$,

$$\|x_\varphi(\cdot, \xi, d)_{[0,t)}\| \leq \max\left\{\sigma_0(|\xi|),\ \sigma_y(\|(y_\varphi)_{[0,t)}\|)\right\},$$

and in particular,

$$|x_\varphi(t, \xi, d)| \leq \max\left\{\sigma_0(|\xi|),\ \sigma_y\left(\|(y_\varphi)_{[0,t)}\|\right)\right\} \tag{56}$$

for all $0 \leq t < \tau_{\xi,d}$. This shows that property (40) holds. Below we show that the attractivity property as in (41) holds for the system (55). By (56), the system (55) satisfies the UO property. Pick any ξ, d. Suppose $\tau_{\xi,d} = \infty$ and $\|y\| < \infty$. Then, again, by (56),

$$\limsup_{t\to\infty} |x_\varphi(t, \xi, d)| < \infty. \tag{57}$$

By the IO-AG property (6), with Lemma 2.10,

$$\begin{aligned}\limsup_{t\to\infty}|x_\varphi(t,\xi,d)| &\le \max\left\{\gamma_u\left(\limsup_{t\to\infty}|u_d(t)|\right),\ \gamma_y\left(\limsup_{t\to\infty}|y_\varphi(t)|\right)\right\}\\ &\le \max\left\{\gamma_u\left(\varphi\left(\limsup_{t\to\infty}|x_\varphi(t,\xi,d)|\right)\right),\ \gamma_y\left(\limsup_{t\to\infty}|y_\varphi(t)|\right)\right\}\\ &\le \max\left\{\frac{1}{2}\limsup_{t\to\infty}|x_\varphi(t,\xi,d)|\,,\ \gamma_y\left(\limsup_{t\to\infty}|y_\varphi(t)|\right)\right\}.\end{aligned}$$

Combining this with (57), we get

$$\limsup_{t\to\infty}|x_\varphi(t,\xi,d)| \le \gamma_y\left(\limsup_{t\to\infty}|y_\varphi(t)|\right).$$

Again, with the UO property, we see that the attractivity property (41) holds for the system (55). Thus, the system is globally detectable. ∎

Proof of Proposition 3.4. Suppose that a system as in (1) satisfies the IO-AG and the IO-GS properties. By Lemma 6.5, there exists some locally Lipschitz $\varphi \in \mathcal{K}_\infty$ such that the corresponding system (55) is globally detectable. By Theorem 4, the system (55) is uniformly output-to-state stable. Applying Theorem 2.16 together with Corollary 3.6 of [16], one sees that the system (1) is IOSS. (Following the work in [16], if a system (1) admits a locally Lipschitz $\varphi \in \mathcal{K}_\infty$ such that the corresponding system (55) is uniformly OSS, then the system (1) is called robustly OSS.) ∎

Appendix A. Characterizations of IO-LIM. Consider the following variations of the IO-LIM property.

Definition A.1. *For system* (1)*, we say that*

- *the asymptotic IO-LIM property holds if for some* $\gamma_u, \gamma_y \in \mathcal{K}$*,*

$$\liminf_{t\to T_{\xi,u}}|x(t,\xi,u)| \le \limsup_{t\to T_{\xi,u}}\max\{\gamma_u(|u(t)|),\gamma_y(|y(t)|)\} \tag{58}$$

for all $\xi \in \mathbb{R}^n$*, all* $u(\cdot)$*;*

- *the causal IO-LIM property holds if for some* $\gamma_u, \gamma_y \in \mathcal{K}$*,*

$$\inf_{t\in[0,T_{\xi,u})}\left\{|x(t,\xi,u)| - \max\{\gamma_u(\|u_{[0,t]}\|),\gamma_y(\|y_{[0,t]}\|)\}\right\} \le 0 \tag{59}$$

for all $\xi \in \mathbb{R}^n$*, all* $u(\cdot)$*;*

- *the asymptotic causal IO-LIM property holds if*

$$\liminf_{t\to T_{\xi,u}}\left\{|x(t,\xi,u)| - \max\{\gamma_u(\|u_{[0,t]}\|),\gamma_y(\|y_{[0,t]}\|)\}\right\} \le 0 \tag{60}$$

for all $\xi \in \mathbb{R}^n$*, all* $u(\cdot)$*.* ∎

Proposition A.2. *For a system as in* (1)*, the following properties are equivalent:*

1. *IO-LIM,*
2. *asymptotic IO-LIM,*
3. *causal IO-LIM,*
4. *asymptotic causal IO-LIM.* ∎

Proof. [1 ⇒ 2]: Suppose the IO-LIM property holds for system (1) with $\gamma_u, \gamma_y \in \mathcal{K}_\infty$ as in (5). Fix ξ, u. Pick any $T \in [0, T_{\xi,u})$. Let $\xi_T = x(T,\xi,u)$ and $u_T(t) = u(t+T)$. Applying (5) to ξ_T with u_T, we get

$$\inf_{t\ge T}|x(t,\xi,u)| = \inf_{t\ge 0}|x(t,\xi_T,u_T)| \le \max\left\{\gamma_u(\|u_{[T,T_{\xi,u})}\|),\ \gamma_y(\|y_{[T,T_{\xi,u})}\|)\right\}.$$

This implies that

$$\liminf_{t\to T_{\xi,u}} |x(t,\xi,u)| \le \lim_{T\to T_{\xi,u}} \left\{\max\left\{\gamma_u(\|u_{[T,T_{\xi,u})}\|),\ \gamma_y(\|y_{[T,T_{\xi,u})}\|)\right\}\right\}.$$

With (2), we get (58).

[2 ⇒ 3]: Assume that (58) holds for some $\gamma_u, \gamma_y \in \mathcal{K}$. Below we show that this will imply (59) with the same γ_u, γ_y functions. Suppose that this fails for some trajectory $x(t,\xi,u)$. It then follows that

$$|x(t,\xi,u)| \ge \max\{\gamma_u(\|u_{[0,t]}\|),\, \gamma_y(\|y_{[0,t]}\|)\} \quad \forall\, t \in [0, T_{\xi,u}).$$

Taking the lim inf on both sides of the inequality, one gets

$$\begin{aligned}\liminf_{t\to T_{\xi,u}} |x(t,\xi,u)| &\ge \liminf_{t\to T_{\xi,u}} \max\{\gamma_u(\|u_{[0,t]}\|),\, \gamma_y(\|y_{[0,t]}\|)\}\\ &= \max\{\gamma_u(\|u_{[0,T_{\xi,u})}\|),\, \gamma_y(\|y_{[0,T_{\xi,u})}\|)\},\end{aligned}$$

which contradicts (58).

[3 ⇒ 4]: The proof of this portion is trivial.

[4 ⇒ 1]: An estimate as in (60) in particular implies that

$$\liminf_{t\to T_{\xi,u}} \Big[|x(t,\xi,u)| - \max\{\gamma_u(\|u\|),\ \gamma_y(\|y\|)\}\Big] \le 0,$$

which is the same as (8). By Remark 2.5, IO-LIM holds. □

Appendix B. The error detectability properties. In some of the proofs in this paper, we have used some results from the recent work [3]. They are stated here, in the interest of making this paper self-contained. The first statement is a special case of Corollary 4.3 in [3].

LEMMA B.1. *For a given unboundedness observable system of the form* (1), *assume*

- *a compact subset $\mathcal{O}$ of the input set $\mathbb{U}$;*
- *subsets K, C, and Ω of the state space $\mathbb{R}^n$ such that K and C are compact, Ω is open, and $K \subset \Omega \subset C$;*
- *subsets $\mathcal{Y}$ and $\mathcal{Y}_0$ of the output space $\mathbb{R}^p$ such that $\mathcal{Y}$ is compact, $\mathcal{Y}_0$ is open, and $\mathcal{Y} \subset \mathcal{Y}_0$*

such that, for all $\xi \in C$ and all inputs $u \in \mathcal{M}_{\mathcal{O}}$, there exists $t \in [0, T_{\xi,u})$ for which

$$x(t,\xi,u) \in K \quad \text{or} \quad h(x(t,\xi,u)) \notin \mathcal{Y}_0. \tag{61}$$

Then $\mathcal{R}_{\mathcal{O}/\mathcal{Y}}(C)$ is bounded. □

Considered a system as in the following:

$$\dot{x}(t) = f(x(t),u(t)),\ z(t) = \varphi(x(t)),\ w(t) = k(x(t)), \tag{62}$$

where $\varphi(\cdot), k(\cdot)$ are continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively, for some k and l, and where f satisfies the same assumptions as were made for system (1). Following the work in [3], we call z the output signal and w the measurement signal. The inputs are functions in $\mathcal{M}_{\mathcal{O}}$ for some $\mathcal{O} \subset \mathbb{U}$.

DEFINITION B.2. *The system* (62) *is said to satisfy the* unboundedness observability property through w *(UO through w) if, for each ξ and u such that $T_{\xi,u} < \infty$, it holds that*

$$\limsup_{t\to T_{\xi,u}} |w(t)| = \infty.$$

The following definitions were proposed in [3].

DEFINITION B.3. *We say that system* (62) *is* globally error-detectable *if there exists some* $\gamma \in \mathcal{K}$ *such that for the map* $\hat{z}(\cdot)$ *defined by*

$$\hat{z}(t) = \max\{|z(t)| - \gamma(|w(t)|),\ 0\} \tag{63}$$

the following hold:

- *(Local uniform output-stability modulo measurements): for some* $\sigma_1, \sigma_2 \in \mathcal{K}$ *and some* $\delta > 0$*, it holds that*

  $$|\hat{z}(0)| < \delta \Longrightarrow |\hat{z}(t)| \leq \sigma_1(|\hat{z}(0)|) + \sigma\left(\|w\|_{[0,t]}\right) \qquad \forall\, 0 \leq t < T_{\xi,u} \tag{64}$$

  for all ξ *and* $u \in \mathcal{M}_{\mathcal{O}}$.
- *(Asymptotic-detectablity):* $\inf_{0 \leq t < T_{\xi,u}} |\hat{z}(t)| = 0$ *for all* ξ *and* $u \in \mathcal{M}_{\mathcal{O}}$. □

DEFINITION B.4. *We say that system* (1) *is* uniformly globally error-detectable *if there exists some* $\gamma \in \mathcal{K}$ *such that for the map* $\hat{z}(\cdot)$ *defined by* (63) *the following hold:*

- *the local uniform output-stability modulo measurements property as in Definition* B.3 *holds, and*
- *(uniform asy-detectability): for any* $\varepsilon > 0$ *and any* $\kappa > 0$*, there exists* $T_{\varepsilon,\kappa}$ *so that for any* $\xi \in \mathbb{R}^n$ *with* $|\xi| \leq \kappa$ *and for any* u *there exists some* $\tau < \min\{T_{\varepsilon,\kappa},\ T_{\xi,u}\}$ *such that*

  $$|\hat{z}(\tau)| \leq \varepsilon. \tag{65}$$

Remark B.5. The uniformly global error-detectability implies the following property: there exists some $\gamma_1 \in \mathcal{K}$ such that for all $\varepsilon > 0$ and all $\kappa > 0$ there exists $T_{\varepsilon,\kappa}$ so that for any $\xi \in \mathbb{R}^n$ with $|\xi| \leq \kappa$ and for any u, if $T_{\varepsilon,\kappa} < T_{\xi,u}$, then

$$|z(t)| \leq \varepsilon + \gamma_1\left(\|w\|_{[0,t)}\right)$$

for all $t \in [T_{\varepsilon,\kappa}\,,\ T_{\xi,u})$. □

A main result in [3] is the following.

LEMMA B.6. *Let* $\mathcal{O}$ *be compact. Assume that system* (62) *satisfies the UO property through the measurement* w*. Then the system is globally error-detectable in* $u \in \mathcal{M}_{\mathcal{O}}$ *if and only if it is uniformly globally error-detectable in* $u \in \mathcal{M}_{\mathcal{O}}$. □

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