Ranks of Fixed Point Measures

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Abstract

H. Woodin in [6] studied a class of fixed point measures in order to define a representation for subsets of $V_{\lambda+1}$ called a $U(j)$-representation in $L(V_{\lambda+1})$ under the large cardinal hypothesis I_0. The propagation of these representations in [6] depended on a particular assumed property of these fixed point measures which we demonstrate below.

The structure $L(V_{\lambda+1})$ under the assumption of I_0 at λ has been shown to have many structural similarities to $L(\mathbb{R})$ assuming AD holds in $L(\mathbb{R})$ (see [6] and[1]). Along these lines, H. Woodin defined a representation for subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$, called a $U(j)$-representation, which is very similar to a weakly-homogeneously Suslin representation in the context of $L(\mathbb{R})$. The propagation of these representations in $L(V_{\lambda+1})$ has been shown to have many important consequences (see [6]). However it is still unclear which subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ have such a representation.

We demonstrate a property below of certain fixed point measures which by results in [6] and [1] implies that every subset of $V_{\lambda+1}$ in $L_\kappa(V_{\lambda+1})$ has a $U(j)$-representation in $L(V_{\lambda+1})$, where κ is the least $\Sigma_1(V_{\lambda+1} \cup \{V_{\lambda+1}\})$-gap. This extends results of [2] and [6] which imply that every subset (a bit beyond) $L_{\lambda^+}(V_{\lambda+1})$ has a $U(j)$-representation in $L(V_{\lambda+1})$. It is still an open question whether every subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ has a $U(j)$-representation in $L(V_{\lambda+1})$, a result which would have many interesting consequences (see [6] and [1]). Our results also show that, at least in terms of the ranks of the fixed point measures which constitute $U(j)$-representations, there seems to be no barrier to every subset of $V_{\lambda+1}$ having such a representation in $L(V_{\lambda+1})$.

1 $U(j)$-representations

We define the notion of a $U(j)$-representation and the fixed point measures which are needed in the definition. Our main concern will be the properties of these measures. For a complete introduction to $U(j)$-representations see [6], Chapter 7.

We fix λ and $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ elementary with $\operatorname{crit}(j) < \lambda$ for the rest of the paper. We will use the notation $j_{(i)}$ to denote the i-th iterate of j to distinguish it from our inverse limit notation. First define

$$\mathcal{F}_\kappa(j) = \{a \in L_\kappa(V_{\lambda+1}) |\, j(a) = a\}$$

and let

$$\mathcal{F}^\omega_\kappa(j) = \bigcup_{n<\omega} \mathcal{F}_\kappa(j_{(n)}).$$

Set $N_j(a)$ to be the least n such that $j_{(n)}(a) = a$.

Definition 1 (Woodin)**.** Let $U(j)$ be the set of $U \in L(V_{\lambda+1})$ such that in $L(V_{\lambda+1})$ the following hold:

1. U is a λ^+-complete ultrafilter.

2. For some $\gamma < \Theta$, U is generated by $U \cap L_\gamma(V_{\lambda+1})$.

3. For all sufficiently large $n < \omega$, $j_{(n)}(U) = U$ and for some $A \in U$,

$$\{a \in A |\, j_{(n)}(a) = a\} \in U.$$

For each ordinal κ, let $\Theta^{L_\kappa(V_{\lambda+1})}$ denote the supremum of the ordinals α such that there is a surjection $\rho : V_{\lambda+1} \to \alpha$ such that $\{(a, b) |\, \rho(a) < \rho(b)\} \in L_\kappa(V_{\lambda+1})$. Suppose that $\kappa < \Theta$ and $\kappa \le \Theta^{L_\kappa(V_{\lambda+1})}$. Then $\mathcal{E}(j, \kappa)$ is the set of all elementary embeddings $k : L_\kappa(V_{\lambda+1}) \to L_\kappa(V_{\lambda+1})$ such that there exists $n, m < \omega$ such that $k_{(n)} = j_{(m)} \restriction L_\kappa(V_{\lambda+1})$.

Note that if $j(\kappa) = \kappa$, then $j(\mathcal{E}(j, \kappa)) = \mathcal{E}(j, \kappa)$.

Suppose that $\kappa < \Theta$ and that $\kappa \le \Theta^{L_\kappa(V_{\lambda+1})}$. For each $\delta \le \lambda$ let $\mathcal{F}^\delta(\mathcal{E}(j, \kappa))$ be the filter on $P(\kappa) \cap L(V_{\lambda+1})$ generated by the sets

$$\{D_\sigma |\, \sigma \in [\mathcal{E}(j, \kappa)]^\delta\},$$

where for each $\sigma \in [\mathcal{E}(j, \kappa)]^\delta$,

$$D_\sigma = \{b \in L_\kappa(V_{\lambda+1}) |\, k(b) = b \text{ for all } k \in \sigma\},$$

the common fixed points of elements of σ.

Lemma 2 (Woodin)**.** *Suppose $\kappa < \Theta$, $\kappa \le \Theta^{L_\kappa(V_{\lambda+1})}$ and that $j(\kappa) = \kappa$. Then there is $\delta < crit(j)$ and a partition $\{S_\alpha |\, \alpha < \delta\} \in L(V_{\lambda+1})$ of $L_\kappa(V_{\lambda+1})$ into $\mathcal{F}^\lambda(\mathcal{E}(j, \kappa))$-positive sets such that for each $\alpha < \delta$,*

$$\mathcal{F}^\lambda(\mathcal{E}(j, \kappa)) \restriction S_\alpha \in U(j).$$

Proof. First, we have that since $j(\kappa) = \kappa$ that

$$j(\mathcal{E}(j, \kappa)) = \mathcal{E}(j, \kappa) \text{ and } j(\mathcal{F}^\lambda(\mathcal{E}(j, \kappa))) = \mathcal{F}^\lambda(\mathcal{E}(j, \kappa)).$$

Now we show that there is no sequence $\langle S_\alpha | \, \alpha < \text{crit}\,(j)\rangle \in L(V_{\lambda+1})$ of pairwise disjoint $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$-positive sets. This follows since

$$\{a \in L_\kappa(V_{\lambda+1}) | \, j(a) = a\} \in \mathcal{F}^\lambda(\mathcal{E}(j,\kappa)),$$

and hence if

$$j(\langle S_\alpha | \, \alpha < \text{crit}\,(j)\rangle) = \langle T_\alpha | \, \alpha < j(\text{crit}\,(j))\rangle\,,$$

then there exists a β such that $\beta \in T_{\text{crit}\,(j)}$ and $j(\beta) = \beta$. But then by elementarity, there exists an $\alpha < \text{crit}\,(j)$ such that $\beta \in S_\alpha$. But then $j(\beta) = \beta \in T_\alpha$, a contradiction.

Now, since $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$ is λ^+-complete, there must exists a $\delta < \text{crit}\,(j)$ and a partition $\{S_\alpha | \, \alpha < \delta\} \in L(V_{\lambda+1})$ of $L_\kappa(V_{\lambda+1})$ into $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$-positive sets such that for each $\alpha < \delta$, $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa)) \restriction S_\alpha$ is an ultrafilter.

For $\alpha < \delta$, let U_α be the ultrafilter given by $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa)) \restriction S_\alpha$. We have that U_α is λ^+-complete since $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$ is λ^+-complete. Furthermore we have that

$$B_\alpha := \{a \in S_\alpha | \, j(a) = a)\} \in U_\alpha.$$

And hence we have that $j(U_\alpha) = U_\alpha$, since for all $\beta \in B_\alpha$, $\beta \in S_\alpha \iff \beta \in j(S_\alpha)$. So we have that for all $\alpha < \delta$, $U_\alpha \in U(j)$. $\square$

Suppose that $\kappa < \Theta$ and $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$. Suppose that $\langle a_i | \, i < \omega\rangle$ is a sequence of elements of $L_\kappa(V_{\lambda+1})$ such that for all $i < \omega$, there exists an $n < \omega$ such that $j_{(n)}(a_i) = a_i$. Let $U(j, \kappa, \langle a_i | \, i < \omega\rangle)$ denote the set of $U \in U(j)$ such that there exists $n < \omega$ such that for all $k \in \mathcal{E}(j,\kappa)$, if $k(a_i) = a_i$ for all $i \leq n$, then

$$\{a \in L_\kappa(V_{\lambda+1}) | \, k(a) = a\} \in U.$$

The proof of Lemma 2 easily generalizes to obtain measures in $U(j, \kappa, \langle a_i | \, i < \omega\rangle)$. We can now define $U(j)$-representations for subsets of $V_{\lambda+1}$.

We now define $U(j)$-representations, which are very similar to weakly-homogeneously Suslin representations, but made specific to the particularities of working at λ instead of ω.

Definition 3 (Woodin)**.** Suppose $\kappa < \Theta$, κ is weakly inaccessible in $L(V_{\lambda+1})$, and $\langle a_i | \, i < \omega\rangle$ is an ω-sequence of elements of $L_\kappa(V_{\lambda+1})$ such that for all $i < \omega$ there is an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.

Suppose that $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$. Then Z is $U(j, \kappa, \langle a_i | \, i < \omega\rangle)$*-representable* if there exists an increasing sequence $\langle \lambda_i | \, i < \omega\rangle$, cofinal in λ and a function

$$\pi : \bigcup\{V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} | \, i < \omega\} \to U(j, \kappa, \langle a_i | \, i < \omega\rangle)$$

such that the following hold:

1. For all $i < \omega$ and $(a, b, i) \in \text{dom}(\pi)$ there exists $A \subseteq (L(V_{\lambda+1}))^i$ such that $A \in \pi(a, b, i)$.

2. For all $i < \omega$ and $(a, b, i) \in \mathrm{dom}(\pi)$, if $m < i$ then

$$(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m) \in \mathrm{dom}(\pi)$$

and $\pi(a, b, i)$ projects to $\pi(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m)$.

3. For all $x \subseteq V_\lambda$, $x \in Z$ if and only if there exists $y \subseteq V_\lambda$ such that

   (a) for all $m < \omega$, $(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) \in \mathrm{dom}(\pi)$,

   (b) the tower
   $$\langle \pi(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) | \, m < \omega \rangle$$
   is well founded.

For $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$ we say that Z is *$U(j)$-representable* if there exists $(\kappa, \langle a_i | \, i < \omega \rangle)$ such that Z is $U(j, \kappa, \langle a_i | \, i < \omega \rangle)$-representable.

2 Inverse Limits

In this section we give a very brief outline of the theory of inverse limits. These structures were originally used for reflecting large cardinal hypotheses of the form: there exists an elementary embedding $L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$. The use of inverse limits in reflecting such large cardinals is originally due to Laver [4]. For an introduction to the theory of inverse limits see [4], [5], and [2].

Suppose that $\langle j_i | \, i < \omega \rangle$ is a sequence of elementary embeddings such that the following hold:

1. For all i, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.

2. There exists $\bar{\lambda} < \lambda$ such that $\mathrm{crit}\, j_0 < \mathrm{crit}\, j_1 < \cdots < \bar{\lambda}$ and $\lim_{i<\omega} \mathrm{crit}\, j_i = \bar{\lambda} =: \bar{\lambda}_J$.

Then we can form the inverse limit

$$J = j_0 \circ j_1 \circ \cdots : V_{\bar{\lambda}} \to V_\lambda$$

by setting
$$J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a)$$
for any $a \in V_{\bar{\lambda}}$. $J : V_{\bar{\lambda}} \to V_\lambda$ is elementary, and can be extended to a Σ_0-embedding $J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ by $J(A) = \bigcup_i J(A \cap V_{\bar{\lambda}_i})$ for $\langle \bar{\lambda}_i | \, i < \omega \rangle$ any cofinal sequence in $\bar{\lambda}$.

Suppose $J = j_0 \circ j_1 \circ \cdots$ is an inverse limit. Then for $i < \omega$ we write $J_i := j_i \circ j_{i+1} \circ \cdots$, the inverse limit obtained by 'chopping off' the first i embeddings. For $i < \omega$ we write

$$J^{(i)} := (j_0 \circ \cdots \circ j_i)(J)$$

and for $n < \omega$,

$$J_n^{(i)} := (j_0 \circ \cdots \circ j_i)(J_n),\ j_n^{(i)} := (j_0 \circ \cdots \circ j_i)(j_n).$$

Then we can rewrite J in the following useful ways:

$$\begin{aligned} J = j_0 \circ j_1 \circ \cdots &= \cdots (j_0 \circ j_1)(j_2) \circ j_0(j_1) \circ j_0 \\ &= \cdots j_2^{(1)} \circ j_1^{(0)} \circ j_0 \end{aligned}$$

and

$$\begin{aligned} J &= j_0 \circ J_1 = j_0(J_1) \circ j_0 = J_1^{(0)} \circ j_0 \\ &= (j_0 \circ \cdots \circ j_{i-1})(J_i) \circ j_0 \circ \cdots \circ j_{i-1} = J_i^{(i-1)} \circ j_0 \circ \cdots \circ j_{i-1} \end{aligned}$$

for any $i > 0$. Hence we can view an inverse limit J as a direct limit.

We let $\mathcal{E}$ be the set of inverse limits. So

$$\mathcal{E} = \{(J, \langle j_i |\, i < \omega\rangle) |\, J = j_0 \circ j_1 \circ \cdots : V_{\bar{\lambda}_J} \to V_\lambda \text{ is elementary}\}.$$

This is a slightly larger collection than is defined in for instance in [2]. Note that we will many times be sloppy and refer to an inverse limit as 'J', '$(J, \vec{j})$' or '$(J, \langle j_i\rangle)$' instead of '$(J, \langle j_i |\, i < \omega\rangle)$'.

Define

$$\mathcal{E}_\alpha = \{(J, \vec{j}) \in \mathcal{E} |\, \forall i < \omega\, (j_i \text{ extends to an elementary embedding } L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

We say that α is *good* if every element of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from elements of $V_{\lambda+1}$. Note that the good ordinals are cofinal in Θ.

Lemma 4 (Laver)**.** *Suppose there exists an elementary embedding*

$$j : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

where α is good. Then $\mathcal{E}_\alpha \neq \emptyset$.

An important property of inverse limits is to what extend they extend beyond $V_{\lambda+1}$ (see [2]). However, in the next section we will consider a different type of extension, where we use inverse limits more as operators than embeddings. With that in mind we make the following definition.

Definition 5. For $\alpha < \Theta$ set

$$\mathcal{E}_\alpha^e = \{(J, \vec{j}) |\, (J, \vec{j} \restriction V_{\lambda+1}) \in \mathcal{E}, \forall i (j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

Suppose that $(J, \vec{j}) \in \mathcal{E}_\alpha^e$. Then we say that $a \in L_\alpha(V_{\lambda+1})$ is in the *extended range of* J if for all $i < \omega$, $a \in \operatorname{rng}(j_0 \circ \cdots \circ j_i)$. Also suppose that $(K, \vec{k}) \in \mathcal{E}_\alpha^e$ for some α. We put

$$K^{\text{ext}}(a) = \lim_{i \to \omega} (k_0 \circ \cdots \circ k_i)(a)$$

if this limit exists (in the sense that for all large enough i, $k_i(a) = a$).

Lemma 6. *Suppose that $(J, \vec{j}) \in \mathcal{E}^e_{\alpha+1}$ for α good. Then for all $i < \omega$, $J_i^{(i-1)} \in \mathcal{E}^e_\alpha$.*

Proof. This follows immediately by elementarity, and the fact that α is good. □

Definition 7. Suppose

$$(J, \langle j_i \rangle), (K, \langle k_i \rangle) \in \mathcal{E}.$$

Then we say that K *is a limit root of* J if there is $n < \omega$ such that $\bar{\lambda}_J = \bar{\lambda}_K$ and

$$\forall i < n\,(k_i = j_i) \text{ and } \forall i \geq n\,(k_i(k_i) = j_i).$$

We say K *is an n-close limit root of* J if n witnesses that K is a limit root of J. We also say that K and J *agree up to* n if for all $i < n$, $j_i = k_i$.

Also for $j : V_{\lambda+1} \to V_{\lambda+1}$ elementary and $(K, \vec{k}) \in \mathcal{E}$ we say that K *is a limit root of* j if for all $i < \omega$, $k_i(k_i) = j$ and for all $n < i$, $k_n \in \operatorname{rng} k_i$.

3 The rank game for fixed point measures

We now introduce the game on fixed points of elementary embeddings which we will be working with for the rest of the paper. Below we will give some motivation for how this game proceeds.

Definition 8. Suppose $\gamma < \Theta^{L(V_{\lambda+1})}$ and

$$\langle a_i |\, i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$$

and we have:

1. $\gamma \leq \Theta^{L_\gamma(V_{\lambda+1})}$,
2. for all $i < \omega$, $a_i \subseteq a_{i+1} \subseteq \gamma$ and $|a_i| < \lambda$,
3. for all $i < \omega$, there exists an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.

Then let $G(j, \gamma, \langle a_i |\, i < \omega \rangle)$ denote the following game. Player I plays a sequence

$$\big\langle (\gamma_i, \big\langle b^i_m : m < \omega \big\rangle) : i < \omega \big\rangle$$

and player II plays a sequence $\langle \mathcal{E}_i : i < \omega \rangle$ such that the following hold:

1. $\mathcal{E}_i \subseteq \operatorname{Emb}(j, \gamma_i)$, $|\mathcal{E}_i| \leq \lambda$, and for each $k \in \mathcal{E}_i$ there exists $m < \omega$ such that $k(b^i_m) = b^i_m$.
2. $\gamma_0 = \gamma$, $\gamma_{i+1} < \gamma_i$ and there exists $m < \omega$ such that

   $$k(b^i_m) = b^i_m \Rightarrow k(\gamma_{i+1}) = \gamma_{i+1}$$

   for all $k \in \mathcal{E}_i$.

3. for all $i < \omega$, $\gamma_i \leq \Theta^{L_{\gamma_i}(V_{\lambda+1})}$,

4. $\langle b^0_m : m < \omega \rangle = \langle a_m : m < \omega \rangle$.

5. for all $m < \omega$, $b^i_m \subseteq b^i_{m+1} \subseteq \gamma_i$ and $|b^i_m| < \lambda$

6. for all $m < \omega$ there exists $m^* < \omega$ such that

$$k(b^i_{m^*}) = b^i_{m^*} \Rightarrow k(b^{i+1}_m) = b^{i+1}_m$$

for all $k \in \mathcal{E}_i$.

Of course II always wins this game, but we are interested in the rank of this game, which we define as follows.

Definition 9. Let $G_\delta(j, \gamma, \langle a_i | \, i < \omega \rangle)$ have the same definition as $G(j, \gamma, \langle a_i | \, i < \omega \rangle)$ except that II must also play ordinals $\delta_0 > \delta_1 > \cdots$ such that $\delta_0 < \delta$. Then if δ is least such that II has a quasi-winning strategy in $G_\delta(j, \gamma, \langle a_i | \, i < \omega \rangle)$, then we set $\delta = \text{rank}(j, \gamma, \langle a_i | \, i < \omega \rangle)$.

Our main goal (see Theorem 21) is to show that for any $\delta < \Theta$ we can find γ and $\langle a_i | \, i < \omega \rangle$ such that $\text{rank}(j, \gamma, \langle a_i | \, i < \omega \rangle) \geq \delta$. That is, the rank of this game can be made arbitrarily large by an appropriate choice of parameters.

Definition 10. Suppose $\gamma < \Theta^{L(V_{\lambda+1})}$, $S \subseteq L_\gamma(V_{\lambda+1})$, and $\langle a_i | \, i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$ and we have:

1. $\gamma \leq \Theta^{L_\gamma(V_{\lambda+1})}$,

2. for all $i < \omega$, $a_i \subseteq a_{i+1} \subseteq \gamma$ and $|a_i| < \lambda$,

3. for all $i < \omega$, $a_i \in \mathcal{F}^\omega_{\gamma+1}(j)$.

4. $S = \bigcup_{i<\omega} a_i$.

Then we say that $\langle a_i | \, i < \omega \rangle$ is a *j-stratification of S*.

Suppose that $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Note that for all $S \subseteq \mathcal{F}^\omega_\Theta(j)$ such that $|S| \leq \lambda$, there is a $\gamma < \Theta$ and a $\langle a_i | \, i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$ such that $\langle a_i | \, i < \omega \rangle$ is a j-stratification of S. Hence for any $\gamma < \lambda^+$, if $\langle a_i | \, i < \omega \rangle$ is a j-stratification of γ, then $\text{rank}(j, \gamma, \langle a_i | \, i < \omega \rangle) = \gamma$. An instructive example then is to show that $\text{rank}(j, \lambda^+, \emptyset) = \lambda^+$, which we leave to the reader.

We define some more terminology for the objects which appear in the game $G(j, \gamma, \vec{a})$.

Definition 11. Suppose that $\vec{\mathcal{E}}$ and $S \subseteq \text{Ord}$ are such that for all $\alpha \in S$ there exists an i such that for all $k \in \mathcal{E}^i$, $k(\alpha) = \alpha$. Let $\vec{a}$ be defined by

$$a_i = \{\alpha \in S | \, \forall k \in \mathcal{E}_i \, (k(\alpha) = \alpha)\}.$$

We say that $\vec{a}$ is the *stratification of S with respect to $\vec{\mathcal{E}}$.*

Suppose $\vec{a}$ is such that $a_0 \subseteq a_1 \subseteq \cdots$,

$$a_i \subseteq \{\alpha \in S | \forall k \in \mathcal{E}_i \, (k(\alpha) = \alpha)\}$$

and for all i, $|a_i| < \operatorname{crit} j_{(i)}$. Then we say that $\vec{a}$ is a *j-layering of S with respect to $\vec{\mathcal{E}}$.*

Our main strategy in playing the above game is to guide Player I using inverse limits. In order to do this, we first give a lemma whose proof gives an outline for working with inverse limits in this context. We will then consider a certain simple decomposition of ordinals which our embeddings preserves, and then we will use these decomposition in order to ensure that we have provided enough room for Player I to continue playing.

Lemma 12. *Suppose that α is good and $(J, \vec{j}) \in \mathcal{E}^e_\alpha$. Then if $\langle \kappa_i | \, i < \omega \rangle$ is defined by $\kappa_i = crit\,(J_i^{(i-1)})$ for $i < \omega$, and $\langle a_i | \, i < \omega \rangle$ is such that for all $i < \omega$, $|a_i| < \kappa_i$ and $a_i \subseteq V_{\lambda+1}$, then there is an inverse limit $(K, \vec{k}) \in \mathcal{E}^e_\alpha$ such that for all $i < \omega$, $a_i \subseteq rng\, K_i^{ext,(i-1)}$.*

Proof. By basic facts about inverse limits (see for instance [1]), there is $(K, \vec{k})$ satisfying the following.

1. $(K, \vec{k}) \in \mathcal{E}^e_\alpha$.
2. For all $i < \omega$, $a_i \in \operatorname{rng} k_n$ for all $i, n < \omega$.
3. For all $i < \omega$, $k_0 \upharpoonright V_\lambda, \ldots, k_i \upharpoonright V_\lambda \in \operatorname{rng} k_{i+1}$.
4. For all $i < \omega$, $\kappa_i > \operatorname{crit}(K_i^{(i-1)}) > |a_i|$.

Now we claim that $(K, \vec{k})$ satisfies the lemma. To see this, note that clearly $a_0 \subseteq \operatorname{rng} K$ since conditions 2 and 3 imply that for all $i < \omega$, $a_0 \in \operatorname{rng} k_0 \circ \cdots k_i$, and since $\operatorname{crit}(K) > \kappa_0 > |a_0|$, we must have that $a_0 \subseteq \operatorname{rng} K$. Hence to see that the lemma holds, it is enough to see that for all $i < \omega$, $a_i \in \operatorname{rng} k_n^{(i-1)}$ for all $n \geq i$. But for any $i < \omega$,

$$(k_0 \circ \cdots \circ k_{i-1})^{-1}(a_i) \in \operatorname{rng} k_n$$

for all $n \geq i$. And hence applying $k_0 \circ \cdots \circ k_{i-1}$, by elementarity we have the desired result. □

Lemma 13. *Suppose that $\langle a_i | \, i < \omega \rangle$ is a j-layering of S with $S \subseteq \gamma$ for some γ. Suppose that $(J, \vec{j}) \in \mathcal{E}^e_{\eta+2}$ is such that $\eta \geq \gamma$ is good and for all $i < \omega$,*

$$J_i^{ext,(i-1)}(a_i) = a_i.$$

Then there exists a $(K, \vec{k}) \in \mathcal{E}^e_{\eta+1}$ a limit root of J such that for all $i < \omega$,

$$K_i^{ext,(i-1)}(a_i) = a_i.$$

Proof. This follows as in the proof of Lemma 12 by noticing that if γ is good and $j(\gamma) = k(\gamma) = \gamma$, then for $\alpha < \gamma$ an ordinal, if $j_i(\alpha) = \alpha$, k_i is a square root of j_i and $\alpha \in \operatorname{rng} k_i$ then $k_i(\alpha) = \alpha$. Hence defining $(K, \vec{k}) \in \mathcal{E}^e_{\eta+1}$ as in the proof of Lemma 12, we have that for all $i < \omega$ and $\xi \in a_i$, $\xi \in \operatorname{rng} K_i^{\mathrm{ext},(i-1)}$. But since $J_i^{\mathrm{ext},(i-1)}(\xi) = \xi$, we have that $K_i^{\mathrm{ext},(i-1)}(\xi) = \xi$. □

Definition 14. Fix $\kappa < \Theta$ good with $\operatorname{cof}(\kappa) > \lambda$. Let $S \subseteq \kappa$ such that $|S| \leq \lambda$. Then we say that S is λ-*threaded* if the following hold:

1. Suppose $\alpha < \sup S$ is such that there exists $\vec{\beta} \in S^{<\omega}$ and $a \in V_\lambda$ such that α is definable over $L_\kappa(V_{\lambda+1})$ from $\vec{\beta}$ and a. Then $\alpha \in S$.

2. Suppose $\alpha \in S$ is a limit and $\operatorname{cof}(\alpha) < \lambda$. Then $S \cap \alpha$ is cofinal in α.

We say that S is *definably closed* if S satisfies (1).

Since λ-DC holds in $L(V_{\lambda+1})$, we have that for every $S \subseteq \kappa$ with $|S| \leq \lambda$, there is $S' \subseteq \kappa$ with $S' \supseteq S$ and $|S'| \leq \lambda$ such that S' is λ-threaded.

We put for E a set such that for all $k \in E$, $k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is elementary for some $\alpha < \Theta$,

$$\mathcal{F}(E) = \{\beta | \, \forall k \in E \, (k(\beta) = \beta)\}.$$

We now need to make some technical definitions involving ordinals. The main point is that we want to decompose an ordinal into basic components where we understand enough of how our elementary embeddings behave on these components. We start with a lemma which allows us to make our definitions.

Lemma 15. *Suppose α is a limit ordinal. Then there exists a $\gamma < \alpha$ such that for all $\beta \in [\gamma, \alpha)$ if β_0 is such that $\beta = \gamma + \beta_0$, then for all $\delta < \alpha$, $\delta + \beta_0 < \alpha$.*

Proof. We prove this by induction on α. Suppose that α is such that there exists a $\beta < \alpha$ such that for some $\delta < \alpha$, $\beta + \delta \geq \alpha$. Let $\alpha^* \leq \alpha$ be the sup of ordinals $\gamma < \alpha$ such that for all $\beta, \delta < \gamma$, $\delta + \beta < \gamma$. Call the set of such ordinals A. Then clearly $\alpha^* \in A$. So $\alpha^* < \alpha$. Let α_0 be such that $\alpha^* + \alpha_0 = \alpha$.

We claim that $\alpha_0 < \alpha$. If not, then $\alpha^* + \alpha = \alpha$. But then $\alpha^* \cdot \omega \leq \alpha$, and $\alpha^* \cdot \omega \in A$, a contradiction.

But then by applying the lemma to α_0, we have that there exists a $\gamma < \alpha_0$ such that for all $\beta \in [\gamma, \alpha_0)$ if β_0 is such that $\beta = \gamma + \beta_0$, then for all $\delta < \alpha_0$, $\delta + \beta_0 < \alpha_0$. But if $\beta \in [\gamma, \alpha_0)$ then $\alpha^* + \beta < \alpha$ and for some β_0,

$$\alpha^* + \beta = \alpha^* + \gamma + \beta_0.$$

Hence $\gamma_0 = \alpha^* + \gamma$ witnesses the lemma for α. To see this let $\beta \in [\gamma_0, \alpha)$ and let β_0 be such that $\alpha^* + \gamma + \beta_0 = \beta$. Suppose $\delta \in [\alpha^*, \alpha)$. Let δ^* be such that $\alpha^* + \delta^* = \delta$. Then we have that $\delta^* < \alpha_0$, and hence $\delta^* + \beta_0 < \alpha_0$. But then

$$\alpha^* + \delta^* + \beta_0 = \delta + \beta_0 < \alpha^* + \alpha_0 = \alpha,$$

which proves the lemma. □

From the previous lemma, given a limit ordinal α there is a decomposition,

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_n$$

for some n with $\alpha_0 > \alpha_1 > \cdots > \alpha_n$, and such that for all for all $i < n$, for all

$$\beta \in [\alpha_0 + \cdots + \alpha_i, \alpha_0 + \cdots + \alpha_{i+1})$$

if β_0 is such that

$$\beta = \alpha_0 + \cdots + \alpha_i + \beta_0$$

then for all $\delta < \alpha_0 + \cdots + \alpha_{i+1}$, $\delta + \beta_0 < \alpha_0 + \cdots + \alpha_{i+1}$. To see this, let $\alpha = \alpha_0^* + \delta_0$ be given by the lemma such that α_0^* is as small as possible. Then if $\alpha_0^* \neq 0$, apply the lemma to α_0^* to obtain $\alpha_0^* = \alpha_1^* + \delta_1$ where α_1^* is as small as possible, and so forth to obtain $\alpha = \delta_n + \delta_{n-1} + \cdots + \delta_0$. We then set $\alpha_i = \delta_{n-i}$. Note that $\delta_n > \delta_{n-1} > \cdots > \delta_0$ as, for instance, if $\delta_1 \leq \delta_0$, then $\alpha = \alpha_1^* + \delta_1 + \delta_0$. But if $\beta \in [\alpha_1^*, \alpha)$ and $\beta = \alpha_1^* + \beta_0$, either $\beta_0 < \delta_1$ or $\beta_0 = \delta_1 + \beta_1$ where $\beta_1 < \delta_0$. But then in the latter case for any $\gamma < \alpha$, we have that $\gamma + \delta_1 < \alpha$ since $\delta_1 < \delta_0$, and hence $\gamma + \delta_1 + \beta_1 = \gamma + \beta_0 < \alpha$. Hence in either case $\gamma + \beta_0 < \alpha$ for any $\gamma < \alpha$, and so $\alpha = \alpha_1^* + (\delta_1 + \delta_0)$ is a decomposition of α which satisfies the lemma, contradicting the definition of α_0^*, since $\alpha_1^* < \alpha_0^*$.

We call $\langle \alpha_0, \ldots, \alpha_n \rangle$ the *addition decomposition of* α.

We define the function $c(\alpha, \beta)$ for $\beta < \alpha$ as follows. Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be the addition decomposition of α. Let i be largest such that $\alpha_0 + \cdots + \alpha_i \leq \beta$, and let β_0 be such that

$$\alpha_0 + \cdots + \alpha_i + \beta_0 = \beta.$$

Set $c(\alpha, \beta) = \beta_0$.

Also define the following functions:

$$\mathrm{ld}(\alpha) := \alpha_0 + \cdots + \alpha_{n-1}$$

if $n > 0$ and $\mathrm{ld}(\alpha) = 0$ otherwise, where $\langle \alpha_0, \ldots, \alpha_n \rangle$ is the addition decomposition of α, and

$$\mathrm{rd}(\alpha) := \alpha_n.$$

Lemma 16. *Suppose that $\beta + \gamma = \alpha$. Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be the addition decomposition of α. Then for some i, $\gamma = \alpha_i + \cdots + \alpha_n$.*

Proof. Note that we have for some i that $\beta = \alpha_0 + \cdots + \alpha_{i-1} + \beta'$ and $\gamma = \gamma' + \alpha_{i+1} + \cdots + \alpha_n$ for some β' and γ' such that $\beta' + \gamma' = \alpha_i$. Furthermore, if β and γ were a contradiction to the lemma, we would have that $\beta', \gamma' < \alpha_i$ and $\beta', \gamma' \neq 0$. But then $\beta' + \gamma' < \alpha_i$ by definition of the addition decomposition, a contradiction. □

Lemma 17. *Suppose that $\beta + \gamma = \alpha$ and $\gamma \neq \emptyset$. Then $rd(\gamma) = rd(\alpha)$.*

Proof. By the previous lemma, if $\langle \alpha_0, \dots, \alpha_n \rangle$ is the addition decomposition of α, we have that $\gamma = \alpha_i + \dots + \alpha_n$ for some i. So it is enough to see that $\langle \alpha_i, \dots, \alpha_n \rangle$ is the addition decomposition of γ. But this is basically immediate by the definition. $\square$

We need some notation and a theorem from [3] which give us a useful structure of inverse limits.

Definition 18. Let $\kappa < \Theta$ be good and let $\bar{\lambda}, \bar{\kappa} < \lambda$. For $\beta < \kappa$ we define by induction a set $\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$ of inverse limits as follows.

$$\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(0) = \{(J, \vec{j}) \in \mathcal{E}_\kappa |\, J \text{ extends to } \hat{J} : L_{\bar{\kappa}}(V_{\bar{\lambda}+1}) \to L_\kappa(V_{\lambda+1}) \text{ which is elementary}\}.$$

Then for any β such that $0 < \beta < \kappa$ we set

$$\begin{aligned}\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta) = \{(J, \vec{j}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(0) \cap \mathcal{E}_{\kappa+\beta} |\, \forall \gamma < \beta\, (\text{if } (J, \vec{j}) \in \mathcal{E}_{\kappa+\gamma} \text{ then}\\ \forall a \in V_{\bar{\lambda}+1}\, \forall b \in V_{\lambda+1}\, \exists (K, \vec{k}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\gamma)\\ (K(a) = J(a) \wedge b \in \operatorname{rng} K \wedge K \text{ is a 0-close limit root of } J))\}.\end{aligned}$$

Theorem 19 ([3])**.** *Suppose that there exists an elementary embedding*

$$j : L_\Theta(V_{\lambda+1}) \to L_\Theta(V_{\lambda+1}).$$

Let κ be good. Then there exists $\bar{\kappa}, \bar{\lambda} < \lambda$ such that for all $\beta < \kappa$, $\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta) \neq \emptyset$. Furthermore for all $\beta < \kappa$, $\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$ is definable over $L_{\kappa+\beta+1}(V_{\lambda+1})$ from $\bar{\lambda}, \bar{\kappa}$ and κ.

We fix a good limit ordinal $\kappa < \Theta$, and ordinals $\bar{\kappa}, \bar{\lambda} < \lambda$ for the rest of the section which are given by Theorem 19. Furthermore for any β and $J \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$, we let $\hat{J}$ be the unique extension of J to an elementary embedding $\hat{J} : L_{\bar{\kappa}}(V_{\bar{\lambda}+1}) \to L_\kappa(V_{\lambda+1})$. Similarly we let J^{ext} be the natural extension of J, considering it as extending to an element of $\mathcal{E}^e_\kappa$.

We first consider a more restrictive version of the above game. This game, in some sense, captures a version of $G(j, \gamma, \langle a_i |\, i < \omega \rangle)$ where only the 'local largeness' of the γ_i matter. Later on when we play $G(j, \gamma, \langle a_i |\, i < \omega \rangle)$, we will do so by playing many versions of this more restrictive game.

Lemma 20. *Suppose that $\alpha_0 < \kappa$ is an ordinal with cof$(\alpha_0) > \lambda$, $(J^0, \vec{j}^0) \in \mathcal{E}_{\kappa+\alpha_0+2}$, $\hat{J}^0$ exists, and $\alpha_0 \in \operatorname{rng} \hat{J}^0$. Then II has a quasi-winning strategy in the following game $G(\alpha_0, J^0)$*

$$\begin{array}{lccccc} I & \beta_0, \gamma_0 & & \beta_1, \gamma_1 & & \cdots \\ II & & \alpha_1, (J^1, \vec{j}^1) & & \alpha_2, (J^2, \vec{j}^2) & \cdots \end{array}$$

which has the following rules.

1. *For all i, $\alpha_i, \beta_i, \gamma_i < \kappa$.*

2. *$\beta_0 > \beta_1 > \beta_2 > \cdots$ and $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$. Also $\alpha_0 > \beta_0$.*

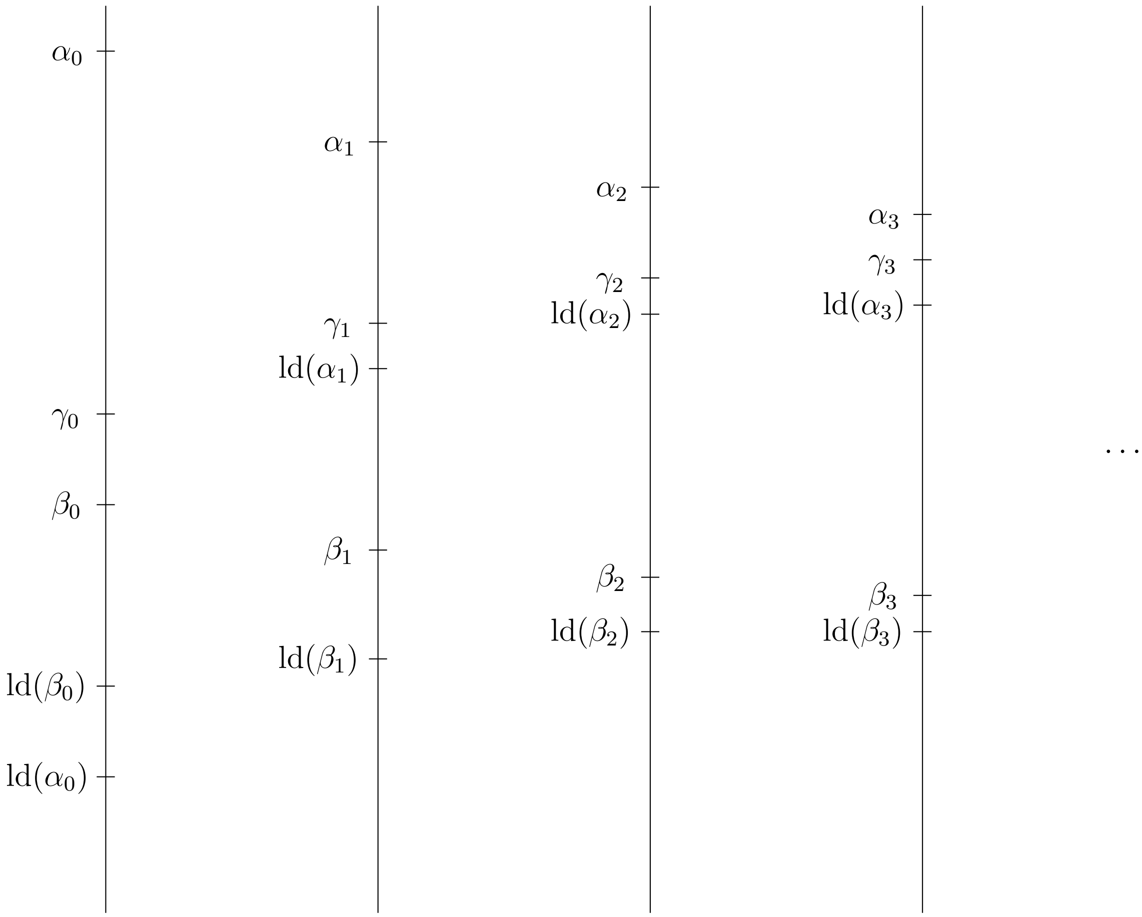

Figure 1: Typical play of $G(\alpha_0, J^0)$.

3. *For all i, if cof(β_i) $> \lambda$ then cof(α_{i+1}) $> \lambda$.*

4. *For all i, $(J^i, \vec{j}^i) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\kappa + \alpha_i + 2)$, $\hat{J}^i$ exists and $\alpha_i \in rng\, \hat{J}^i$.*

5. *For all i, $\gamma_i \in (ld(\alpha_i), \alpha_i)$, $(J^i)^{ext}(\gamma_i) = \gamma_i$ and $\alpha_{i+1} \geq \gamma_i$.*

6. *For all i, α_{i+1} is definable over $L_{\kappa+\alpha_i+2}(V_{\lambda+1})$ from parameters in*

$$\{\alpha_0, \dots, \alpha_i\} \cup \{\gamma_i, \kappa\} \cup \lambda,$$

and $(J^{i+1})^{ext}(\alpha_{i+1}) = \alpha_{i+1}$.

7. *For all i, $\beta_{i+1} > ld(\beta_i)$ and $\beta_i > ld(\alpha_0)$.*

The first player to violate one of the rules loses.

Proof. We describe a quasi-winning strategy for II. First suppose that I plays β_0, γ_0. Let

$$K^0 \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\kappa + \alpha_0 + 1)$$

be a 0-close limit root of J^0 such that $\beta_0 \in \operatorname{rng} \hat{K}^0$, $(K^0)^{\text{ext}}(\gamma_0) = \gamma_0$, and

$$\hat{J}^0(\bar{\alpha}_0) = \alpha_0 = \hat{K}^0(\bar{\alpha}_0)$$

for some $\bar{\alpha}_0$. Let $\bar{\beta}_0$ be such that $\hat{K}^0(\bar{\beta}_0) = \beta_0$. Now we have by elementarity that $\operatorname{ld}(\bar{\alpha}_0) < \bar{\beta}_0$. So let $\bar{\beta}_0^*$ be such that

$$\operatorname{ld}(\bar{\alpha}_0) + \bar{\beta}_0^* = \bar{\beta}_0.$$

Let β_0^- be the least β such that there exists $(K, \vec{k}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\kappa + \alpha_0 + 1)$ with

$$\hat{K}(\bar{\beta}_0^*) = \beta, \qquad K^{\text{ext}}(\beta) = \beta, \text{ and } K^{\text{ext}}(\gamma_0) = \gamma_0.$$

Let $\alpha_1 = \gamma_0 + \beta_0^-$. Clearly we have that α_1 is definable over $L_{\kappa+\alpha_0+2}(V_{\lambda+1})$ from $\bar{\lambda}, \bar{\kappa}, \kappa, \gamma_0$ and $\bar{\beta}_0^*$. Furthermore we have that $\alpha_1 < \alpha_0$ since $\beta_0^- \leq \hat{K}^0(\bar{\beta}_0^*)$, and for all $\delta < \alpha_0$,

$$\delta + \hat{K}^0(\bar{\beta}_0^*) < \alpha_0.$$

To see that $\beta_0^- \leq \hat{K}^0(\bar{\beta}_0^*)$, note that in fact

$$\beta_0^- \leq (K^{\text{ext}})^{-1}(\hat{K}^0(\bar{\beta}_0^*))$$

since for large enough i, K_i^0 satisfies the above conditions in the definition of β_0^-, and hence gives this inequality, since for all large enough $i < \omega$,

$$\hat{K}_i^0(\bar{\beta}_0^*) = (K^{\text{ext}})^{-1}(\hat{K}^0(\bar{\beta}_0^*)).$$

Let $(J^1, \vec{j}^1) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\kappa + \alpha_0 + 1)$ be such that

$$\hat{J}^1(\bar{\beta}_0^*) = \beta_0^-, \ (J^1)^{\text{ext}}(\beta_0^-) = \beta_0^- \text{ and } (J^1)^{\text{ext}}(\gamma_0) = \gamma_0.$$

Then clearly $(J^1)^{\text{ext}}(\alpha_1) = \alpha_1$. Also, if $\operatorname{cof}(\beta_0) > \lambda$, then $\operatorname{cof}(\bar{\beta}_0^*) > \bar{\lambda}$ and hence $\operatorname{cof}(\alpha_1) > \lambda$, all by elementarity.

Now suppose that I has played $\beta_0, \ldots, \beta_i$ and $\gamma_0, \ldots, \gamma_i$ satisfying the rules and II has responded with $\alpha_1, \ldots, \alpha_i$ and $(J^1, \vec{j}^1), \ldots, (J^i, \vec{j}^i)$ satisfying the rules. Also assume that II has chosen $\bar{\beta}_0, \ldots, \bar{\beta}_{i-1}$ and $K^0, \ldots, K^{i-1}$ satisfying that for all $n < i$, $\hat{K}^n(\bar{\beta}_n) = \beta_n$ and $\hat{J}^{n+1}(\bar{\beta}_n^*) = \beta_n^-$ where β_n^- is such that $\gamma_n + \beta_n^- = \alpha_{n+1}$.

Let $(K^i, \vec{k}^i) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\kappa + \alpha_i + 1)$ be such that for some $\bar{\beta}_i$

$$\hat{K}^i(\bar{\beta}_i) = \beta_i, \qquad \hat{K}^i(\bar{\beta}_{i-1}) = \beta_{i-1}, \text{ and } \hat{K}^i(\bar{\beta}_{i-1}^*) = \beta_{i-1}^-,$$

and

$$(K^i)^{\text{ext}}(\gamma_i, \gamma_{i-1}, \beta_{i-1}^-) = (\gamma_i, \gamma_{i-1}, \beta_{i-1}^-).$$

Now we have $\operatorname{ld}(\bar{\beta}_{i-1}) < \bar{\beta}_i$. So let $\bar{\beta}_i^*$ be such that $\operatorname{ld}(\bar{\beta}_{i-1}) + \bar{\beta}_i^* = \bar{\beta}_i$.

Let β_i^- be the least β such that there exists $(K, \vec{k}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\kappa + \alpha_i + 1)$ with

$$\hat{K}(\bar{\beta}_i^*) = \beta, \qquad K^{\text{ext}}(\beta) = \beta, \text{ and } K^{\text{ext}}(\gamma_i) = \gamma_i.$$

Let $\alpha_{i+1} = \gamma_i + \beta_i^-$. Clearly we have that α_{i+1} is definable over $L_{\kappa+\alpha_i+2}(V_{\lambda+1})$ from $\bar{\lambda}$, $\bar{\kappa}$, κ, γ_i and $\bar{\beta}_i^*$. Furthermore we have that $\alpha_{i+1} < \alpha_i$ since

$$\beta_i^- \leq \hat{K}^i(\bar{\beta}_i^*),$$

and $\bar{\beta}_{i-1} = \gamma + \bar{\beta}_{i-1}^*$ for some γ implies by Lemmas 16 and 17 that

$$\mathrm{rd}(\bar{\beta}_{i-1}^*) = \mathrm{rd}(\bar{\beta}_{i-1}) > \bar{\beta}_i^*$$

by definition of $\bar{\beta}_i^*$ and the fact that $\bar{\beta}_i > \bar{\beta}_{i-1}$. But applying $\hat{K}^i$ we get that

$$\mathrm{rd}(\beta_{i-1}^-) > \hat{K}^i(\bar{\beta}_i^*) \geq \beta_i^-$$

which is enough to show that

$$\alpha_{i+1} = \gamma_i + \beta_i^- < \alpha_i = \gamma_{i-1} + \beta_{i-1}^-.$$

To see that $\beta_i^- \leq \hat{K}^i(\bar{\beta}_i^*)$, as above note that in fact

$$\beta_i^- \leq ((K^i)^{\mathrm{ext}})^{-1}(\hat{K}^i(\bar{\beta}_i^*))$$

since for large enough m, K_m^i satisfies the above conditions in the definition of β_i^-, and hence gives this inequality, since for all large enough $m < \omega$,

$$\hat{K}_m^i(\bar{\beta}_m^*) = (K^{\mathrm{ext}})^{-1}(\hat{K}^0(\bar{\beta}_0^*)).$$

Let $(J^{i+1}, \vec{j}^{i+1}) \in \mathcal{E}_{\bar{\lambda},\bar{\kappa}}^{\kappa}(\kappa + \alpha_i + 1)$ be such that

$$\hat{J}^{i+1}(\bar{\beta}_i^*) = \beta_i^-,\ (J^{i+1})^{\mathrm{ext}}(\beta_i^-) = \beta_i^- \text{ and } (J^{i+1})^{\mathrm{ext}}(\gamma_i) = \gamma_i.$$

Then clearly $(J^{i+1})^{\mathrm{ext}}(\alpha_{i+1}) = \alpha_{i+1}$. Also, if $\mathrm{cof}(\beta_i) > \lambda$, then $\mathrm{cof}(\bar{\beta}_i^*) > \bar{\lambda}_{J^{i+1}}$ and hence $\mathrm{cof}(\alpha_{i+1}) > \lambda$.

We have described a quasi-winning strategy for II, which proves the lemma. □

Theorem 21. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be elementary. Fix $\kappa < \Theta$ good and regular in $L(V_{\lambda+1})$. Suppose that S has a largest element α_0, S is λ-threaded, and $\langle a_i | \, i < \omega \rangle$ is a j-stratification of S. Then rank$(j, \kappa + \alpha_0, \vec{a}) \geq \alpha_0$.*

Proof. We prove this by induction on α_0. Clearly, if $\alpha_0 = \alpha_0' + 1$ then $S \cap \alpha_0$ is λ-threaded and has largest element α_0', hence the induction is immediate.

Now assume that α_0 is a limit. There are two cases. Either $\mathrm{cof}(\alpha) < \lambda$ or $\mathrm{cof}(\alpha) > \lambda$.

First assume that $\mathrm{cof}(\alpha_0) < \lambda$. Then there must be a sequence $\langle \beta_i | \, i < \mathrm{cof}(\alpha_0) \rangle$ cofinal in α_0 such that for all $i < \mathrm{cof}(\alpha_0)$, $\beta_i \in S$. Hence we have that $S \cap \beta_i + 1$ is λ-threaded and has largest element β_i. So by induction we have that

$$\mathrm{rank}(j, \kappa + \beta_i, \langle a_i \cap \beta_i + 1 | \, i < \omega \rangle) \geq \beta_i.$$

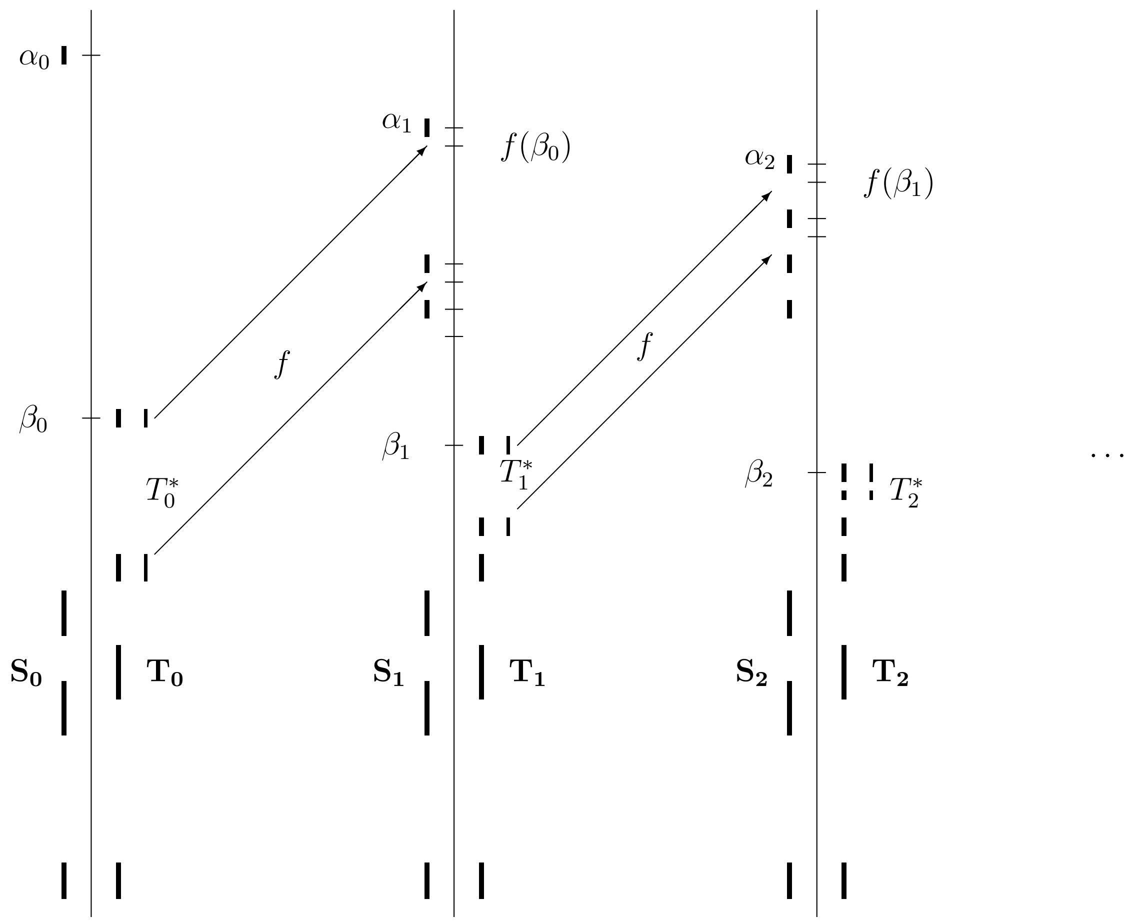

Figure 2: Strategy for $G(j, \kappa + \alpha_0, \vec{a})$.

But then clearly we have that $\operatorname{rank}(j, \kappa + \alpha_0, \vec{a}) \geq \sup_i \beta_i = \alpha_0$ since for all $i < \omega$ we have that for some n, $\beta_i \in a_n$.

Now assume that $\operatorname{cof}(\alpha_0) > \lambda$. Based on an arbitrary sequence $\beta_0 > \beta_1 > \cdots$ with $\beta_0 < \alpha_0$ we will choose responses α_i and $\vec{a}^i$ which are legal plays against a play by II in $G(j, \kappa + \alpha_0, \vec{a})$.

Let $\beta_0 < \alpha_0$. Let $\vec{\mathcal{E}}^0$ be a first play by II in $G(j, \kappa + \alpha_0, \vec{a})$ and set $S_0 = S$.

Let $(J^0, \vec{j}^0) \in \mathcal{E}^{\kappa}_{\bar{\lambda}, \bar{\kappa}}(\kappa + \alpha_0 + \omega)$ be such that $\alpha_0 \in \operatorname{rng} \hat{J}^0$ and $(J^0)^{\text{ext}}(\alpha_0) = \alpha_0$. Let $T_0 \subseteq \beta_0 + 1$ be λ-threaded with $\beta_0 \in T_0$.

For each $\beta \in T_0 \setminus \sup S_0$, we play a version of $G(\alpha_0, J^0)$ and define $f(\beta)$ by induction on the order of $T_0 \setminus \sup S_0$. We call this game $G(\alpha_0, J^0)[\beta]$ and let $\alpha[\beta]$ be a winning response by II to the play $\beta, f(\beta)$ by I. Let i be least such that

$$\alpha_0 \in \bigcap_{n \geq i} \mathcal{F}(\mathcal{E}^0_n).$$

Assume we have defined $f(\beta')$ and $\alpha[\beta']$ for all $\beta' \in \beta \cap (T_0 \setminus \sup S_0)$. Let γ be least such that for all $\beta' \in \beta \cap (T_0 \setminus \sup S_0)$,

$$\gamma > \alpha[\beta'], \qquad \forall n \geq i \, (\gamma \in \mathcal{F}(\mathcal{E}^0_n)), \text{ and } \qquad (J^0)^{\text{ext}}(\gamma) = \gamma.$$

Set $f(\beta) = \gamma$.

Let

$$S_1 = \{\alpha[\beta] | \, \beta \in T_0 \setminus \sup S_0\} \cup (S_0 \cap \alpha_0)$$

and let $\vec{a}^1$ be a j-layering of S_1 with respect to $\mathcal{E}^0$. Note that for all $\alpha \in S_1 \setminus \sup(S_0 \cap \alpha_0)$, there exists an i such that for some $\gamma \in \bigcap_{n \geq i} \mathcal{F}(\mathcal{E}^0_n)$, α is definable from parameters in $\{\alpha_0, \gamma, \kappa\} \cup \lambda$ over $L_{\kappa+\alpha_0+2}(V_{\lambda+1})$. Hence there exists an i' such that for all $n \geq i'$, $\alpha \in \mathcal{F}(\mathcal{E}^0_n)$. I then plays $(\vec{a}^1, \alpha[\beta_0])$.

Now assume that I has played

$$(\vec{a}, \alpha_0), (\vec{a}^1, \alpha[\beta_0]), \ldots, (\vec{a}^n, \alpha[\beta_0, \ldots, \beta_{n-1}])$$

against $\vec{\mathcal{E}}^0, \vec{\mathcal{E}}^1, \ldots, \vec{\mathcal{E}}^{n-1}$ and $\beta_0 > \beta_1 > \cdots > \beta_{n-1}$. Assume we have defined the following as well.

1. $T_0, \ldots, T_{n-1}$ such that for $i < n$, $T_i \subseteq \beta_i + 1$ is λ-threaded and $\beta_i \in T_i$. Let

$$T^*_i = T_i \setminus (\sup(T_{i-1} \cap \beta_i)),$$

where $T_{-1} = S_0$.

2. Suppose that $\delta_0 > \cdots > \delta_{m-1}$ is such that $m \leq n$ and the following hold: $\delta_0 \in T^*_0$, and for all $i < m - 1$, there is an i' such that $\beta_{i'} = \delta_i$, and $\delta_{i+1} \in T^*_{i'}$. Then

$$G(\alpha_0, J^0)[\delta_0, \ldots, \delta_{m-1}]$$

is an instance of $G(\alpha_0, J^0)$ with

$$f(\delta_0), f(\delta_0, \delta_1), \dots, f(\delta_0, \dots, \delta_{m-1})$$

defined and with $\alpha[\delta_0] > \cdots > \alpha[\delta_0, \dots, \delta_{m-1}]$ a winning response by II against the play

$$(\delta_0, f(\delta_0)), (\delta_1, f(\delta_0, \delta_1)), \dots, (\delta_{m-1}, f(\delta_0, \dots, \delta_{m-1})).$$

3. For W_n the set of such tuples $(\delta_0, \dots, \delta_{m-1})$ the function f is defined on W such that it is order preserving from lexicographically ordered tuples to ordinals. Furthermore for all $(\delta_0, \dots, \delta_{m-1}) \in W_n$, if s is such that $\delta_{m-1} \in T^*_s$, then there is an i such that for all $n' \geq i$
$$f(\delta_0, \dots, \delta_{m-1}) \in \mathcal{F}(\mathcal{E}^s_{n'}).$$

Now let $\beta_n < \beta_{n-1}$ and let $\vec{\mathcal{E}}^n$ be a play by II. We can assume without loss of generality that if

$$T_{n-1} \cap [\beta_n, \beta_{n-1}) \neq \emptyset$$

then $\beta_n \in T_{n-1}$.

Suppose first that $\beta_n \notin T_{n-1}$. Let $T_n \subseteq \beta_n + 1$ be λ-threaded such that $\beta_n \in T_n$ and $T_{n-1} \cap \beta_{n-1} \subseteq T_n$. For each $\delta \in T_n \setminus (\sup T_{n-1} \cap \beta_n)$ we define $f(\beta_{s(0)}, \dots, \beta_{s(m-1)}, \delta)$ by induction, where s is longest such that for all $i < m-1$, there exists an i' such that $\beta_{s(i)} \in T_{i'}$ but $\beta_{s(i+1)} \notin T_{i'}$ and $s(m-1) = n-1$:

First we know that $\beta_{n-1} \in T_{n-1}$ and it is the least element of T_{n-1} greater than β_n. Hence $\mathrm{cof}(\beta_{n-1}) > \lambda$ since T_{n-1} is λ-threaded. Hence by definition of the game $G(\alpha_0, J^0)$, $\alpha^* = \alpha[\beta_{s(0)}, \dots, \beta_{s(m-1)}]$ is such that $\mathrm{cof}(\alpha^*) > \lambda$. Let i be least such that for all $i' \geq i$, $\alpha^* \in \mathcal{F}(\mathcal{E}^n_{i'})$. Let γ be least in $\bigcap_{i' \geq i} \mathcal{F}(\mathcal{E}^n_i) \cap \alpha^*$ such that for all

$$\delta' \in \delta \cap (T_n \setminus (\sup(T_{n-1} \cap \beta_n)))$$

we have

$$\alpha[\beta_{s(0)}, \dots, \beta_{s(m-1)}, \delta'] > \gamma.$$

Set $f(s(0), \dots, s(m-1), \delta) = \gamma$.

Now let

$$S_n = (\{\alpha[\delta_0, \dots, \delta_{m'-1}] \,|\, (\delta_0, \dots, \delta_{m'-1}) \in \mathrm{dom}(f)\} \cap \alpha_{n-1}) \cup (S_{n-1} \cap \alpha_{n-1}).$$

Set

$$\alpha_n = \alpha[\beta_{s(0)}, \dots, \beta_{s(m-1)}, \beta_n],$$

and let $\vec{a}^n$ be a j-layering of S_n with respect to $\vec{\mathcal{E}}^n$. I then plays $(\vec{a}^n, \alpha_n)$.

Now suppose that $\beta_n \in T_{n-1}$. Then we simply let $T_n = T_{n-1} \cap \beta_n + 1$ and we set

$$\alpha_n = \alpha[\delta_0, \dots, \delta_{m-1}]$$

where $(\delta_0, \ldots, \delta_{m-1}) \in W_n$ is the unique sequence satisfying that $\delta_{m-1} = \beta_n$. We set $S_n = S_{n-1} \cap \alpha_n + 1$ and let $\vec{a}^n$ be a j-layering of S_n with respect to $\vec{\mathcal{E}}^n$. I then plays $(\vec{a}^n, \alpha_n)$.

Clearly we have shown legal plays by I based on any finite sequence $\beta_0 > \beta_1 > \cdots$. Hence the induction is complete. $\square$

We immediately have the following theorems, which are our main results.

Theorem 22. *Suppose there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Then the supremum of rank$(j, \kappa, \vec{a})$ for all possible κ and $\vec{a}$ is Θ.*

Theorem 23. *Assume there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Let κ be least such that*

$$L_\kappa(V_{\lambda+1}) \not\prec_1^{V_{\lambda+1} \cup \{V_{\lambda+1}\}} L_{\kappa+1}(V_{\lambda+1}).$$

Then for all sets $X \subseteq V_{\lambda+1}$ such that $X \in L_\kappa(V_{\lambda+1})$, X is $U(j)$-representable in $L(V_{\lambda+1})$.

Proof. The theorem immediately follows by combining Theorem 148 of [6], Theorem 22, and the fact that the Tower Condition holds (see [1]). $\square$

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