SOLUTIONS—ASSIGNMENT 9

Chapter 6: P 1. The sample space is $S = \{(m, n) : m, n \text{ integers } 1, 2, \ldots, 6\}$, so $|S| = 36$.

(a) Here $X(m, n) = \max\{m, n\}$ and $Y(m, n) = m + n$. The joint range of (X, Y) is all pairs of integers (x, y) with $x+1 \le y \le 2x$ and $1 \le x \le 6$, $2 \le y \le 12$. If $y = 2x$, then only one point $(x, x) \in S$ gives this value, so $p(x, 2x) = 1/36$. If $x < y < 2x$ then the points $(x, y-x)$ and $(y-x, x)$ in S both give this value for (X, Y). So $p(x, y) = 2/36$ in this case. Here is a table:

$X\backslash Y$	2	3	4	5	6	7	8	9	10	11	12
1	1/36										
2		1/18	1/36								
3			1/18	1/18	1/36						
4				1/18	1/18	1/18	1/36				
5					1/18	1/18	1/18	1/18	1/36		
6						1/18	1/18	1/18	1/18	1/18	1/36

(b) Now $X(m, n) = m$ and $Y(m, n) = \max\{m, n\}$. The joint range of (X, Y) is all pairs of integers (x, y) with $x \le y$ and $x, y = 1, \ldots, 6$. For $x = y$, the points $(x, 1), (x, 2), \ldots, (x, x)$ of S give this value for (X, Y), so $p(x, x) = x/36$. For $x < y$, the only point of S giving this value is (x, y). So $p(x, y) = 1/36$. See the table below.

(c) Now $X(m, n) = \min\{m, n\}$ and $Y(m, n) = \max\{m, n\}$. The joint range of (X, Y) is the same as in (b). If $x = y$, then the only point of S giving this value for (X, Y) is (x, y). So $p(x, y) = 1/36$. If $x < y$ then the points (x, y) and (y, x) of S both give this value of (X, Y). Here is the table:

$Y\backslash X$	1	2	3	4	5	6
1	1/36					
2	1/36	2/36				
3	1/36	1/36	3/36			
4	1/36	1/36	1/36	4/36		
5	1/36	1/36	1/36	1/36	5/36	
6	1/36	1/36	1/36	1/36	1/36	6/36

Table for 1(b)

$Y\backslash X$	1	2	3	4	5	6
1	1/36					
2	2/36	1/36				
3	2/36	2/36	1/36			
4	2/36	2/36	2/36	1/36		
5	2/36	2/36	2/36	2/36	1/36	
6	2/36	2/36	2/36	2/36	2/36	1/36

Table for 1(c)

6. Think of testing all 5 transistors in order. There are $\binom{5}{2}$ ways to choose the two positions in the testing order for the defective transistors, and all of these choices are equally likely; each choice corresponds to a possible value of N_1 and N_2, with N_1 the position of the first defective in the testing order and $N_1 + N_2$ the position of the second. These numbers must satisfy $1 \le N_1 < N_1 + N_2 \le 5$ or $1 \le N_1 \le 4$, $1 \le N_2 \le 5 - N_1$. All $\binom{5}{2}$ pairs (N_1, N_2) satisfying these inequalities are equally likely; each has probability $1/\binom{5}{2} = 1/10$.

8. (a) As usual, we use the principle that the total probability is 1:

$$1 = \iint f(x, y)\,dx\,dy = c\int_0^\infty \int_{-y}^{y} (y^2 - x^2)e^{-y}\,dx\,dy$$
$$= c\int_0^\infty e^{-y}\left[y^2x - x^3/3\right]_{-y}^{y} dy = \frac{4c}{3}\int_0^\infty y^3e^{-y}dy = 8c\ .$$

So $c = 1/8$.

(b) Since $Y > 0$, $f_Y(y) = 0$ for $y < 0$. For $y \ge 0$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \frac{1}{8}\int_{-y}^{y}(y^2 - x^2)e^{-y}\,dx = \frac{1}{6}y^3e^{-y}\ .$$

For any x, $f(x,y) = 0$ unless $y > |x|$. So

$$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}(y^2 - x^2)e^{-y}\,dy = \frac{1}{8}\Big[(-y^2 - 2y - 2 + x^2)e^{-y}\Big]_{|x|}^{\infty} = \frac{1}{4}(|x|+1)e^{-|x|}\ .$$

(c) Since f_X is even ($f_X(x) = f_X(-x)$), $E[X] = 0$ by symmetry:

$$E[X] = \int_{-\infty}^{0} xf(x)\,dx + \int_0^{\infty} xf(x)\,dx = \int_{\infty}^{0}(-y)f(-y)\,(-dy) + \int_0^{\infty} xf(x)\,dx$$
$$= -\int_0^{\infty} yf(y)\,dy + \int_0^{\infty} xf(x)\,dx.$$

where we have made the substitution $y = -x$.

9. (a) $f(x,y)$ is clearly non-negative. The other requirement for a density is that it be normalized, which we can check:

$$\frac{6}{7}\int_0^1\int_0^2\left(x^2 + \frac{xy}{2}\right)\,dy\,dx = \frac{6}{7}\int_0^1(2x^2 + x)\,dx = \frac{6}{7}\left(\frac{2}{3} + \frac{1}{2}\right) = 1.$$

(b) For $0 \le x \le 1$, $f_X(x) = \frac{6}{7}\int_0^2\left(x^2 + \frac{xy}{2}\right)\,dy = \frac{6}{7}(2x^2 + x)$.

(c)

$$P\{X > Y\} = \frac{6}{7}\int_0^1\int_0^x\left(x^2 + \frac{xy}{2}\right)\,dy\,dx = \frac{6}{7}\int_0^1\frac{5x^3}{4}\,dx = \frac{15}{56}.$$

(d)

$$P\{X < \frac{1}{2}\} = \int_0^{1/2} f_X(x) = \frac{6}{7}\Big[\frac{2x^3}{3} + \frac{x^2}{2}\Big]_0^{1/2} = \frac{5}{28}.$$

So

$$P\Big(\{Y > \frac{1}{2}\}\Big|\{X < \frac{1}{2}\}\Big) = \frac{28}{5}P\Big(\{Y > \frac{1}{2}\}\text{ and }\{X < \frac{1}{2}\}\Big)$$
$$= \frac{28}{5}\frac{6}{7}\int_0^{1/2}\int_{1/2}^2(x^2 + \frac{xy}{2})\,dy\,dx$$
$$= \frac{24}{5}\int_0^{1/2}\left(\frac{3x^2}{2} + \frac{15x}{16}\right)\,dx$$
$$= \frac{24}{5}\left(\frac{1}{16} + \frac{15}{128}\right) = \frac{69}{80}.$$

(d)

$$E[X] = \int_{\infty}^{\infty} xf_X(x)\,dx = \frac{6}{7}\int_0^1 x(2x^2 + x)\,dx = \frac{5}{7}.$$

(e)

$$f_Y(y) = \frac{6}{7}\int_0^1\left(x^2 + \frac{xy}{2}\right)\,dx = \frac{2}{7} + \frac{3y}{14}; \qquad E[Y] = \int_0^2 y\left(\frac{2}{7} + \frac{3y}{14}\right)\,dy = \frac{8}{7}$$

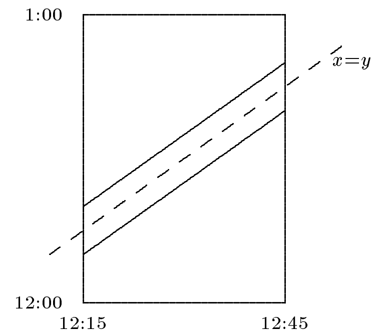
10. (a) $P\{X < Y\} = \int_0^{\infty}\int_0^y e^{-(x+y)}\,dx\,dy = \int_0^{\infty} e^{-y}[1 - e^{-y}]\,dy = 1 - 1/2 = 1/2$ (this could be seen without calculation by symmetry in X and Y).

(b) Since $f(x,y) = e^{-x}\cdot e^{-y}$, X and Y are independent and each has exponential distribution with parameter $\lambda = 1$. So $P\{X < a\} = 1 - e^{-a}$.

13. If X and Y are the arrival times of the man and woman, respectively, in minutes after noon, then (X,Y) is uniformly distributed on the rectangle $15 \le x \le 45$, $0 \le y \le 60$. We can thus determine probabilities geometrically, as ratios of areas.

(a) The set in which the early arriver must wait at most 5 minutes, $\{|x-y| \le 5\}$, is a parallelogram of altitude 30 and base 10 (shown in the figure), so

$$P\{|X-Y| \le 5\} = (10 \times 30)/(30 \times 60) = 1/6.$$



(b) It seems clear from the symmetry between the man and the woman that we should have $P\{X<Y\} = 1/2$, and this is easily verified by computing the area of the trapezoid lying below the dashed line in the figure.

15. (a) Since $\int\int_R 1\,dx\,dy = (\text{area of } R)$, we have $\int\int_R \frac{1}{(\text{area of } R)}\,dx\,dy = 1$.

(b) Here R has area 4 so the joint density is $f(x,y)=1/4$ if $-1 \le x,y \le 1$, $f(x,y)=0$ otherwise. Thus

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} (1/4)\int_{-1}^{1} dy = 1/2, & \text{if } -1 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

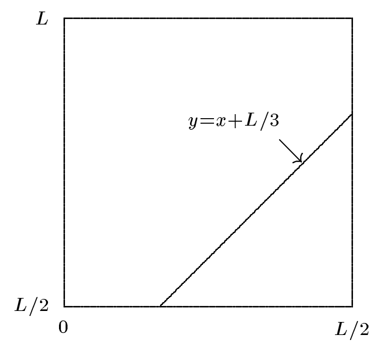
Clearly X is uniform on $(-1,1)$. Similarly, Y is uniform on $(-1,1)$. Moreover, X and Y are independent because $f(x,y) = f_X(x)f_Y(y)$.

(c) Let A be the region inside the circle of radius 1, centered at the origin. Then

$$P\{X^2+Y^2 \le 1\} = P\{(X,Y) \in A\} = \frac{|A|}{4} = \pi/4.$$

18. X and Y are uniformly distributed on the square $0 \le X \le L/2$, $L/2 \le Y \le L$, so we compute probabilities as ratios of areas. The set where $Y - X > L/3$ is the complement of a triangle with base and height $L/3$, so

$$P\{Y > X + L/3\} = 1 - \frac{(L/3)^2(1/2)}{(L/2)^2} = \frac{7}{9}.$$



See the Figure.

20. (a) Integrating with respect to y yields $f_X(x) = xe^{-x}$ for $x \ge 0$, and similarly $f_Y(y) = e^{-y}$ for $y \ge 0$; f_X and f_Y are zero if their arguments are negative. Since $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$, X and Y are independent.

(b) Here

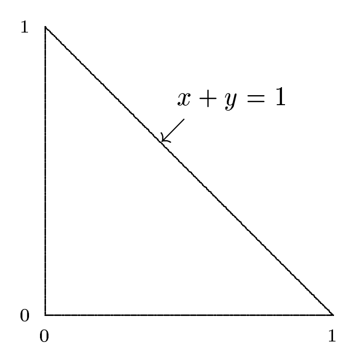
$$f_X(x) = \begin{cases} 2(1-x) & 0<x<1, \\ 0 & \text{otherwise;} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & 0<y<1, \\ 0 & \text{otherwise,} \end{cases}$$

so X and Y are not independent.

21. (a) Clearly $f(x,y) \ge 0$; we check normalization:

$$\int_0^1 \int_0^{1-x} 24xy\,dy\,dx = 12\int_0^1 x(1-x)^2\,dx = 1.$$



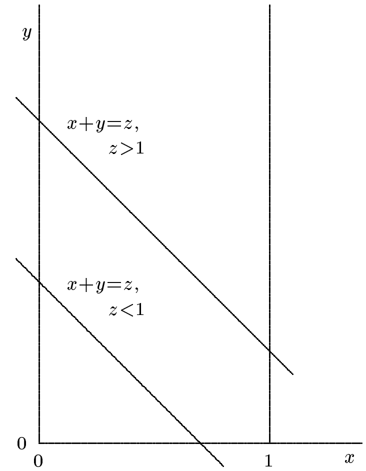
(b) $f_X(x) = \int_0^{1-x} 24xy\,dy = 12x(1-x)^2$, $0 \le x \le 1$;

$E[X] = \int_0^1 xf(x)\,dx = 2/5.$

(c) Just as in (b) (or by symmetry): $E[Y] = 2/5$.

25. For each individual, the probability that he or she arrives in the first hour is 10^{-6}. Thus the number N of people that arrive in the first hour is binomial with parameters $n = 10^6$, $p = 10^{-6}$. This is well approximated by a Poisson distribution with $\lambda = np = 1$, so $P(N = i) \approx e^{-1}/i!$.

27. By independence, X and Y have joint density



$$f(x,y) = f_X(x)f_Y(y) = \begin{cases} e^{-y} & \text{if } 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) If $Z = X + Y$, then

$$F_Z(z) = P\{X + Y \le z\} = P\{Y \le z - X\}$$
$$= \iint_{y \le z-x} f(x,y)\, dy\, dx.$$

Setting up the limits on this integral depends on whether $z > 1$ or $z < 1$. If $z < 1$,

$$F_Z(z) = \int_0^z \int_0^{z-x} e^{-y}\, dy\, dx = \int_0^z (1 - e^{x-z})\, dx = z - 1 + e^{-z},$$

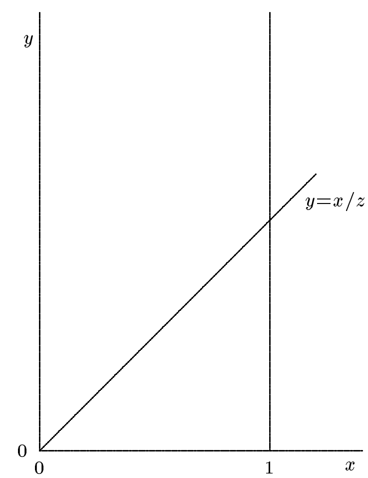
while if $z > 1$,

$$F_Z(z) = \int_0^1 \int_0^{z-x} e^{-y}\, dy\, dx = \int_0^1 (1 - e^{x-z})\, dx = 1 - e^{1-z} + e^{-z}.$$

Thus

$$f_Z(z) = \begin{cases} 1 - e^{-z}, & \text{for } 0 < z < 1; \\ e^{-(z-1)} - e^{-z} & \text{for } z \ge 1. \end{cases}$$

(b) If $Z = X/Y$, then



$$F_Z(z) = P\{Y > X/z\} = \int_0^1 \int_{x/z}^{\infty} e^{-y}\, dy\, dx$$
$$= \int_0^1 (e^{-x/z})\, dx = z\left[1 - e^{-1/z}\right].$$

So

$$f_Z(z) = \frac{d}{dz} P(Z \le z)$$
$$= \begin{cases} (1 - e^{-1/z}) + \dfrac{e^{-1/z}}{z}, & \text{for } 0 < z, \\ 0, & \text{otherwise.} \end{cases}$$