Due date: March 25 (in class)

Collaboration is encouraged. However the writeup should be your own. If you use any resources such as books or the internet, you should make sure you mention this in the homework. If you collaborate with other students then make sure you acknowledge your collaborators. Some of the questions asked might be “classic” and hence their solutions might be found on the internet or in research papers. However, you should not refer to such sources.

1. (a) If every point of the two-dimensional integer plane ($\mathbb{Z}^2$) is colored one of two colors, then show that there must be a 4 points that form a monochromatic rectangle.

(b) Let $r$ be any positive integer. If every point of the two-dimensional integer plane ($\mathbb{Z}^2$) is colored one of $r$ colors, then show that there must be a 4 points that form a monochromatic rectangle.

(c) For every integer $r$ and every pair of positive integers $m, n$, if every point of the two-dimensional integer plane ($\mathbb{Z}^2$) is colored one of $r$ colors, then show that there must be two subsets $S, T \subset \mathbb{Z}$ such that $|S| \geq m$, $|T| \geq n$, and the grid $S \times T \subseteq \mathbb{Z}^2$ is monochromatic.

(d) Prove or disprove: If every point of the two-dimensional integer plane ($\mathbb{Z}^2$) is colored one of 2 colors, then there must be two infinite subsets $S, T \subset \mathbb{Z}$ such that the grid $S \times T \subseteq \mathbb{Z}^2$ is monochromatic.

2. Prove that the following two versions of van der Waerden’s theorem are equivalent.

**Version 1**: If the positive integers are partitioned into two classes, then one of the classes must contain arbitrarily long arithmetic progressions.

**Version 2**: For every positive integer $k$, there exists an integer $W(k)$ such that if $\{1, 2, \ldots, W(k)\}$ is partitioned into two classes then at least one of the classes contains a $k$-term arithmetic progression.

3. Show that for every positive integer $k$, there exists an integer $N(k)$ such that if $\{1, 2, \ldots, N(k)\}$ is partitioned into two sets, then one of the two sets contains a set of the form $\{x_1, x_2, \ldots, x_k, x_1 + x_2 + \cdots + x_k\}$. The $x_i$s don’t have to be distinct.

4. Let $f(c)$ be the $c$-color Ramsey number $R(3, 3, 3, 3, \ldots, 3)$ (i.e. the smallest integer $n$ such that any $c$-coloring of the edges of a $K_n$ contains a monochromatic triangle. Show that $f(c_1 + c_2) - 1 \geq (f(c_1) - 1) \cdot (f(c_2) - 1)$. Thus show that $f(c) \geq 5^{\lceil c/2 \rceil}$. Recall that the proof of Ramsey’s Theorem shows that $f(c) \leq O(c!) = \exp(c \log c)$. It is still unknown what the correct behaviour of $f(c)$ is.
5. Let $X, Y$ and $Z$ be three random variables. For $H$ being the binary entropy function, prove that $H(X \mid Y, Z) \leq H(X \mid Y)$.

6. Let $G = (V,E)$ be an undirected graph. Let $n_a$ be the number of cliques of size $a$ in $G$, and let $n_b$ be the number of cliques of size $b$ in $G$. Suppose that $a < b$. Then show that $(b! \cdot n_b)^a \leq (a! \cdot n_a)^b$. 