Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate $P$, and a domain $D$, we define the truth set of $P$ to be the set of elements $x$ in $D$ for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

**Example 23**

What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$", $Q(x)$ is "$x^2 = 2$", and $R(x)$ is "$|x| = x$".

**Solution:** The truth set of $P$, $\{x \in \mathbb{Z} \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers $x$, we see that the truth set of $P$ is the set $\{-1, 1\}$.

The truth set of $Q$, $\{x \in \mathbb{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers $x$ for which $x^2 = 2$.

The truth set of $R$, $\{x \in \mathbb{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of $R$ is $\mathbb{N}$, the set of nonnegative integers.

Note that $\forall x P(x)$ is true over the domain $U$ if and only if the truth set of $P$ is the set $U$. Likewise, $\exists x P(x)$ is true over the domain $U$ if and only if the truth set of $P$ is nonempty.

**Exercises**

1. List the members of these sets.
   a) $\{x \mid x$ is a real number such that $x^2 = 1\}$
   b) $\{x \mid x$ is a positive integer less than 12\}
   c) $\{x \mid x$ is the square of an integer and $x < 100\}$
   d) $\{x \mid x$ is an integer such that $x^2 = 2\}$

2. Use set builder notation to give a description of each of these sets.
   a) $\{0, 3, 6, 9, 12\}$
   b) $\{-3, -2, -1, 0, 1, 2, 3\}$
   c) $\{m, n, o, p\}$

3. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
   a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi
   b) the set of people who speak English, the set of people who speak Chinese
   c) the set of flying squirrels, the set of living creatures that can fly

4. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
   a) the set of people who speak English, the set of people who speak English with an Australian accent
   b) the set of fruits, the set of citrus fruits
   c) the set of students studying discrete mathematics, the set of students studying data structures

5. Determine whether each of these pairs of sets are equal.
   a) $\{1, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$
   b) $\{\{1\}, \{1\}, \{1\}\}$
   c) $\emptyset$, $\emptyset$

6. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.

7. For each of the following sets, determine whether 2 is an element of that set.
   a) $\{x \in \mathbb{R} \mid x$ is an integer greater than 1\}
   b) $\{x \in \mathbb{R} \mid x$ is the square of an integer\}
   c) $\{2, \{2\}\}$
   d) $\{\{2\}, \{\{2\}\}\}$
   e) $\{\{2\}, \{2, \{2\}\}\}$
   f) $\{\{2\}\}$

8. For each of the sets in Exercise 7, determine whether $\{2\}$ is an element of that set.

9. Determine whether each of these statements is true or false.
   a) $0 \in \emptyset$
   b) $\emptyset \in \{0\}$
   c) $\{0\} \subset \emptyset$
   d) $\emptyset \subset \{0\}$
   e) $\{0\} \in \{0\}$
   f) $\emptyset \in \emptyset$
   g) $\{\emptyset\} \subset \{\emptyset, \{0\}\}$

10. Determine whether these statements are true or false.
    a) $\emptyset \notin \{\emptyset\}$
    b) $\emptyset \in \emptyset$
    c) $\{\emptyset\} \notin \{\emptyset\}$
    d) $\emptyset \in \{\emptyset\}$
    e) $\{\emptyset\} \subset \emptyset$
    f) $\{\emptyset\} \subset \{\emptyset\}$
    g) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

11. Determine whether each of these statements is true or false.
    a) $x \in \{x\}$
    b) $\{y\} \subseteq \{x\}$
    c) $\{x\} \subseteq \{x\}$
    d) $\{x\} \subseteq \{x\}$
    e) $\emptyset \subseteq \{x\}$
    f) $\emptyset \in \{x\}$

12. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.
13. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all months of the year.

14. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

15. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

16. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

17. Suppose that $A$, $B$, and $C$ are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

18. Find two sets $A$ and $B$ such that $A \in B$ and $A \subseteq B$.

19. What is the cardinality of each of these sets?
   - a) $\{a\}$
   - b) $\{\{a\}\}$
   - c) $\{a, \{a\}\}$
   - d) $\{a, \{a\}, \{\{a\}\}\}$

20. What is the cardinality of each of these sets?
   - a) $\emptyset$
   - b) $\{\emptyset\}$
   - c) $\{\emptyset, \{\emptyset\}\}$
   - d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

21. Find the power set of each of these sets, where $a$ and $b$ are distinct elements.
   - a) $\{a\}$
   - b) $\{a, b\}$
   - c) $\{\emptyset, \{\emptyset\}\}$

22. Can you conclude that $A = B$ if $A$ and $B$ are two sets with the same power set?

23. How many elements does each of these sets have where $a$ and $b$ are distinct elements?
   - a) $P(\{a, b, \{a, b\}\})$
   - b) $P(\{\emptyset, a, \{a\}\})$
   - c) $P(P(\emptyset))$

24. Determine whether each of these sets is the power set of a set, where $a$ and $b$ are distinct elements.
   - a) $\emptyset$
   - b) $\{\emptyset, \{a\}\}$
   - c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$
   - d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

25. Prove that $P(A) \subseteq P(B)$ if and only if $A \subseteq B$.

26. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

27. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find
   - a) $A \times B$.
   - b) $B \times A$.

28. What is the Cartesian product $A \times B$, where $A$ is the set of courses offered by the mathematics department at a university and $B$ is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

29. What is the Cartesian product $A \times B \times C$, where $A$ is the set of all airlines and $B$ and $C$ are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

30. Suppose that $A \times B = \emptyset$, where $A$ and $B$ are sets. What can you conclude?

31. Let $A$ be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

32. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find
   - a) $A \times B \times C$.
   - b) $C \times B \times A$.
   - c) $C \times A \times B$.
   - d) $B \times B \times B$.

33. Find $A^2$ if
   - a) $A = \{0, 1, 3\}$.
   - b) $A = \{1, 2, a, b\}$.

34. Find $A^3$ if
   - a) $A = \{a\}$.
   - b) $A = \{0, a\}$.

35. How many different elements does $A \times B$ have if $A$ has $m$ elements and $B$ has $n$ elements?

36. How many different elements does $A \times B \times C$ have if $A$ has $m$ elements, $B$ has $n$ elements, and $C$ has $p$ elements?

37. How many different elements does $A^n$ have when $A$ has $m$ elements and $n$ is a positive integer?

38. Show that $A \times B \neq B \times A$, when $A$ and $B$ are nonempty, unless $A = B$.

39. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

40. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

41. Translate each of these quantifications into English and determine its truth value.
   - a) $\forall x \in R \ (x^2 \neq -1)$
   - b) $\exists x \in \mathbb{Z} \ (x^2 = 2)$
   - c) $\forall x \in \mathbb{Z} \ (x^2 > 0)$
   - d) $\exists x \in R \ (x^2 = x)$

42. Translate each of these quantifications into English and determine its truth value.
   - a) $\exists x \in R \ (x^2 = -1)$
   - b) $\exists x \in \mathbb{Z} \ (x + 1 > x)$
   - c) $\forall x \in \mathbb{Z} \ (x - 1 \in \mathbb{Z})$
   - d) $\forall x \in \mathbb{Z} \ (x^2 \in \mathbb{Z})$

43. Find the truth set of each of these predicates where the domain is the set of integers.
   - a) $P(x) : x^2 < 3$
   - b) $Q(x) : x^2 > x$
   - c) $R(x) : 2x + 1 = 0$

44. Find the truth set of each of these predicates where the domain is the set of integers.
   - a) $P(x) : x^3 \geq 1$
   - b) $Q(x) : x^2 = 2$
   - c) $R(x) : x < x^2$

45. The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair $(a, b)$ to be $\{(a), \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [Hint: First show that $\{(a), \{a, b\}\} = \{(c), \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

46. *46. This exercise presents Russell’s paradox. Let $S$ be the set that contains a set $x$ if the set $x$ does not belong to itself, so that $S = \{x | x \notin x\}$.

   a) Show the assumption that $S$ is a member of $S$ leads to a contradiction.
   b) Show the assumption that $S$ is not a member of $S$ leads to a contradiction.

   By parts (a) and (b) it follows that the set $S$ cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

47. Describe a procedure for listing all the subsets of a finite set.
Exercises

1. Let $A$ be the set of students who live within one mile of school and let $B$ be the set of students who walk to classes. Describe the students in each of these sets.
   a) $A \cap B$
   b) $A \cup B$
   c) $A - B$
   d) $B - A$

2. Suppose that $A$ is the set of sophomores at your school and $B$ is the set of students in discrete mathematics at your school. Express each of these sets in terms of $A$ and $B$.
   a) the set of sophomores taking discrete mathematics in your school
   b) the set of sophomores at your school who are not taking discrete mathematics
   c) the set of students at your school who either are sophomores or are taking discrete mathematics
   d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B - A$

4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find
   a) $A \cup B$
   b) $A \cap B$
   c) $A - B$
   d) $B - A$

In Exercises 5–10 assume that $A$ is a subset of some underlying universal set $U$.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.

6. Prove the identity laws in Table 1 by showing that
   a) $A \cup \emptyset = A$
   b) $A \cap \emptyset = A$

7. Prove the domination laws in Table 1 by showing that
   a) $A \cup U = U$
   b) $A \cap \emptyset = \emptyset$

8. Prove the idempotent laws in Table 1 by showing that
   a) $A \cap A = A$
   b) $A \cup A = A$

9. Prove the complement laws in Table 1 by showing that
   a) $A \cap \overline{A} = \emptyset$
   b) $A \cap \overline{A} = \emptyset$

10. Show that
    a) $A - \emptyset = A$
    b) $\emptyset - A = \emptyset$

11. Let $A$ and $B$ be sets. Prove the commutative laws from Table 1 by showing that
    a) $A \cup B = B \cup A$
    b) $A \cap B = B \cap A$

12. Prove the first absorption law from Table 1 by showing that if $A$ and $B$ are sets, then $A \cup (A \cap B) = A$.

13. Prove the second absorption law from Table 1 by showing that if $A$ and $B$ are sets, then $A \cap (A \cup B) = A$.

14. Find the sets $A$ and $B$ if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.

15. Prove the second De Morgan law in Table 1 by showing that if $A$ and $B$ are sets, then $A \cup B = \overline{\overline{A \cap B}}$
    a) by showing each side is a subset of the other side.
    b) using a membership table.

16. Let $A$ and $B$ be sets. Show that
    a) $(A \cap B) \subseteq A$
    b) $A \subseteq (A \cup B)$
    c) $A - B \subseteq A$
    d) $A \cap (B - A) = \emptyset$
    e) $A \cup (B - A) = A \cup B$

17. Show that if $A$, $B$, and $C$ are sets, then $A \cap B \cap C = A \cup B \cup C$
    a) by showing each side is a subset of the other side.
    b) using a membership table.

18. Let $A$, $B$, and $C$ be sets. Show that
    a) $(A \cup B) \subseteq (A \cup B \cup C)$
    b) $(A \cap B \cap C) \subseteq (A \cap B)$
    c) $(A - B) - C \subseteq A - C$
    d) $(A - C) \cap (C - B) = \emptyset$
    e) $(B - A) \cup (C - A) = (B \cup C) - A$

19. Show that if $A$ and $B$ are sets, then
    a) $A - B = A \cap B$
    b) $(A \cap B) \cap (A \cap B) = A$

20. Show that if $A$ and $B$ are sets with $A \subseteq B$, then
    a) $A \cup B = B$
    b) $A \cap B = A$

21. Prove the first associative law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.

22. Prove the second associative law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.

23. Prove the first distributive law from Table 1 by showing that if $A$, $B$, and $C$ are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

24. Let $A$, $B$, and $C$ be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

25. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find
    a) $A \cap B \cap C$
    b) $A \cup B \cup C$
    c) $(A \cup B) \cap C$
    d) $(A \cap B) \cup C$

26. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, and $C$.
    a) $A \cup (B \cap C)$
    b) $A \cap B \cap C$
    c) $(A - B) \cap (A - C) \cap (B - C)$

27. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, and $C$.
    a) $(A \cup B) \cap (A \cup C)$
    b) $(A \cap B) \cap (A \cap C)$
    c) $(A \cap B) \cap (A \cap C)$

28. Draw the Venn diagrams for each of these combinations of the sets $A$, $B$, $C$, and $D$.
    a) $(A \cup B) \cup (C \cup D)$
    b) $(A \cap B) \cup (C \cap D)$
    c) $(A \cap B) \cup (C \cap D)$

29. What can you say about the sets $A$ and $B$ if we know that
    a) $A \cup B = A$
    b) $A \cap B = A$
    c) $A - B = A$
    d) $A \cap B = B \cap A$
    e) $A - B = B - A$
30. Can you conclude that $A = B$ if $A$, $B$, and $C$ are sets such that
   a) $A \cup C = B \cup C$?
   b) $A \cap C = B \cap C$?
   c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

31. Let $A$ and $B$ be subsets of a universal set $U$. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

The symmetric difference of $A$ and $B$, denoted by $A \mathbin{\triangle} B$, is the set containing those elements in either $A$ or $B$, but not in both $A$ and $B$.

32. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

33. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

34. Draw a Venn diagram for the symmetric difference of the sets $A$ and $B$.

35. Show that $A \triangle B = (A \cup B) - (A \cap B)$.

36. Show that $A \triangle B = (A - B) \cup (B - A)$.

37. Show that if $A$ is a subset of a universal set $U$, then
   a) $A \cup A = \emptyset$.
   b) $A \cup \emptyset = A$.
   c) $A \cup U = U$.
   d) $A \cup \overline{A} = U$.

38. Show that if $A$ and $B$ are sets, then
   a) $A \triangle B = B \triangle A$.
   b) $(A \triangle B) \triangle B = A$.

39. What can you say about the sets $A$ and $B$ if $A \triangle B = A$?

40. Determine whether the symmetric difference is associative; that is, if $A$, $B$, and $C$ are sets, does it follow that $A \triangle (B \triangle C) = (A \triangle B) \triangle C$?

41. Suppose that $A$, $B$, and $C$ are sets such that $A \triangle C = B \triangle C$. Must it be the case that $A = B$?

42. If $A$, $B$, $C$, and $D$ are sets, does it follow that $(A \triangle B) \triangle (C \triangle D) = (A \triangle C) \triangle (B \triangle D)$?

43. If $A$, $B$, $C$, and $D$ are sets, does it follow that $(A \triangle B) \triangle (C \triangle D) = (A \triangle C) \triangle (B \triangle D)$?

44. Show that if $A$ and $B$ are finite sets, then $A \cup B$ is a finite set.

45. Show that if $A$ is an infinite set, then whenever $B$ is a set, $A \cup B$ is also an infinite set.

46. Show that if $A$, $B$, and $C$ are sets, then
   \[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \]
   (This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

47. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$ Find
   a) $\bigcup_{i=1}^{n} A_i$.
   b) $\bigcap_{i=1}^{n} A_i$.

48. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find
   a) $\bigcup_{i=1}^{n} A_i$.
   b) $\bigcap_{i=1}^{n} A_i$.

49. Let $A_i$ be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding $i$.
   Find
   a) $\bigcup_{i=1}^{n} A_i$.
   b) $\bigcap_{i=1}^{n} A_i$.

50. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer $i$,
   a) $A_i = \{i, i+1, i+2, \ldots\}$.
   b) $A_i = \{0, i\}$.
   c) $A_i = (0, i)$, that is, the set of real numbers $x$ with $0 < x < i$.
   d) $A_i = (i, \infty)$, that is, the set of real numbers $x$ with $x > i$.

51. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer $i$,
   a) $A_i = \{-i, -i+1, \ldots, -1, 0, 1, \ldots, i-1, i\}$.
   b) $A_i = \{-i, i\}$.
   c) $A_i = [-i, i]$, that is, the set of real numbers $x$ with $-i \leq x \leq i$.
   d) $A_i = [i, \infty)$, that is, the set of real numbers $x$ with $x \geq i$.

52. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the $i$th bit in the string is 1 if $i$ is in the set and 0 otherwise.
   a) $[3, 4, 5]$
   b) $[1, 3, 6, 10]$
   c) $[2, 3, 4, 7, 8, 9]$

53. Using the same universal set as in the last problem, find the set specified by each of these bit strings.
   a) 11 1100 1111
   b) 01 0111 1000
   c) 10 0000 0001

54. What subsets of a finite universal set do these bit strings represent?
   a) the string with all zeros
   b) the string with all ones

55. What is the bit string corresponding to the difference of two sets?

56. What is the bit string corresponding to the symmetric difference of two sets?

57. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, f, u, x, y\}$.
   a) $A \cup B$
   b) $A \cap B$
   c) $(A \cup D) \cap (B \cup C)$
   d) $A \cup B \cup C \cup D$

58. How can the union and intersection of $n$ sets that all are subsets of the universal set $U$ be found using bit strings?

The successor of the set $A$ is the set $A \cup \{A\}$.

59. Find the successors of the following sets.
   a) $\{1, 2, 3\}$
   b) $\emptyset$
   c) $\{\emptyset\}$
   d) $\{\emptyset, \{\emptyset\}\}$
60. How many elements does the successor of a set with \( n \) elements have?

Sometimes the number of times that an element occurs in an unordered collection matters. **Multisets** are unordered collections of elements where an element can occur as a member more than once. The notation \( \{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\} \) denotes the multiset with element \( a_1 \) occurring \( m_1 \) times, element \( a_2 \) occurring \( m_2 \) times, and so on. The numbers \( m_i, i = 1, 2, \ldots, r \) are called the **multiplicities** of the elements \( a_i, i = 1, 2, \ldots, r \).

Let \( P \) and \( Q \) be multisets. The **union** of the multisets \( P \) and \( Q \) is the multiset where the multiplicity of an element is the maximum of its multiplicities in \( P \) and \( Q \). The **intersection** of \( P \) and \( Q \) is the multiset where the multiplicity of an element is the minimum of its multiplicities in \( P \) and \( Q \). The **difference** of \( P \) and \( Q \) is the multiset where the multiplicity of an element is the multiplicity of the element in \( P \) less its multiplicity in \( Q \) unless this difference is negative, in which case the multiplicity is 0. The **sum** of \( P \) and \( Q \) is the multiset where the multiplicity of an element is the sum of multiplicities in \( P \) and \( Q \). The union, intersection, and difference of \( P \) and \( Q \) are denoted by \( P \cup Q \), \( P \cap Q \), and \( P - Q \), respectively (where these operations should not be confused with the analogous operations for sets). The sum of \( P \) and \( Q \) is denoted by \( P + Q \).

61. Let \( A \) and \( B \) be the multisets \( \{3 \cdot a, 2 \cdot b, 1 \cdot c\} \) and \( \{2 \cdot a, 3 \cdot b, 4 \cdot d\} \), respectively. Find

   a) \( A \cup B \). b) \( A \cap B \). c) \( A - B \).
   d) \( B - A \). e) \( A + B \).

62. Suppose that \( A \) is the multiset that has as its elements the types of computer equipment the university and the multiplicities are the number of pieces of each type needed, and \( B \) is the analogous multiset for a second department of the university. For instance, \( A \) could be the multiset \( \{107 \cdot \text{personal computers}, 44 \cdot \text{routers}, 6 \cdot \text{servers}\} \) and \( B \) could be the multiset \( \{14 \cdot \text{personal computers}, 6 \cdot \text{routers}, 2 \cdot \text{mainframes}\} \).

   a) What combination of \( A \) and \( B \) represents the equipment the university should buy assuming both departments use the same equipment?

b) What combination of \( A \) and \( B \) represents the equipment that will be used by both departments if both departments use the same equipment?

c) What combination of \( A \) and \( B \) represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?

d) What combination of \( A \) and \( B \) represents the equipment that the university should purchase if both departments do not share equipment?

**Fuzzy sets** are used in artificial intelligence. Each element in the universal set \( U \) has a **degree of membership**, which is a real number between 0 and 1 (including 0 and 1), and a fuzzy set \( S \). The fuzzy set \( S \) is denoted by listing the elements with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write \( \{0.6 \cdot \text{Alice}, 0.9 \cdot \text{Brian}, 0.4 \cdot \text{Fred}, 0.1 \cdot \text{Oscar}, 0.5 \cdot \text{Rita}\} \) for the set \( F \) of famous people to indicate that Alice has a 0.6 degree of membership in \( F \), Brian has a 0.9 degree of membership in \( F \), Fred has a 0.4 degree of membership in \( F \), Oscar has a 0.1 degree of membership in \( F \), and Rita has a 0.5 degree of membership in \( F \) (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that \( R \) is the set of people with \( R = \{0.4 \cdot \text{Alice}, 0.8 \cdot \text{Brian}, 0.2 \cdot \text{Fred}, 0.9 \cdot \text{Oscar}, 0.7 \cdot \text{Rita}\} \).

63. The **complement** of a fuzzy set \( S \) is the set \( \complement S \), with a degree of membership of an element in \( \complement S \) equal to 1 minus the degree of membership of this element in \( S \). Find \( \complement F \) (the fuzzy set of people who are not famous) and \( \complement R \) (the fuzzy set of people who are not rich).

64. The union of two fuzzy sets \( S \) and \( T \) is the fuzzy set \( S \cup T \), where the degree of membership of an element in \( S \cup T \) is the maximum of the degrees of membership of this element in \( S \) and in \( T \). Find the fuzzy set \( F \cup R \) of rich and famous people.

65. The intersection of two fuzzy sets \( S \) and \( T \) is the fuzzy set \( S \cap T \), where the degree of membership of an element in \( S \cap T \) is the minimum of the degrees of membership of this element in \( S \) and in \( T \). Find the fuzzy set \( F \cap R \) of rich and famous people.

## 2.3 Functions

### Introduction

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first). For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set \( \{A, B, C, D, F\} \). And suppose that the grades for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment of grades is illustrated in Figure 1.

This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science. For example, in discrete mathematics functions are used in the definition of such discrete structures as sequences and strings. Functions are also used to represent how long it takes a computer to solve problems of a given size. Many computer programs and subroutines are designed to calculate values of functions. Recursive functions