OPEN STOCHASTIC SYSTEMS
&
THEIR INTERCONNECTION

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SontagFest

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In honor of Eduardo Sontag
on the occasion of his 60-th birthday.
When & where & how we first met
Outline

- Motivation
- Definitions
- Interconnection
- [Variable sharing versus input/output]
- [Identification]
- Conclusions
Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

- probability distributions, probability density functions;
- means that the event $\sigma$-algebra consists of the Borel sets.

$\sim$ ‘Every’ subset of $\mathbb{R}^n$ is assigned a probability.
Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

- probability distributions, probability density functions;
- means that the event $\sigma$-algebra consists of the Borel sets.

$\sim$ ‘Every’ subset of $\mathbb{R}^n$ is assigned a probability.

**Thesis:**

This is unduly restrictive, even for elementary applications.
Motivating examples
Noisy resistor

\[ V = RI + \epsilon \]

\( \epsilon \) gaussian

zero mean

variance \( \sigma \sim \sqrt{RT} \)

‘Johnson-Nyquist resistor’
What is \[
\frac{V}{I}
\] as a mathematical object?

\[ V = RI + \varepsilon \]

\( \varepsilon \) gaussian
zero mean
variance \( \sigma \sim \sqrt{RT} \)

'Johnson-Nyquist resistor'
How do we deal with interconnection?
Deterministic price/demand/supply

- Demand curve: decreasing function of price
- Supply curve: increasing function of price
Stochastic price/demand/supply

Only certain regions of the \([\text{price demand}]\) and \([\text{price supply}]\) planes are assigned a probability.
Only certain regions of the [demand:price] and [supply:price] planes are assigned a probability.

How do we deal with equilibrium supply = demand?
Formal definitions
Definition

A *stochastic system* is a probability triple \((\mathbb{W}, \mathcal{E}, P)\)

- \(\mathbb{W}\) a non-empty set, the *outcome space*,
- \(\mathcal{E}\) a \(\sigma\)-algebra of subsets of \(\mathbb{W}\): the *events*,
- \(P : \mathcal{E} \to [0, 1]\) a *probability measure*.

\(\mathcal{E}\): the subsets that are assigned a probability. Probability that outcomes \(\in E, E \in \mathcal{E}\), is \(P(E)\).

Model \(\cong \mathcal{E} \text{ and } P\); \(\mathcal{E}\) is an essential part.
Definition

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‘Classical’ stochastic system:

\(\mathcal{W} = \mathbb{R}^n\) and \(\mathcal{E}\) = the Borel subsets of \(\mathbb{R}^n\).

\(\mathcal{E}\) is inherited from the topology on \(\mathbb{R}^n\).

\(P\) can then be specified by a probability distribution.
\[ V = RI + \varepsilon: \text{stoch. system, } \mathbb{W} = \mathbb{R}^2, \text{outcomes } \left[ \begin{array}{c} V \\ I \end{array} \right]. \]

Events: \( \left\{ \left[ \begin{array}{c} V \\ I \end{array} \right] \in \mathbb{R}^2 \mid V - RI \in A \text{ with } A \text{ a Borel subset of } \mathbb{R} \right\}. \)

\( P(\text{event}) = \text{gaussian measure of } A. \)

\( V \) and \( I \) are not classical real random variables.
\( \mathcal{E} = \) the regions that are assigned a probability.

\( p, d, \) and \( s \) are not classical real random variables.
Linearity

linear \Leftrightarrow \text{Borel probability on } \mathbb{R}^n / \mathbb{L}, \mathbb{L} \text{ linear, ‘fiber’}.
Linearity

linear :⇔ Borel probability on \( \mathbb{R}^n / \mathbb{L} \), \( \mathbb{L} \) linear, ‘fiber’.

Borel probability on \( \mathbb{M} \cong \mathbb{R}^n / \mathbb{L} \).

gaussian :⇔ linear, Borel probability gaussian.

Classical \( \Rightarrow \) linear.
Deterministic

\((\mathcal{W}, \mathcal{E}, P)\) is said to be \textit{deterministic} if

\[\mathcal{E} = \{\emptyset, \mathcal{B}, \mathcal{B}^{\text{complement}}, \mathcal{W}\}\] and \(P(\mathcal{B}) = 1\).

If \(\mathcal{B} = \mathcal{W}\), the variables are said to be \textit{free}.

noisy resistor: linear, gaussian, fiber \(V = RI\).
\(w = V - RI\) is a classical random variable.
\(V\) and \(I\) are free.
Only statements \(P(\{V \in \mathbb{R}\}) = 1\), \(P(\{I \in \mathbb{R}\}) = 1\).
\([V, I]\) no pdf, no cumulative, no conditional distr’ions.
Interconnection
Interconnection
Can we impose two distinct probabilistic laws on the same set of variables?
Complementarity

$\Sigma_1 = (\mathbb{W}, \mathcal{E}_1, P_1)$ and $\Sigma_2 = (\mathbb{W}, \mathcal{E}_2, P_2)$ are said to be complementary $\iff$ for $E_1, E'_1 \in \mathcal{E}_1$ and $E_2, E'_2 \in \mathcal{E}_2$:

$$[E_1 \cap E_2 = E'_1 \cap E'_2] \Rightarrow [P_1(E_1)P_2(E_2) = P_1(E'_1)P_2(E'_2)].$$
Interconnection of complementary systems

Let $\Sigma_1 = (\mathcal{W}, \mathcal{E}_1, P_1)$ and $\Sigma_2 = (\mathcal{W}, \mathcal{E}_2, P_2)$ be complementary stochastic systems (assumed stochastically independent). Their \textit{interconnection} is

$$(\mathcal{W}, \mathcal{E}, P)$$

with $\mathcal{E} := \sigma$-algebra generated by the ‘rectangles’

$$\{E_1 \cap E_2 \mid E_1 \in \mathcal{E}_1, E_2 \in \mathcal{E}_2\},$$

and $P$ defined through the rectangles by

$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).$$
Noisy resistor terminated by voltage source
Noisy resistor terminated by voltage source

\[ P(E) = P_1(E_1)P_2(E_2) \]
Equilibrium price/demand/supply

\[ P(E) = P_1(E_1)P_2(E_2). \]
Open stochastic systems
$\Sigma_1 = (\mathbb{R}^n, \mathcal{E}_1, P_1)$.

If $\mathcal{E}_1$ is the Borel $\sigma$-algebra, and $\text{support}(P_1) = \mathbb{R}^n$, then $\Sigma_1$ interconnectable only with the free system $\Sigma_2 = (\mathbb{R}^n, \mathcal{E}_2, P_2)$, $\mathcal{E}_2 = \{\emptyset, \mathbb{R}^n\}$.

$\implies$ classical = ‘closed’ system.
Open versus closed

\[ \Sigma_1 = (\mathbb{R}^n, \mathcal{E}_1, P_1). \]

If \( \mathcal{E}_1 = \) the Borel \( \sigma \)-algebra, and \( \text{support}(P_1) = \mathbb{R}^n \), then \( \Sigma_1 \) interconnectable only with the free system \( \Sigma_2 = (\mathbb{R}^n, \mathcal{E}_2, P_2), \mathcal{E}_2 = \{\emptyset, \mathbb{R}^n\}. \)

\[ \Rightarrow \text{classical} = \text{‘closed’ system}. \]

Parsimonious \( \mathcal{E}_1 \)

\[ \Rightarrow \Sigma_1 \text{ is interconnectable}. \]

\[ \Rightarrow \text{‘open’ system}. \]
Interconnection ⇔ variable sharing
Variable sharing

\[ w_1 = w_2 \]
Output-to-input assignment

\[ u_1 = y_2, \quad u_2 = y_1 \]
Resistor interconnection

\[ V_1 = V_2, \quad I_1 = I_2 \]
Resistor interconnection

\[ V_1 = V_2, \quad I_1 = I_2 \]
Price/demand/supply interconnection

\[ p_1 = p_2, \quad d = s \]
Price/demand/supply interconnection

\[ p_1 = p_2, \quad d = s \]
Price/demand/supply interconnection

$p_1 = p_2, \quad d = s$

Marketing

Manufacturing

price

demand

price

supply
Identification
Data collection requires observing a stochastic system *in interaction with an environment*. Is it possible to disentangle the laws of a system from the laws of the environment?
Measurements

Data collection requires observing a stochastic system *in interaction with an environment*.

*Is it possible to disentangle the laws of a system from the laws of the environment?*

In engineering, it may be possible to set the experimental conditions. In economics and the social sciences (and biology?), data often gathered passively ‘*in vivo*’. 
Disentangling

Can $R$ and $\sigma$ be deduced by sampling $(V, I)$?
Can the price/demand characteristic be deduced by sampling \((p, d)\) in equilibrium?
Let $\Sigma_1$ and $\Sigma_2$ be complementary gaussian systems and assume that the interconnection $\Sigma_1 \wedge \Sigma_2$ is a classical random system.

Sampling $\sim$ the mean and covariance of $\Sigma_1 \wedge \Sigma_2$. 
Let \( \Sigma_1 \) and \( \Sigma_2 \) be complementary gaussian systems and assume that the interconnection \( \Sigma_1 \land \Sigma_2 \) is a classical random system.

Sampling \( \sim \) the mean and covariance of \( \Sigma_1 \land \Sigma_2 \).

Given the fiber of \( \Sigma_1 \) or \( \Sigma_2 \), all the other parameters of \( \Sigma_1 \) and \( \Sigma_2 \) can be deduced from \( \Sigma_1 \land \Sigma_2 \).

The fiber of \( \Sigma_1 \) or \( \Sigma_2 \) can be chosen freely.
Linearized gaussian price/demand/supply

Identifiability provided one of the fibers is known.

Sampling alone does not give the elasticities.
The Borel $\sigma$-algebra is inadequate even for elementary applications.
The Borel $\sigma$-algebra is inadequate even for elementary applications.

Complementary stochastic systems can be interconnected:

- two distinct laws imposed on one set of variables.

Open stochastic systems require a parsimonious $\sigma$-algebra.

Classical stochastic systems are closed systems.
Measurements are the result of interaction with an environment. Modeling from data requires disentanglement. The data alone are insufficient for identifiability.
Happy birthday, Eduardo!
Ad multos annos felices!

Copies of the lecture frames available from/at
http://www.esat.kuleuven.be/~jwillems