Homework-1 (Graph Theory)

Due date: February 20 (in class)

Collaboration is encouraged. However the writeup should be your own.

1. (a) A graph is $k$-regular if every vertex of the graph has degree $k$. Show that every $k$-regular bipartite graph has a perfect matching.

(b) Let $A$ be a square $n \times n$ matrix of nonnegative integers, in which each row and column sum up to the positive integer $m$. Prove that $A$ can be expressed as a sum of $m$ permutation matrices $P_1, P_2, \ldots, P_m$. Here, a permutation matrix is an $n \times n$ matrix of zeros and ones, such that each row contains a single 1, and each column contains a single 1.

2. Show that Hall’s condition does not guarantee a matching in an infinite bipartite graph. Show however that it will do so if $G$ is countable and every vertex in $X$ has finite degree.

3. A regular polyhedron is a polyhedron where all faces are bounded by the same number of edges, and all vertices have the same number of incident edges. Every polyhedron naturally gives rise to a planar graph (with the same vertex, edge and face structure). Using Euler’s formula on this planar graph, show that there are only 5 regular polyhedra.

4. Prove the following lines version of the Sylvester-Gallai theorem: Given a set $L$ of lines in the 2-dimensional Euclidean plane, such that through the point of intersection of every 2 lines in $L$, there is a third line in $L$ that also passes through that same point of intersection; then it must hold that all lines in $L$ pass through a single point $P$. (Hint: consider the planar graph defined by the lines).

5. (a) Prove that a graph $G$ is $k$-connected iff $|G| \geq k + 1$ and for any set $U \subset V(G)$ with $|U| \geq k$ and for any vertex $x \not\in U$, there are $k$ paths from $x$ to $U$, any pair of paths having only the vertex $x$ in common.

(b) Prove that if $G$ is $k$-connected $k \geq 2$ and $\{x_1, x_2, \ldots, x_k\} \subset V(G)$ then there is a cycle in $G$ of length at least $k + 1$ that contains all $x_i$, $i \leq i \leq k$.

6. (Optional) In this problem you will show that every planar graph has a drawing in the plane in which every edge is a straight line segment.

(a) A maximal planar graph is a planar graph such that the addition of any edge makes it nonplanar. Show that every maximal planar graph $G = (V, E)$ has $|E| = 3|V| - 6$, and every face is bounded by exactly 3 edges.
(b) Show that in any maximally planar graph, there are at least 4 vertices of degree at most 5.

(c) Show that for any pentagon (not necessarily convex), there is a point in its interior that can “see” all the 5 vertices (i.e., the interiors of the line segments connecting the point to the vertices of the pentagon do not intersect the boundary of the pentagon).

(d) Show that every planar graph has a drawing in the plane in which every edge is a straight line segment. (Hint. Apply induction on the order of maximal planar graphs by omitting a suitable vertex.)