Factoring elementary embeddings

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1 Factoring embeddings

Definition 1. We say that $j \in \mathcal{E}$ is *irreducible* if there is no $j_1, j_2 \in \mathcal{E} \setminus \{\mathrm{id}\}$ such that $j = j_1 \circ j_2$. We call such an equation $j = j_1 \circ j_2$ a *reduction of* j.

Lemma 2. *Suppose that $j_1, \ldots, j_n \in \mathcal{E}$. Then there is $k_1, \ldots, k_n \in \mathcal{E}$ such that.*

1. $j_n \circ \cdots \circ j_1 = k_n \circ \cdots \circ k_1$.
2. $crit(k_1) < crit(k_2) < \cdots < crit(k_n)$.
3. *If* $crit(j_1) < crit(j_i)$ *for all* $1 < i \leq n$, *then* $j_1 = k_1$.

Proof. We prove this by induction on n. It is immediate for $n = 1$. For $n > 1$, if $\mathrm{crit}\,(j_1) < \mathrm{crit}\,(j_i)$ for all $1 < i \leq n$, then applying the lemma to $j_2, \ldots, j_n$, to get $k_2, \ldots, k_n$, then clearly $j_1, k_2, k_3, \ldots, k_n$, witnesses the lemma.

If not, then let m be largest such that for all $1 \leq i \leq n$, $\mathrm{crit}\,(j_i) \geq \mathrm{crit}\,(j_m)$. We then have that

$$j_m \circ j_{m-1} \circ \cdots \circ j_1 = j_m(j_{m-1}) \circ j_m(j_{m-2}) \circ \cdots \circ j_m(j_1) \circ j_m$$

and we have for all $i < m$, that $\mathrm{crit}\,(j_m(j_{m-1})) > \mathrm{crit}\,(j_m)$. Combining with the fact that for all i with $m < i \leq n$, $\mathrm{crit}\,(j_i) > \mathrm{crit}\,(j_m)$, if we apply the lemma to

$$j_m(j_1), j_m(j_2), \ldots, j_m(j_{m-1}), j_{m+1}, j_{m+2}, \ldots, j_n$$

to get $k_2, \ldots, k_n$, we have that $j_m, k_2, \ldots, k_n$ witnesses the lemma. $\square$

Definition 3. Suppose that $j_1, j_2, \ldots \in \mathcal{E}$ and for all $\alpha < \lambda$ there exists an n such that for all $m \geq n$,

$$(j_n \circ j_{n-1} \circ \cdots \circ j_1)(\alpha) = (j_m \circ j_{m-1} \circ \cdots \circ j_1)(\alpha).$$

Then we define

$$k = \cdots \circ j_2 \circ j_1 = \lim n \to \omega j_n \circ j_{n-1} \circ \cdots j_1 \in \mathcal{E}$$

to be the direct limit of this system of embeddings. So for any $a \in V_\lambda$

$$k(a) = (j_n \circ j_{n-1} \circ \cdots \circ j_1)(a)$$

for n large enough.

Lemma 4. *For all $j \in \mathcal{E}$, there is $j_1, k \in \mathcal{E}$ such that we have*

$$j = k \circ j_1 = k(j_1) \circ k,$$

$k \neq id$ is irreducible and $crit(k) = crit(j)$.

Proof. We define by induction $\langle k_i, j_i | \, i < \omega \rangle$ such that $k_i, j_i \in \mathcal{E}$, $k_i, j_i \neq \mathrm{id}$ and

$$j = k_i \circ j_i \circ j_{i-1} \circ \cdots \circ j_1$$

for all $i < \omega$. Our induction proceeds as long as the lemma has not been satisfied so far. Suppose that j is not irreducible. Then let $j = k_1 \circ j_1$ be a reduction such that $(\mathrm{crit}\,(k_1), \mathrm{crit}\,(j_1))$ is lexicographically least among all such reductions. We must have then that $\mathrm{crit}\,(k_1) = \mathrm{crit}\,(j)$ since if not then $\mathrm{crit}\,(j_1) = \mathrm{crit}\,(j)$, and we have

$$j = k_1 \circ j_1 = k_1(j_1) \circ k_1.$$

Hence since $\mathrm{crit}\,(k_1(j_1)) = \mathrm{crit}\,(j_1) = \mathrm{crit}\,(j) < \mathrm{crit}\,(k_1)$, we would have a contradiction to our choice of k_1, j_1.

Now assume we have chosen $k_1, \ldots, k_n$ and $j_1, \ldots, j_n$ such that the following hold.

1. For all $i \leq n$, $j = k_i \circ j_i \circ j_{i-1} \circ \cdots \circ j_1$.
2. For all $i \leq n$, $(\mathrm{crit}\,(k_i), \mathrm{crit}\,(j_i))$ is lexicographically least among all reductions of k_{i-1}.
3. For all $i \leq n$, $\mathrm{crit}\,(k_i) = \mathrm{crit}\,(j) \leq \mathrm{crit}\,(j_1) \leq \mathrm{crit}\,(j_2) \leq \cdots, \leq \mathrm{crit}\,(j_n)$.

We let $k_{n+1}, j_{n+1} \in \mathcal{E}$ be such that $k_n = k_{n+1} \circ j_{n+1}$ is a reduction of k_n (if one exists–otherwise the lemma is satisfied) such that $(\mathrm{crit}\,(k_{n+1}), \mathrm{crit}\,(j_{n+1}))$ is lexicographically least among all such reductions. We have again that $\mathrm{crit}\,(k_{n+1}) = \mathrm{crit}\,(k_n) = \mathrm{crit}\,(j)$ as before. Also, $\mathrm{crit}\,(j_{n+1}) \geq \mathrm{crit}\,(j_n)$, since if not we would have that

$$k_{n-1} = k_n \circ j_n = k_{n+1} \circ j_{n+1} \circ j_n = k_{n+1} \circ (j_{n+1} \circ j_n)$$

is a reduction which has

$$(\mathrm{crit}\,(k_{n+1}), \mathrm{crit}\,(j_{n+1} \circ j_n)) <_{\mathrm{lex}} (\mathrm{crit}\,(k_n), \mathrm{crit}\,(j_n))$$

a contradiction.

Having defined these sequences, we claim that

$$\lim_{n \to \omega} \mathrm{crit}\,(j_n) = \lambda.$$

To see this, note that for all $\alpha < \lambda$ and $n < \omega$ we have that if $\mathrm{crit}\,(j_{n+1}) < \alpha$ then

$$k_n(\alpha) = (k_{n+1} \circ j_{n+1})(\alpha) > k_{n+1}(\alpha).$$

Hence there are only finitely many $n < \omega$ such that $\operatorname{crit}(j_n) < \alpha$. Which shows that $\lim_{n\to\omega} \operatorname{crit}(j_n) = \lambda$.

In fact for any $\alpha < \lambda$ there is an n such that for all $m \geq n$,

$$(j_n \circ j_{n-1} \circ \cdots \circ j_1)(\alpha) = (j_m \circ j_{m-1} \circ \cdots \circ j_1)(\alpha).$$

This follows since for all $n < \omega$,

$$j(\alpha) = (k_n \circ j_n \circ j_{n-1} \circ \cdots \circ j_1)(\alpha) \geq (j_n \circ j_{n-1} \circ \cdots \circ j_1)(\alpha).$$

And hence for n such that $\operatorname{crit}(j_n) > j(\alpha)$, this n has the desired property.

Now let $\ell = \cdots \circ j_2 \circ j_1$ and let $k = \lim_{n\to\omega} k_n$. Note that k_n and k_{n+1} agree up to $\operatorname{crit}(j_{n+1})$, and hence this limit makes sense. We have

$$k \circ \ell = (\lim_{n\to\omega} k_n) \circ (\lim_{n\to\omega} j_n \circ j_{n-1} \circ \cdots \circ j_1) = \lim_{n\to\omega} k_n \circ j_n \circ j_{n-1} \circ \cdots \circ j_1 = \lim_{n\to\omega} j = j.$$

Furthermore for $m \geq 1$ and $\ell_m = \cdots \circ j_{m+1} \circ j_m$ we have that

$$k \circ \ell_m = (\lim_{n\to\omega} k_n) \circ (\lim_{n\to\omega} j_n \circ j_{n-1} \circ \cdots \circ j_m) = \lim_{n\to\omega} k_n \circ j_n \circ j_{n-1} \circ \cdots \circ j_m = \lim_{n\to\omega} k_m = k_m.$$

We claim that k is irreducible. To see this, suppose not and let $k = k^1 \circ k^2$ be a reduction with $(\operatorname{crit}(k^1), \operatorname{crit}(k^2))$ lexicographically least. Let n_0 be the least n such that $\operatorname{crit}(j_n) > \operatorname{crit}(k^2)$. Then we have that

$$\begin{aligned} j = k \circ \ell &= k^1 \circ k^2 \circ \lim_{n\to\omega} k_n \circ j_n \circ j_{n-1} \circ \cdots \circ j_1 \\ &= k^1 \circ k^2 \circ (\lim_{n\to\omega} j_n \circ j_{n-1} \circ \cdots \circ j_{n_0}) \circ j_{n_0-1} \circ j_{n_0-2} \circ \cdots \circ j_1 \end{aligned}$$

But then for

$$k^* = k^2 \circ (\lim_{n\to\omega} j_n \circ j_{n-1} \circ \cdots \circ j_{n_0})$$

we have $k_{n_0-1} = k^1 \circ k^*$ and

$$(\operatorname{crit}(k^1), \operatorname{crit}(k^*)) <_{\text{lex}} (\operatorname{crit}(k_{n+1}), \operatorname{crit}(j_{n_0})),$$

a contradiction.

Hence k satisfies the lemma. □

Corollary 5. *Suppose $j \in \mathcal{E}$. Then there is a sequence $\langle k_n | n < \omega \rangle$ such that*

$$j = k_1 \circ k_2 \circ \cdots$$

and for all $n < \omega$, $k_n \neq id$ is irreducible or $k_n = id$.

Proof. We repeatedly apply the previous lemma by induction. First let $j = k_1 \circ j_1$ be a reduction of j such that k_1 is irreducible and $(\mathrm{crit}\,(k_1), \mathrm{crit}\,(j_1))$ is lexicographically least among all such reductions. Then by induction after defining $k_1, \dots, k_n$ and $j_1, \dots, j_n$, such that for all $i \leq n$,

$$j = k_1 \circ \cdots \circ k_i \circ j_i$$

and k_i is irreducible, let $k_{n+1} \circ j_{n+1} = j_n$ be a reduction such that k_{n+1} is irreducible (if j_n is not irreducible–otherwise we are done) and $(\mathrm{crit}\,(k_{n+1}), \mathrm{crit}\,(j_{n+1}))$ is lexicographically least.

Having defined $k_1, k_2, \dots$, we claim that

$$j = k_1 \circ k_2 \circ \cdots .$$

This follows since we must have

$$\lim_{n \to \omega} \mathrm{crit}\,(j_n) = \lim_{n \to \omega} \mathrm{crit}\,(k_n) = \lambda$$

since we chose the lexicographically least pairs and if the critical points of k_n were bounded below λ, they would form an inverse limit K with $\bar{\lambda}_K < \lambda$. But then clearly $j = k_1 \circ k_2 \circ \cdots$ by continuity. So this is the decomposition we wanted. □

We also isolate the following from the previous proof:

Lemma 6. *Suppose that $j = k_i \circ j_i \circ j_{i-1} \circ \cdots \circ j_1$ where $j, k_i, j_i \in \mathcal{E}$ for all i. Then $\lim_{n \to \omega}$ crit $(j_i) = \lambda$.*

Definition 7. We say that j and k are *right-relatively prime* if there is no $\ell \in \mathcal{E}$, $\ell \neq \mathrm{id}$ such that $j = j' \circ \ell$ and $k = k' = \ell$ for some $j', k' \in \mathcal{E}$.

Lemma 8. *Suppose that $j, k \in \mathcal{E}$. Then there is $j_1, k_1, \ell \in \mathcal{E}$ such that $j = j_1 \circ \ell$, $k = k_1 \circ \ell$ and j_1 and k_1 are relatively prime.*

Proof. We define by induction $\langle k_i, j_i, \ell_i | \, i < \omega \rangle$ with the following properties for all $i < \omega$.

1. $j = j_i \circ \ell_i \circ \ell_{i-1} \circ \cdots \circ \ell_1$ and $k = k_i \circ \ell_i \circ \ell_{i-1} \circ \cdots \circ \ell_1$.
2. $j_i = j_{i+1} \circ \ell_{i+1}$ and $k_i = k_{i+1} \circ \ell_{i+1}$.
3. $\mathrm{crit}\,\ell_{i+1}$ is the least $\mathrm{crit}\,(j')$ among all j', k', ℓ' with $\ell' \neq \mathrm{id}$ such that $j_i = j' \circ \ell'$ and $k_i = k' \circ \ell'$.

If we can only define this for finitely many i, then clearly the lemma can be satisfied at the first point where we can't continue. Otherwise we have defined k_i, j_i, ℓ_i for $i < \omega$ satisfying the above properties.

We have that

$$\lim_{n \to \omega} \mathrm{crit}\,(\ell_n) = \lambda$$

by the above lemma. Hence we can define

$$\ell = \lim_{n\to\omega}(\ell_n \circ \ell_{n-1} \circ \cdots \circ \ell_1), \qquad j^* = \lim_{n\to\omega} j_n, \qquad k^* = \lim_{n\to\omega} k_n.$$

And we have by continuity that

$$j = j^* \circ \ell \text{ and } k = k^* \circ \ell.$$

We claim that j^* and k^* are relatively prime. To see this, suppose that $j^* = \bar{j} \circ \ell^*$ and $k^* = \bar{k} \circ \ell^*$ for $\ell^* \neq \mathrm{id}$. Then let $n_0 < \omega$ be such that $\mathrm{crit}\,(\ell_{n_0}) > \mathrm{crit}\,(\ell^*)$. Then we have that

$$\begin{aligned} j = j^* \circ \ell &= \bar{j} \circ \ell^* \circ \lim_{n\to\omega} \ell_n \circ \ell_{n-1} \circ \cdots \circ \ell_1 \\ &= \bar{j} \circ \ell^* \circ (\lim_{n\to\omega} \ell_n \circ \ell_{n-1} \circ \cdots \circ \ell_{n_0}) \circ \ell_{n_0-1} \circ \cdots \circ \ell_1 \end{aligned}$$

and similarly for k. But setting

$$\ell^*_{n_0} = \ell^* \circ (\lim_{n\to\omega} \ell_n \circ \ell_{n-1} \circ \cdots \circ \ell_{n_0})$$

we have that

$$j = j^* \circ \ell^*_{n_0} \circ \ell_{n_0-1} \circ \ell_{n_0-2} \circ \cdots \circ \ell_1$$

and

$$k = k^* \circ \ell^*_{n_0} \circ \ell_{n_0-1} \circ \ell_{n_0-2} \circ \cdots \circ \ell_1.$$

Since $\mathrm{crit}\,(\ell^*_{n_0}) < \mathrm{crit}\,(\ell_{n_0})$, this is a contradiction to the way in which we chose ℓ_{n_0}. □

Definition 9. We say that j and k are *left-relatively prime* if there is no $\ell \in \mathcal{E}$, $\ell \neq \mathrm{id}$ such that $j = \ell \circ j'$ and $k = \ell \circ k'$ for some $j', k' \in \mathcal{E}$.

Lemma 10. *Suppose that $j = j_1 \circ j_2 \circ \cdots \circ j_i \circ k_i$ and $k_i = j_{i+1} \circ k_{i+1}$ where $j, k_i, j_i \in \mathcal{E}$ for all $i < \omega$. Then $\lim_{n\to\omega} crit\,(j_i) = \lambda$.*

Proof. For any $\alpha < \lambda$ we have that

$$k_i(\alpha) = (j_{i+1} \circ k_{i+1})(\alpha).$$

And hence if $\mathrm{crit}\,(j_{i+1}) \leq k_{i+1}(\alpha)$, then $k_i(\alpha) > k_{i+1}(\alpha)$. And so for large enough i we must have $\mathrm{crit}\,(j_{i+1}) > k_{i+1}(\alpha) \geq \alpha$. So since this is true for arbitrary $\alpha < \lambda$, we must have $\lim_{n\to\omega} \mathrm{crit}\, j_i = \lambda$. □

Lemma 11. *Suppose that $j, k \in \mathcal{E}$. Then there is $j_1, k_1, \ell \in \mathcal{E}$ such that $j = \ell \circ j_1$, $k = \ell \circ k_1$ and j_1 and k_1 are left-relatively prime.*

Proof. We define by induction $\langle k_i, j_i, \ell_i | \, i < \omega\rangle$ with the following properties for all $i < \omega$.

1. $j = \ell_1 \circ \ell_2 \circ \cdots \circ \ell_i \circ j_i$ and $k = \ell_1 \circ \ell_2 \circ \cdots \circ \ell_i \circ k_i$.

2. $j_i = \ell_{i+1} \circ j_{i+1}$ and $k_i = \ell_{i+1} \circ k_{i+1}$.

3. $\operatorname{crit} \ell_{i+1}$ is the least $\operatorname{crit}(\ell')$ among all j', k', ℓ' with $\ell' \neq \mathrm{id}$ such that $j_i = \ell' \circ j'$ and $k_i = \ell' \circ k'$.

If we can only define this for finitely many i, then clearly the lemma can be satisfied at the first point where we can't continue. Otherwise we have defined k_i, j_i, ℓ_i for $i < \omega$ satisfying the above properties.

We have that

$$\lim_{n \to \omega} \operatorname{crit}(\ell_n) = \lambda$$

by the above lemma. Hence we can define

$$\ell = \lim_{n \to \omega} (\ell_n \circ \ell_{n-1} \circ \cdots \circ \ell_1), \qquad j^* = \lim_{n \to \omega} j_n, \qquad k^* = \lim_{n \to \omega} k_n.$$

And we have by continuity that

$$j = \ell \circ j^* \text{ and } k = \ell \circ k^*.$$

We claim that j^* and k^* are left-relatively prime. To see this, suppose that $j^* = \ell^* \circ \bar{j}$ and $k^* = \ell^* \circ \bar{k}$ for $\ell^* \neq \mathrm{id}$. Then let $n_0 < \omega$ be such that $\operatorname{crit}(\ell_{n_0}) > \operatorname{crit}(\ell^*)$. Then we have that

$$\begin{aligned} j = \ell \circ j^* &= (\lim_{n \to \omega} \ell_1 \circ \ell_2 \circ \cdots \circ \ell_n) \circ \ell^* \circ \bar{j} \\ &= \ell_1 \circ \ell_2 \circ \cdots \ell_{n_0 - 1} \circ (\lim_{n \to \omega} \ell_{n_0} \circ \ell_{n_0+1} \circ \cdots \circ \ell_n) \circ \ell^* \circ \bar{j} \end{aligned}$$

and similarly for k. But setting

$$\ell^*_{n_0} = (\lim_{n \to \omega} \ell_{n_0} \circ \ell_{n_0+1} \circ \cdots \circ \ell_n) \circ \ell^*$$

we have that

$$j = \ell_1 \circ \ell_2 \circ \cdots \ell_{n_0-1} \circ \ell^*_{n_0} \circ \bar{j}$$

and

$$k = \ell_1 \circ \ell_2 \circ \cdots \ell_{n_0-1} \circ \ell^*_{n_0} \circ \bar{k}.$$

Since $\operatorname{crit}(\ell^*_{n_0}) < \operatorname{crit}(\ell_{n_0})$, this is a contradiction to the way in which we chose ℓ_{n_0}. $\square$

Lemma 12. *Suppose that $j, k, \ell \in \mathcal{E}$ and $j = k \circ \ell$. Then for any ℓ' such that $j = k \circ \ell'$, $\ell = \ell'$.*

Proof. This follows from the fact that we can write ℓ and ℓ' as $\ell = k^{-1} \circ j = \ell'$. $\square$