Generic absoluteness from tree representations in $L(V_{\lambda+1})$

Scott Cramer

July 27, 2015

Abstract

We define representations for subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ under the assumption I_0 which are similar to Suslin representations. We show that uniform versions of these representations give generic absoluteness results for $L(V_{\lambda+1})$ similar to generic absoluteness results of H. Woodin [10] from $U(j)$-representations.

A property is generically absolute if it cannot be changed by forcing. One important example of generic absoluteness is the theorem of H. Woodin that if there is a proper class of Woodin cardinals then the theory of $L(\mathbb{R})$ is generically absolute (see [6]). In this paper we consider generic absoluteness for a similar structure $L(V_{\lambda+1})$ under the assumption of I_0 (see [4] for an introduction to $L(V_{\lambda+1})$). While $L(V_{\lambda+1})$ and $L(\mathbb{R})$ have remarkably similar properties (assuming enough large cardinals), the theory of $L(V_{\lambda+1})$ is not generically absolute, as the theory of V_λ is not generically absolute. Nevertheless Woodin [10] formulated a restricted version of generic absoluteness for $L(V_{\lambda+1})$ using a (uniform version of a) representation for subsets of $V_{\lambda+1}$ called a $U(j)$-representation. In order to extend his results in [10] an obvious approach is to extend the subsets of $V_{\lambda+1}$ which have $U(j)$-representations. We take an alternative approach here (although our techniques were originally motivated by this goal) and instead define a different representation which achieves a slightly stronger generic absoluteness result. While $U(j)$-representations are analogous to weakly-homogeneously Suslin representations in the context of $\mathbb{R}$, we define representations which seem analogous to Suslin representations. The main difference is that instead of considering trees of ordinals, we consider trees on fixed points of iterates of our I_0-embedding j.

It can be shown that j-Suslin representations do exist for all subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ assuming I_0, though we leave this verification for the sequel (see [2]). One consequence of our results together with this existence theorem is obtaining the consistency of I_0 at λ together with the failure of the singular cardinal hypothesis at λ from $I_0^\#$. This theorem follows by results independently shown by Dimonte-Friedman and Woodin. In addition, our generic absoluteness result, by a theorem of Shi-Woodin gives a new proof of the perfect set property for subsets of $V_{\lambda+1}$.

1 Inverse Limits

We first recall the definition of I_0: we say that I_0 *holds at* λ if there is a non-trivial elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

such that $\mathrm{crit}\,(j) < \lambda$. We call such a j an I_0 *embedding*. Below we will always assume our elementary embeddings are non-trivial and that their critical points are below λ. We refer the reader to [4] for an introduction to $L(V_{\lambda+1})$. Recall that in this context $\Theta = \Theta_\lambda$ is the sup of ordinal α such that in $L(V_{\lambda+1})$ there is a surjection of $V_{\lambda+1}$ onto α.

We now give a very brief outline of the theory of inverse limits in the context of $L(V_{\lambda+1})$. These structures were originally used for reflecting large cardinal hypotheses of the form: there exists an elementary embedding $L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$, although they now have a variety of uses when working with $L(V_{\lambda+1})$. The use of inverse limits in reflecting such large cardinals is originally due to Laver [7]. For an introduction to the theory of inverse limits see [4], [7], and [8].

Suppose that $\langle j_i |\, i < \omega\rangle$ is a sequence of elementary embeddings such that the following hold:

1. For all i, $j_i : V_\lambda \to V_\lambda$ is elementary.

2. There exists $\bar\lambda < \lambda$ such that $\mathrm{crit}\, j_0 < \mathrm{crit}\, j_1 < \cdots < \bar\lambda$ and $\lim_{i<\omega} \mathrm{crit}\, j_i = \bar\lambda =: \bar\lambda_J$.

Then we can form the inverse limit

$$J = j_0 \circ j_1 \circ \cdots : V_{\bar\lambda} \to V_\lambda$$

by setting

$$J(a) = \lim_{i\to\omega} (j_0 \circ \cdots \circ j_i)(a)$$

for any $a \in V_{\bar\lambda}$. $J : V_{\bar\lambda} \to V_\lambda$ is elementary, and can be extended to a Σ_0-embedding $J^* : V_{\bar\lambda+1} \to V_{\lambda+1}$ by $J(A) = \bigcup_i J(A \cap V_{\bar\lambda_i})$ for $\langle \bar\lambda_i |\, i < \omega\rangle$ any cofinal sequence in $\bar\lambda$.

Note that if $j : V_\lambda \to V_\lambda$ is elementary then it naturally extends to a function $j : V_{\lambda+1} \to V_{\lambda+1}$, which we also refer to as j. Hence we will make statements below such as $k(k) = j$ where $j, k : V_\lambda \to V_\lambda$ are elementary, using this convention.

Suppose $J = j_0 \circ j_1 \circ \cdots$ is an inverse limit. Then for $i < \omega$ we write $J_i := j_i \circ j_{i+1} \circ \cdots$, the inverse limit obtained by 'chopping off' the first i embeddings. For $i < \omega$ we write

$$J^{(i)} := (j_0 \circ \cdots \circ j_i)(J)$$

and for $n < \omega$,

$$J^{(i)}_n := (j_0 \circ \cdots \circ j_i)(J_n),\ j^{(i)}_n := (j_0 \circ \cdots \circ j_i)(j_n).$$

When indexing inverse limits as $\langle J^m |\, m < \omega\rangle$ for instance, we will combine our notations as in

$$J^{m,(i)}_n = (j^m_0 \circ \cdots j^m_i)(J^m_n).$$

We can also rewrite J in the following useful ways:

$$\begin{aligned} J = j_0 \circ j_1 \circ \cdots &= \cdots (j_0 \circ j_1)(j_2) \circ j_0(j_1) \circ j_0 \\ &= \cdots j_2^{(1)} \circ j_1^{(0)} \circ j_0 \end{aligned}$$

and

$$\begin{aligned} J &= j_0 \circ J_1 = j_0(J_1) \circ j_0 = J_1^{(0)} \circ j_0 \\ &= (j_0 \circ \cdots \circ j_{i-1})(J_i) \circ j_0 \circ \cdots \circ j_{i-1} = J_i^{(i-1)} \circ j_0 \circ \cdots \circ j_{i-1} \end{aligned}$$

for any $i > 0$. Hence we can view an inverse limit J as a direct limit.

We let $\mathcal{E}$ be the set of inverse limits. So

$$\begin{aligned} \mathcal{E} = \{(J, \langle j_i | \, i < \omega\rangle) | \, \forall i < \omega (j_i : V_\lambda \to V_\lambda \text{ is elementary}), \operatorname{crit}(j_0) < \operatorname{crit}(j_1) < \cdots, \\ \text{and } J = j_0 \circ j_1 \circ \cdots : V_{\bar\lambda_J} \to V_\lambda \text{ is elementary}\}. \end{aligned}$$

This is a slightly larger collection than is defined in [4], and it has the added benefit of being closed in the natural sense, which we will use to our advantage below.

We will many times be sloppy and refer to an inverse limit as 'J', '$(J, \vec{j})$' or '$(J, \langle j_i \rangle)$' instead of '$(J, \langle j_i | \, i < \omega\rangle)$'.

Define

$$\mathcal{E}_\alpha = \{(J, \vec{j}) \in \mathcal{E} | \, \forall i < \omega \, (j_i \text{ extends to an elementary embedding } L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

We say that α is *good* if every element of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from elements of $V_{\lambda+1}$. Note that the good ordinals are cofinal in Θ.

Lemma 1 (Laver)**.** *Suppose there exists an elementary embedding*

$$j : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

where α is good. Then $\mathcal{E}_\alpha \neq \emptyset$. In fact for any $a \in V_{\lambda+1}$ there is $(K, \vec{k}) \in \mathcal{E}_\alpha$ such that $a \in rng\, K$.

Lemma 2 (Laver)**.** *Suppose $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$ for α good, $\bar a \in V_{\bar\lambda_J+1}$, and $b \in V_{\lambda+1}$. Then there is $(K, \vec{k}) \in \mathcal{E}_\alpha$ such that $\bar\lambda_J = \bar\lambda_K$,*

$$K(\bar a) = J(\bar a) \qquad \textit{and} \qquad b \in rng\, K.$$

An important property of inverse limits is to what extend they extend beyond $V_{\lambda+1}$ (see [4] and [3]), and we will use such extensions below. However, in the next section we will consider a different type of extension, where we use inverse limits more as operators than embeddings. With that in mind we make the following definition.

Definition 3. For $\alpha < \Theta$ set

$$\mathcal{E}^e_\alpha = \{(J, \vec{j}) |\, (J, \langle j_i \restriction V_\lambda |\, i < \omega\rangle) \in \mathcal{E}, \forall i(j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

Suppose that $(J, \vec{j}) \in \mathcal{E}^e_\alpha$. Then we say that $a \in L_\alpha(V_{\lambda+1})$ is in the *extended range of* J if for all $i < \omega$, $a \in \operatorname{rng}(j_0 \circ \cdots \circ j_i)$.

Definition 4. Suppose

$$(J, \langle j_i\rangle), (K, \langle k_i\rangle) \in \mathcal{E}.$$

Then we say that K *is a limit root of* J if there is $n < \omega$ such that $\bar{\lambda}_J = \bar{\lambda}_K$ and

$$\forall i < n\, (k_i = j_i) \text{ and } \forall i \geq n\, (k_i(k_i) = j_i).$$

We say K *is an n-close limit root of* J if n witnesses that K is a limit root of J. We also say that K and J *agree up to* n if for all $i < n$, $j_i = k_i$.

Also for $j : V_{\lambda+1} \to V_{\lambda+1}$ elementary and $(K, \vec{k}) \in \mathcal{E}$ we say that K *is a limit root of* j if for all $i < \omega$, $k_i(k_i) = j$ and for all $n < i$, $k_n \in \operatorname{rng} k_i$.

A basic fact which we will use repeatedly below without comment is the following.

Lemma 5. *Suppose $j, k : V_\lambda \to V_\lambda$ are elementary and $k(k) = j$. Then if $a \in V_{\lambda+1}$ is such that $a \in rng\, k$, then $k(a) = j(a)$.*

Proof. Suppose that $\langle \kappa_i |\, i < \omega\rangle$ is increasing cofinal in λ and for all $i < \omega$, $\kappa_i \in \operatorname{rng} k$. It is enough to see that for all $i < \omega$ that $k(a \cap V_{\kappa_i}) = j(a \cap V_{\kappa_i})$, and for this it is enough to show the lemma holds for $a \in V_\lambda$ arbitrary. Hence we calculate for $k(\bar{a}) = a$,

$$k(a) = k(k(\bar{a})) = k(k \restriction V_\alpha(\bar{a})) = j \restriction V_{k(\alpha)}(k(\bar{a})) = j(a)$$

where $\alpha < \lambda$ is large enough. Hence the lemma follows. □

2 Inverse limit operators

In this section we consider a kind of 'naive extension' of an inverse limit. These extensions are well-defined on elements which are fixed by iterates of j, and they are therefore useful in working with M_ω, the ωth iterate of $L(V_{\lambda+1})$ by j. We prove various properties of these extensions that will be useful in the proofs of Theorems 16 and 19.

We first fix some notation. Fix $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ elementary and iterable and let

$$j_{0,\omega} : L(V_{\lambda+1}) \to M_\omega$$

be the embedding into the ωth iterate of $L(V_{\lambda+1})$ by j. Similarly define $j_{n,\omega}$. Suppose that $(K, \vec{k}) \in \mathcal{E}^e_\alpha$ for some α is a limit root of j. We put

$$K^{\text{ext}}(a) = \lim_{i \to \omega} (k_0 \circ \cdots \circ k_i)(a)$$

if this limit exists (in the sense that for all large enough i, $k_i(a) = a$) and we put

$$K^*(a) = \lim_{i\to\omega}(k_0 \circ \cdots \circ k_i)^{-1}(a)$$

if this limit exists (in a similar sense). Furthermore we use the convention that if $\beta' < \operatorname{crit} K$ and $b = \langle b_\beta | \, \beta < \beta' \rangle$ is a sequence such that for all $\beta < \beta'$, $K^{\text{ext}}(b_\beta)$ exists, then

$$K^{\text{ext}}(b) = \left\langle K^{\text{ext}}(b_\beta) | \, \beta < \beta' \right\rangle .$$

We make a similar convention with K^*.

We first show some calculations involving K^{ext} and K^* as they relate to M_ω, the ωth iterate of $L(V_{\lambda+1})$ by j.

The first basic fact to keep in mind is the following.

Lemma 6. *Suppose that $a \in L_\alpha(V_{\lambda+1})$ for $\alpha < \Theta$. Then $j(j_{0,\omega}(a)) = j_{0,\omega}(a)$. Also if α is a good limit ordinal such that $j_{0,\omega}(\alpha) = \alpha$, $k : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$ is elementary, and $k(k \restriction V_\lambda) = j \restriction V_\lambda$, then we have*

$$a \in rng\, k \Rightarrow k(j_{0,\omega}(a)) = j_{0,\omega}(a).$$

Proof. For the first part we do the following calculation,

$$j(j_{0,\omega}(a)) = j(j_{0,\omega} \restriction L_\alpha(V_{\lambda+1})(a)) = j_{1,\omega} \restriction L_{j(\alpha)}(V_{\lambda+1})(j(a)) = j_{0,\omega}(a).$$

The second part follows by noticing that since $j_{0,\omega}(\alpha) = \alpha$, for arbitrarily large $\beta < \alpha$, $j_{0,\omega} \restriction L_\beta(V_{\lambda+1}) \in \operatorname{rng} k$. And hence $j_{0,\omega}(a) \in \operatorname{rng} k$, which implies that

$$k(j_{0,\omega}(a)) = j(j_{0,\omega}(a)) = j_{0,\omega}(a),$$

which is what we wanted. □

Lemma 7. *Suppose that $(K, \vec{k}) \in \mathcal{E}^e_{\alpha+1}$ for some good limit α is an inverse limit root of j, and $j_{0,\omega}(\alpha) = \alpha$. Let $a \in L_\alpha(V_{\lambda+1})$ and assume that for all $i < \omega$, $a \in rng\, k_i$. Then for all i, $a \in rng\, k_0 \circ \cdots \circ k_i$, and*

$$K^{ext}(j_{0,\omega}(a)) = K^*(j_{0,\omega}(a)) = j_{0,\omega}(a).$$

Proof. First, the fact that for all i, $a \in \operatorname{rng} k_0 \circ \cdots \circ k_i$ follows from the definition of an inverse limit root, the fact that α is good and $(K, \vec{k}) \in \mathcal{E}^e_{\alpha+1}$. Namely, for all i, we have that

$$a, k_0 \restriction V_\lambda, \cdots, k_{i-1} \restriction V_\lambda \in \operatorname{rng} k_i.$$

And hence we must have that $(k_0 \circ \cdots k_{i-1})^{-1}(a) \in \operatorname{rng} k_i$ by induction.

Now to see the latter part of the lemma, we have by Lemma 6

$$\begin{aligned} K^*(j_{0,\omega}(a)) &= \lim_{n\to\omega}(k_0 \circ \cdots \circ k_n)^{-1}(j_{0,\omega}(a)) \\ &= \lim_{n\to\omega}(j \circ \cdots \circ j)^{-1}(j_{0,\omega}(a)) = j_{0,\omega}(a). \end{aligned}$$

And this is what we wanted. Similarly we have $K^{\text{ext}}(j_{0,\omega}(a)) = j_{0,\omega}(a)$. □

We now prove an important lemma which shows how K^* acts on sequences of elements of M_ω. This fact is especially important below when we consider generics over M_ω such as the critical sequence of j.

Lemma 8. *Suppose that $(K, \vec{k}) \in \mathcal{E}^e_{\alpha+1}$ for some good limit α is an inverse limit root of j, and $j_{0,\omega}(\alpha) = \alpha$. Let M_ω be the ωth iterate of $L(V_{\lambda+1})$ by j. Then K^* has the property that*

$$K^* \restriction (M_\omega)^\omega : (M_\omega)^\omega \cap \bigcap_{i<\omega} rng\, k_i \to (j_{0,\omega}[L_\alpha(V_{\lambda+1})])^\omega.$$

Furthermore we have that for all $a \in (M_\omega)^\omega \cap \bigcap_{i<\omega} rng\, k_i$,

$$K^{ext}(K^*(a)) = a.$$

Proof. Suppose that

$$f^\omega \in (M_\omega)^\omega \cap \bigcap_{i<\omega} \operatorname{rng} k_i.$$

Then for all $n < \omega$ there is an i_n such that $f^\omega \restriction n = j_{i_n\omega}(f_n)$ for some $f_n \in L_\alpha(V_{\lambda+1})$. Clearly $f_n \in \bigcap_{i<\omega} \operatorname{rng} k_i$. Now we calculate:

$$\begin{aligned} K^*(f^\omega \restriction n) &= K^*(j_{i_n\omega}(f_n)) \\ &= \lim_{m\to\omega} (k_0 \circ \cdots \circ k_m)^{-1}(j_{i_n\omega}(f_n)) \\ &= \lim_{m\to\omega} (k_{i_n} \circ \cdots \circ k_m)^{-1}(j_{0\omega}((k_0 \circ \cdots \circ k_{i_n-1})^{-1}(f_n))) \\ &= j_{0\omega}((k_0 \circ \cdots \circ k_{i_n-1})^{-1}(f_n)), \end{aligned}$$

where for the third equality we have applied the first i_n-many embeddings and for the fourth equality we used Lemma 7. And since this holds for any $n < \omega$, we have that

$$K^*(f) \in j_{0,\omega}[L_\alpha(V_{\lambda+1})]^\omega,$$

which is what we wanted for the first part of the lemma.

The second part of the lemma follows by basically the same calculation as above, but performed in reverse, applying Lemma 7 once again. $\square$

We need the following technical lemma involving K^{ext} and the direct limit form of K in order to prove Lemma 10. First define for κ an ordinal

$$\mathcal{F}_\kappa(j) = \{a \in L_\kappa(V_{\lambda+1}) |\, j(a) = a\}$$

and let, for $j_{(n)}$ the nth iterate of j,

$$\mathcal{F}^\omega_\kappa(j) = \bigcup_{n<\omega} \mathcal{F}_\kappa(j_{(n)}).$$

Set $N_j(a)$ to be the least n such that $j_{(n)}(a) = a$ if it exists. This lemma shows how K^* acts on elements of $\mathcal{F}^\omega_\kappa(j)$.

Lemma 9. *Suppose that $a \in \mathcal{F}^\omega_\kappa(j)$ for some $\kappa < \Theta$ good. Then for any $(K, \vec{k}) \in \mathcal{E}^e_{\kappa+1}$ a limit root of j such that for all $i < \omega$, $a \in rng\, k_i$, we have that $a \in rng\, K^{ext}$. In particular, let*

$$k_0(a_0) = a, k_1(a_1) = a_0, \ldots.$$

Then there exists an m (in fact $m = N_j(a)$ works) such that for all $n \geq m$, $a_n = a_m$. Furthermore, for such m we have that for all $n \geq m$, $k_n^{(n-1)}(a) = a$.

Proof. Let m be such that $j_{(m-1)}(a) = a$. We prove by induction that for $n \geq m$ we have $k_n(a_n) = a_n$. First suppose that $m = 1$. Then $j(a) = a$. We have that

$$j(a) = a \Rightarrow k_0(k_0)(a) = a \Rightarrow k_0(a_0) = a_0.$$

And hence $a_0 = a$. The fact that $k_n(a_n) = a_n$ follows by induction.

Now suppose that $m > 1$. Assume by induction that we have proved the result for all $m' < m$. Then we have for $n = m - 1$

$$j_{(n)}(a) = a \Rightarrow (k_0(k_0))_{(n)}(a) = a \Rightarrow (k_0)_{(n)}(a_0) = a_0 \Rightarrow j_{(n-1)}(a_0) = a_0.$$

And then using the induction hypothesis on a_0 and $\langle k_i | \, i \geq 1\rangle$ we have the first result.

To see the second result, simply note that $a_{m-1} = k_m(a_m) = a_m$, and hence

$$\begin{aligned}&k_m(a_m) = a_m \Rightarrow k_m(a_{m-1}) = a_{m-1} \Rightarrow \\ &\quad k_m^{(m-1)}((k_0 \circ \cdots \circ k_{m-1})(a_{m-1})) = (k_0 \circ \cdots \circ k_{m-1})(a_{m-1}) \Rightarrow \\ &\quad k_m^{(m-1)}(a) = a,\end{aligned}$$

for any $m \geq n$, for n satisfying the first part of the conclusion (where $a_{-1} = a$). □

The next lemma shows how $j_{0,\omega}$ and K^{ext} commute with one another. Recall the definition of $N_j(a)$ as the least iterate of j which fixes a.

Lemma 10. *Suppose that $a \in \mathcal{F}^\omega_\kappa(j)$ for some good κ a limit such that $j_{0,\omega}(\kappa) = \kappa$. Then for any $(K, \vec{k}) \in \mathcal{E}^e_{\kappa+1}$ such that $a \in rng\, K^{ext}$, we have that*

$$K^{ext}(j_{0,\omega}(K^*(a))) = j_{N_j(a),\omega}(a).$$

Proof. To see this, let

$$k_0(a_0) = a, k_1(a_1) = a_0, k_2(a_2) = a_1, \ldots$$

and let n be large enough so that for all $m \geq n$, $k_m(a_n) = a_n$. Then we have that $K^*(a) = a_n$. Hence we have that for $m \geq n$, $k_m(j_{0,\omega}(a_n)) = j_{0,\omega}(a_n)$. So

$$K^{\mathrm{ext}}(j_{0,\omega}(K^*(a))) = (k_0 \circ k_1 \circ \cdots \circ k_{n-1})(j_{0,\omega}((k_0 \circ k_1 \circ \cdots k_{n-1})^{-1}(a))) = j_{n,\omega}(a),$$

which holds for $n = N_j(a)$ in particular. □

The next lemma gives more information on how $j_{0,\omega}$ and K^{ext} commute.

Lemma 11. *Suppose that $(K,\vec{k}) \in \mathcal{E}^e_{\alpha+1}$ for some good limit α is an inverse limit root of j, and $j_{0,\omega}(\alpha) = \alpha$. Let M_ω be the ωth iterate of $L(V_{\lambda+1})$ by j. Suppose that $a \in M_\omega$, $a \in domK^*$ and*

$$a \in \mathcal{F}^\omega_\alpha(j_{0,\omega}(j))^{M_\omega}.$$

Then we have that $j^{-1}_{0,\omega}(K^*(a)) \in domK^{ext}$.

Proof. To see this, let $\bar{a}$ and n be such that $j_{n,\omega}(\bar{a}) = a$. Then by elementarity we have that $\bar{a} \in \mathcal{F}^\omega_\alpha(j)$. Furthermore since $a \in \text{dom}K^*$, $\bar{a} \in \text{dom}K^*$. So for $\bar{a}^* = K^*(\bar{a})$, we have that $\bar{a}^* \in \text{dom}K^{\text{ext}}$. But $j^{-1}_{0,\omega}(K^*(a)) = \bar{a}^*$ by Lemma 9, since

$$K^*(a) = K^*(j_{n,\omega}(\bar{a})) = j_{0,\omega}((k_0 \circ \cdots \circ k_{n-1})^{-1}(\bar{a}))$$

and

$$\bar{a}^* = K^*(\bar{a}) = (k_0 \circ \cdots \circ k_{n-1})^{-1}(\bar{a}).$$

Hence we have the lemma. □

Finally, we show how $j_{0,\omega}$ and K^{ext} commute on functions.

Lemma 12. *Let κ be a good limit ordinal and $j_{0,\omega}(\kappa) = \kappa$. Suppose that $f \in L_\kappa(V_{\lambda+1})$ is a function such that $j(f) = f$ and $f \subseteq \mathcal{F}^\omega_\kappa(j)$. Then for any $a \in domf$ and $(K,\vec{k}) \in \mathcal{E}^e_{\kappa+1}$ a limit root of j such that $a, f \in rng\,K^{ext}$, we have that*

$$j_{0,\omega}(f)(j_{N_j(a),\omega}(a)) = K^{ext}(j_{0,\omega}(K^*(f(a)))).$$

Proof. This follows by a calculation since $K^{\text{ext}}(f) = K^*(f) = f$ implies that

$$K^*(f(a)) = f(K^*(a))$$

and

$$\begin{aligned} K^{\text{ext}}(j_{0,\omega}(f(K^*(a)))) &= K^{\text{ext}}(j_{0,\omega}(f)(j_{0,\omega}(K^*(a)))) \\ &= j_{0,\omega}(f)(K^{\text{ext}}(j_{0,\omega}(K^*(a)))) \\ &= j_{0,\omega}(f)(j_{N_j(a),\omega}(a)), \end{aligned}$$

which completes the proof. □

3 j-Suslin representations

We introduce some terminology for tree representations of subsets of $V_{\lambda+1}$. These tree representations seem rather similar to Suslin representations, and so we give them names which indicate this fact.

For this section we fix $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ an iterable elementary embedding witnessing I_0 at λ and let

$$j_{0,\omega} : L(V_{\lambda+1}) \to M_\omega$$

be the embedding into the ωth iterate of $L(V_{\lambda+1})$ by j, as in the previous section. For $n < \omega$, let $j_{(n)}$ denote the nth iterate of j. Also define

$$E[j] = \{k : V_\lambda \to V_\lambda | \, \exists n, m(k_{(n)} = j_{(m)})\}.$$

Note that $j(E[j]) = E[j]$ and that for $\kappa < \Theta$ good, $\mathcal{F}^\omega_\kappa(j)$ is definable from κ and $E[j]$ in $L(V_{\lambda+1})$. Hence if $j(\kappa) = \kappa$ then $j(\mathcal{F}^\omega_\kappa(j)) = \mathcal{F}^\omega_\kappa(j)$.

Definition 13. For $\vec{\kappa} = \langle \kappa_i | \, i < \omega \rangle$ increasing below λ, we let $\mathcal{W}^{\vec{\kappa}}$ be the set of sequences s such that

1. for some $n < \omega$, $|s| = n$ and for all $i < n$, $s(i) \subseteq V_{\kappa_i}$,
2. if $i \leq m < |s|$ then $s(i) = s(m) \cap V_{\kappa_i}$.

Also let $\mathcal{W}^{\vec{\kappa}}_n = \{s \in W^{\vec{\kappa}} | \, |s| = n\}$ and

$$\mathcal{W}^{\vec{\kappa}}_\omega = \{x \in V^\omega_\lambda | \, \forall n < \omega (x \restriction n \in \mathcal{W}^{\vec{\kappa}}_n)\}.$$

In this context if $x \in V_{\lambda+1}$, we set

$$\hat{x} = \hat{x}_{\vec{\kappa}} = \langle x \cap V_{\kappa_n} | \, n < \omega \rangle \in \mathcal{W}^{\vec{\kappa}}_\omega.$$

Suppose that $\kappa < \Theta$. Let $X \subseteq V_{\lambda+1}$. We say that T is a (j, κ)-*Suslin representation* for X if for some sequence $\langle \kappa_i | \, i < \omega \rangle$ increasing and cofinal in λ the following hold.

1. T is a (height ω) tree on $V_\lambda \times \mathcal{F}^\omega_\kappa(j)$ such that for all $(s, a) \in T$, $s \in \mathcal{W}^{\vec{\kappa}}_{|s|}$.
2. For all $s \in \mathcal{W}^{\vec{\kappa}}$, $T_s \in \mathcal{F}^\omega_\Theta(j)$.
3. For all $x \in V_{\lambda+1}$, $x \in X$ iff $T_{\hat{x}}$ is illfounded.

We say that X is j-Suslin if for some κ, X has a (j, κ)-Suslin representation.

Remark 14. Note that if we did not require T_s be in $\mathcal{F}^\omega_\Theta(j)$, there would be a seemingly trivial such representation by considering the pointwise image of X under $j_{0\omega}$. On the other hand, we will only be obtaining results from the uniform version of this definition which we make below, and hence it is not necessarily clear that our definition of (j, κ)-Suslin is strong enough to obtain a useful representation for subsets of $V_{\lambda+1}$.

We say that T is a *uniform* (j, κ)-*Suslin representation* for X if the following hold.

1. T is a function on $[\lambda]^{<\omega}$ such that for all $s \in [\lambda]^\omega$, if $T(s)$ is the tree whose nth level is given by $T(s \restriction n)$, then $T(s)$ is a (height ω) tree on $V_\lambda \times \mathcal{F}^\omega_\kappa(j)$.

2. For all $s \in [\lambda]^\omega$ such that s is cofinal in λ, $T(s)$ is a (j,κ)-Suslin representation for X (as witnessed by the sequence s).

Remark 15. Our definition of a j-Suslin representation was heavily motivated by the definition of a $U(j)$-representation[10]. Namely, it can be viewed roughly as a $U(j)$-representation where we have forgotten about the measures. This is not strictly true however, and it is not at all clear that $U(j)$-representations give j-Suslin representations directly. On the other hand j-Suslin representations are motivated by Suslin representations since (for iterable j), $\kappa \subseteq \mathcal{F}^\omega_\kappa(j)$.

We now give some consequences of the existence of uniform j-Suslin representations. First we show that if a set has a uniform j-Suslin representation, then this implies it has a nonempty intersection with $M_\omega[\vec{\kappa}]$ for $\vec{\kappa}$ the critical sequence of j. The following two theorems generalize this result slightly to allow for parameters in $M_\omega[\vec{\kappa}]$ and for replacing $\vec{\kappa}$ with other suitable generics. The proof of Theorem 16 is very similar to the proofs of the latter two theorems, although Theorem 19 is what we will actually apply below.

Theorem 16. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an iterable I_0 embedding and let M_ω be the ωth iterate of $L(V_{\lambda+1})$ by j. Suppose that $T \in L_\Theta(V_{\lambda+1})$ is a uniform (j,κ)-Suslin representation for $X \subseteq V_{\lambda+1}$, $X \neq \emptyset$, and for some n, $j_{(n)}(T) = T$. Then for $\vec{\kappa}$ the critical sequence of j, we have that*

$$M_\omega[\vec{\kappa}] \cap X \neq \emptyset.$$

Proof. Assume without loss of generality that $j(T) = T$ and fix κ good such that $T \in L_\kappa(V_{\lambda+1})$ and $j(\kappa) = \kappa$. Let $j_{0,\omega} : L(V_{\lambda+1}) \to M_\omega$ be the embedding into the ωth iterate given by j. Let $j_{0,\omega}(T) = T^\omega$. And let $T^\omega_\lambda = T^\omega \restriction [\lambda]^{<\omega}$. Now let $T^* = T^\omega_\lambda(\vec{\kappa})$. Note that we have $T^* \in M_\omega[\vec{\kappa}]$.

Claim 17. *T^* is illfounded.*

Proof. To see this, note that since $X \neq \emptyset$, $T(\vec{\kappa})$ is illfounded. So let $b = \langle b_i | \, i < \omega\rangle$ be a branch through $T(\vec{\kappa})$. Now, using Lemma 1, let

$$(K, \langle k_i | \, i < \omega\rangle) \in \mathcal{E}^e_{\kappa+1}$$

be a limit root of j such that $b, T, \vec{\kappa} \in \operatorname{rng} K^{\text{ext}}$. Let $\bar{b}$ and $\vec{\delta}$ be such that $K^{\text{ext}}(\bar{b}) = b$ and $K^{\text{ext}}(\vec{\delta}) = \vec{\kappa}$. Note that we are using here that $b, \vec{\kappa} \in [\mathcal{F}^\omega_\kappa(j)]^\omega$. Now we have that since $j(T) = T$,

$$K^*(T(\kappa_0, \dots, \kappa_{n-1})) = T(\delta_0, \dots, \delta_{n-1}).$$

Hence we have that $\bar{b}$ is a branch through $T(\vec{\delta})$ by elementarity. Applying $j_{0,\omega}$ to this fact we have that $j_{0,\omega}(\bar{b})$ is a branch through $j_{0,\omega}(T(\vec{\delta}))$. We claim that the tree whose nth level is $K^{\text{ext}}(j_{0,\omega}(T(\vec{\delta} \restriction n)))$ is exactly T^*. But this follows from Lemma 12 which implies

$$\begin{aligned} j_{0,\omega}(T)(\vec{\kappa} \restriction n) &= j_{0,\omega}(T)(j_{N_j(\vec{\kappa}\restriction n),\omega}(\vec{\kappa} \restriction n)) \\ &= K^{\text{ext}}(j_{0,\omega}(K^*(T(\vec{\kappa} \restriction n)))) \\ &= K^{\text{ext}}(j_{0,\omega}(T(\vec{\delta} \restriction n))). \end{aligned}$$

And since $\bar{b} \in \text{dom} K^{\text{ext}}$, by Lemma 7 we have $j_{0,\omega}(\bar{b}) \in \text{dom} K^{\text{ext}}$. Hence by applying K^{ext} we have that $K^{\text{ext}}(j_{0,\omega}(\bar{b}))$ is a branch through $T^{\omega}_{\lambda}(\vec{\kappa}) = T^*$, which is therefore illfounded. $\square$

Since T^* is illfounded, by absoluteness there is a branch $b^* = (\hat{x}, a^{\omega})$ in $M_{\omega}[\vec{\kappa}]$. We claim that $x \in X$. This follows basically the same as above, by applying $K^{\text{ext}} \circ j^{-1}_{0,\omega} \circ K^*$. To see this let $(K, \langle k_i | \, i < \omega \rangle) \in \mathcal{E}^{e}_{\kappa+1}$ be a limit root of j such that $b, T, \vec{\kappa}, x, a \in \operatorname{rng} K^{\text{ext}}$. Let $\bar{a}^{\omega}$, y, and $\vec{\delta}$ be such that $K^{\text{ext}}(\bar{a}^{\omega}) = a^{\omega}$, $K^{\text{ext}}(\vec{\delta}) = \vec{\kappa}$ and $K^{\text{ext}}(\hat{y}_{\vec{\delta}}) = \hat{x}_{\vec{\kappa}}$. Note that $\bar{a}^{\omega}$ exists since for any $n < \omega$, $a^{\omega} \restriction n \in M_{\omega}$, and also note that $\bar{a}^{\omega} \in \operatorname{rng} j_{0,\omega}$. So let $\bar{a} = j^{-1}_{0,\omega}(\bar{a}^{\omega})$ and let, using Lemma 11, $a = K^{\text{ext}}(\bar{a})$. We have that the tree whose nth level is

$$K^{\text{ext}}(j^{-1}_{0,\omega}(K^*(T^{\omega}(\vec{\kappa} \restriction n))))$$

is exactly $T(\vec{\kappa})$ since

$$\begin{aligned} K^{\text{ext}}(j^{-1}_{0,\omega}(K^*(T^{\omega}(\vec{\kappa} \restriction n)))) &= K^{\text{ext}}(j^{-1}_{0,\omega}(T^{\omega}(\vec{\delta} \restriction n))) \\ &= K^{\text{ext}}(T(\vec{\delta} \restriction n)) = T(\vec{\kappa} \restriction n). \end{aligned}$$

And hence

$$\begin{aligned} K^{\text{ext}}(j^{-1}_{0,\omega}(K^*(\hat{x}_{\vec{\kappa}}, a^{\omega}))) &= (K^{\text{ext}}(j^{-1}_{0,\omega}(\hat{y}_{\vec{\delta}})), K^{\text{ext}}(j^{-1}_{0,\omega}(\bar{a}^{\omega}))) \\ &= (K^{\text{ext}}(\hat{y}_{\vec{\delta}}), K^{\text{ext}}(\bar{a})) \\ &= (\hat{x}_{\vec{\kappa}}, a) \end{aligned}$$

is a branch through $T(\vec{\kappa})$. So $x \in X$ since T is a uniform (j, κ)-Suslin representation for X. So $x \in M_{\omega}[\vec{\kappa}] \cap X$ as desired. $\square$

The next theorem generalizes the above result to allow parameters in $M_{\omega}[\vec{\kappa}] \cap V_{\lambda+1}$. This is especially useful since it is unclear if uniform j-Suslin representations can be obtained if there is a parameter for defining the set in $V_{\lambda+1}$ which is not fixed by an iterate of j.

Theorem 18. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an iterable I_0 embedding and let M_{ω} be the ωth iterate of $L(V_{\lambda+1})$ by j. Let $\vec{\kappa}$ be the critical sequence of j. Suppose that $T \in L(V_{\lambda+1})$ is a uniform (j, κ)-Suslin representation for $X \subseteq \{a \times b | \, a \in [\lambda]^{\omega}, b \in V_{\lambda+1}\}$. Write for $a \in V_{\lambda+1}$*

$$X_a = \{b \in V_{\lambda+1} | \, a \times b \in X\}.$$

Also assume $X_{\vec{\kappa}} \neq \emptyset$, and $T \in \mathcal{F}^{\omega}_{\kappa}(j)$. Then

$$M_{\omega}[\vec{\kappa}] \cap X_{\vec{\kappa}} \neq \emptyset.$$

The proof is exactly the same as the previous proof, replacing for instance $T^{\omega}_{\lambda}(\vec{\kappa})$ by $T^{\omega}_{\lambda}(\vec{\kappa})_{\vec{\kappa}}$. Note that since $T \in \mathcal{F}^{\omega}_{\kappa}(j)$ that for any $n < \omega$, $T(\vec{\kappa} \restriction n)_{\vec{\kappa} \restriction n} \in \mathcal{F}^{\omega}_{\kappa}(j)$. Instead of giving the details we will prove the next theorem which extends both of the above theorems to a collection of generics $g \in V$, instead of just $\vec{\kappa}$.

Theorem 19. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an iterable I_0 embedding and let M_ω be the ωth iterate of $L(V_{\lambda+1})$ by j. Suppose $g \in V$ is $\mathbb{P}$-generic over M_ω for $\mathbb{P} \in M_\omega$. Also assume that*

$$\mathit{cof}(\lambda)^{M_\omega[g]} = \omega.$$

Suppose that $T \in L(V_{\lambda+1})$ is a uniform (j, κ)-Suslin representation for

$$X \subseteq \{a \times b |\, a \in V_{\lambda+1}, b \in V_{\lambda+1}\},$$

and write for $a \in V_{\lambda+1}$

$$X_a = \{b \in V_{\lambda+1} |\, a \times b \in X\}.$$

Let $x \in M_\omega[g] \cap V_{\lambda+1}$. Assume $X_x \neq \emptyset$, and $T \in \mathcal{F}^\omega_\kappa(j)$. Then

$$M_\omega[g] \cap X_x \neq \emptyset.$$

Proof. The proof is again very similar to the proof of Theorem 16. We give the details since this is the theorem we shall actually use below.

Assume without loss of generality that $j(T) = T$ and fix κ good such that $T \in L_\kappa(V_{\lambda+1})$ and $j(\kappa) = \kappa$. Let $j_{0,\omega} : L(V_{\lambda+1}) \to M_\omega$ be the embedding into the ωth iterate given by j. Let $j_{0,\omega}(T) = T^\omega$. And let $T^\omega_\lambda = T^\omega \restriction [\lambda]^{<\omega}$.

Let $g \in V$ be $\mathbb{P}$-generic over M_ω for $\mathbb{P} \in M_\omega$. Assume $\mathrm{cof}(\lambda)^{M_\omega[g]} = \omega$ and $x \in M_\omega[g] \cap V_{\lambda+1}$ is such that $X_x \neq \emptyset$. Let $\vec{\kappa} \in M_\omega[g]$ be increasing cofinal in λ. Now let $T^* = T^\omega_\lambda(\vec{\kappa})_{\hat{x}}$. Note that we have $T^* \in M_\omega[g]$.

Claim 20. *T^* is illfounded.*

Proof. To see this, note that since $X_x \neq \emptyset$, $T(\vec{\kappa})_{\hat{x}}$ is illfounded. So let $b = \langle b_i |\, i < \omega\rangle$ be a branch through $T(\vec{\kappa})_{\hat{x}}$. Now, using Lemma 1, let

$$(K, \langle k_i |\, i < \omega\rangle) \in \mathcal{E}^{\mathrm{e}}_{\kappa+1}$$

be a limit root of j such that $b, T, \vec{\kappa}, x \in \operatorname{rng} K^{\mathrm{ext}}$. Let $\bar{b}$, $\vec{\delta}$, and $y \in V_{\bar{\lambda}_K+1}$ be such that $K^{\mathrm{ext}}(\bar{b}) = b$, $K^{\mathrm{ext}}(\vec{\delta}) = \vec{\kappa}$, and $K(y) = x$. Now we have that since $j(T) = T$,

$$K^*(T(\kappa_0, \ldots, \kappa_{n-1})_{\hat{x}\restriction n}) = T(\delta_0, \ldots, \delta_{n-1})_{\hat{y}\restriction n}.$$

Hence we have that $\bar{b}$ is a branch through $T(\vec{\delta})_{\hat{y}}$ by elementarity. Applying $j_{0,\omega}$ to this fact we have that $j_{0,\omega}(\bar{b})$ is a branch through $j_{0,\omega}(T(\vec{\delta})_{\hat{y}})$. We claim that the tree whose nth level is $K^{\mathrm{ext}}(j_{0,\omega}(T(\vec{\delta} \restriction n)_{\hat{y}\restriction n}))$ is exactly T^*. But this follows from Lemma 12 which implies

$$\begin{aligned} j_{0,\omega}(T)(\vec{\kappa} \restriction n)_{\hat{x}\restriction n} &= j_{0,\omega}(T)(j_{N_j(\vec{\kappa}\restriction n),\omega}(\vec{\kappa} \restriction n))_{j_{N_j(\vec{\kappa}\restriction n),\omega}(\hat{x}\restriction n)} \\ &= K^{\mathrm{ext}}(j_{0,\omega}(K^*(T(\vec{\kappa} \restriction n)_{\hat{x}\restriction n}))) \\ &= K^{\mathrm{ext}}(j_{0,\omega}(T(\vec{\delta} \restriction n)_{\hat{y}\restriction n})). \end{aligned}$$

Hence by applying K^{ext} we have that $K^{\mathrm{ext}}(j_{0,\omega}(\bar{b}))$ is a branch through $T^\omega_\lambda(\vec{\kappa})_{\hat{x}} = T^*$, which is therefore illfounded. □

Since T^* is illfounded, by absoluteness there is a branch $b^* = (\hat{z}, a^\omega)$ in $M_\omega[g]$. We claim that $z \in X$. We show this by applying $K^{\text{ext}} \circ j_{0,\omega}^{-1} \circ K^*$. To see this let $(K, \langle k_i | \, i < \omega \rangle) \in \mathcal{E}^e_{\kappa+1}$ be a limit root of j such that $b, T, \vec{\kappa}, z, x, a^\omega \in \operatorname{rng} K^{\text{ext}}$. Let $\bar{a}^\omega$, z', y, and $\vec{\delta}$ be such that $K^{\text{ext}}(\bar{a}^\omega) = a^\omega$, $K^{\text{ext}}(z') = z$, $K(y) = x$, and $K^{\text{ext}}(\vec{\delta}) = \vec{\kappa}$ (note that $\bar{a}^\omega$ again exists because for all $n < \omega$, $a^\omega \restriction n \in M_\omega$. As above we have that the tree whose nth level is

$$K^{\text{ext}}(j_{0,\omega}^{-1}(K^*(T^\omega(\vec{\kappa} \restriction n)_{\hat{x} \restriction n})))$$

is exactly $T(\vec{\kappa})_{\hat{x}}$. And hence

$$\begin{aligned} K^{\text{ext}}(j_{0,\omega}^{-1}(K^*(\hat{z}_{\vec{\kappa}}, a^\omega))) &= (K^{\text{ext}}(j_{0,\omega}^{-1}(\hat{z}'_{\vec{\delta}})), K^{\text{ext}}(j_{0,\omega}^{-1}(K^*(a^\omega)))) \\ &= (K^{\text{ext}}(\hat{z}'_{\vec{\delta}}), K^{\text{ext}}(j_{0,\omega}^{-1}(K^*(a^\omega)))) \\ &= (\hat{z}_{\vec{\kappa}}, K^{\text{ext}}(j_{0,\omega}^{-1}(K^*(a^\omega)))) \end{aligned}$$

is a branch through $T(\vec{\kappa})_{\hat{x}}$, and hence $z \in X_x$ since T is a uniform (j, κ)-Suslin representation for X. So $z \in M_\omega[g] \cap X_x$ as desired. $\square$

4 Generic absoluteness from j-Suslin representations

We now come to our generic absoluteness results which follow from the existence of uniform j-Suslin representations. Based on Theorem 18, we make the following conjecture about the existence of uniform j-Suslin representations.

Definition 21. Suppose $Y \subseteq V_{\lambda+1}$ and $j : L(Y, V_{\lambda+1}) \to L(Y, V_{\lambda+1})$ is an $I_0(Y)$ elementary embedding. We say that the *uniform j-Suslin conjecture* holds in $L(Y, V_{\lambda+1})$ if the following holds. Suppose $X \subseteq V_{\lambda+1}$, $X \in L(Y, V_{\lambda+1})$ is such that there is an $a \in \mathcal{F}^\omega_0(j)$ and a δ such that X is definable over $L_\delta(V_{\lambda+1})$ from a (so $X \in \mathcal{F}^\omega_\Theta(j)$). Then for some $\kappa < \Theta$ there exists a uniform (j, κ)-Suslin representation T for X such that $T \in \mathcal{F}^\omega_\Theta(j)$.

The following is proved in [2].

Theorem 22. *Suppose that I_0 holds at λ. Then the uniform j-Suslin conjecture holds in $L(V_{\lambda+1})$. In fact if $I_0^\#$ holds at λ then the uniform j-Suslin conjecture holds in $L(V^\#_{\lambda+1})$.*

This result together with the techniques of the previous section allow us to obtain the following generic absoluteness result.

Theorem 23. *Suppose that I_0 holds at λ, $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then for $\vec{\kappa}$ the critical sequence of j, if $\alpha < \Theta$ is good then for some $\bar{\alpha} < \lambda$ there is an elementary embedding*

$$L_{\bar{\alpha}}(M_\omega[\vec{\kappa}] \cap V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}).$$

Proof. The theorem follows from Theorem 19. To see this, assume that $a \in M_\omega[\vec{\kappa}] \cap V_{\lambda+1}$ and ψ is such that $X \subseteq V_{\lambda+1}$ is definable for some good κ from a over $L_\kappa(V_{\lambda+1})$ by a formula ϕ. So

$$x \in X \iff L_\kappa(V_{\lambda+1}) \models \phi(x, a).$$

Now let $\tilde{X}$ be defined by

$$(b, x) \in \tilde{X} \iff L_\kappa(V_{\lambda+1}) \models \phi(x, b) \wedge x \in V_{\lambda+1}.$$

We have that $\tilde{X}$ is definable over $L_\kappa(V_{\lambda+1})$ and that $\tilde{X}_a = X$. By the uniform j-Suslin conjecture we have that $\tilde{X}$ is uniformly (j, κ')- Suslin for some κ'. And hence by Theorem 19 we have that

$$\tilde{X}_a \cap M_\omega[\vec{\kappa}] = X \cap M_\omega[\vec{\kappa}] \neq \emptyset.$$

Now fix $\kappa < \Theta$ good and let $X \subseteq V_{\lambda+1}$ code $L_\kappa(V_{\lambda+1})$ in the natural way. Then we have that $Y \subseteq V_{\lambda+1}$ is definable from $a \in V_{\lambda+1}$ over $L_\kappa(V_{\lambda+1})$ iff $Y \subseteq V_{\lambda+1}$ is definable from a over $(V_{\lambda+1}, X)$. But by what we showed above, if $Y \subseteq V_{\lambda+1}$, $Y \neq \emptyset$ is definable from a parameter $a \in M_\omega[\vec{\kappa}] \cap V_{\lambda+1}$ over $(V_{\lambda+1}, X)$, then $Y \cap M_\omega[\vec{\kappa}] \neq \emptyset$. Hence we have by Tarski-Vaught

$$(M_\omega[\vec{\kappa}] \cap V_{\lambda+1}, X \cap M_\omega[\vec{\kappa}]) \prec (V_{\lambda+1}, X).$$

Let $\bar{X} = X \cap M_\omega[\vec{\kappa}]$. Then by elementarity, $\bar{X}$ codes $L_{\bar{\alpha}}(M_\omega[\vec{\kappa}] \cap V_{\lambda+1})$ for some $\bar{\alpha}$ and we have an elementary embedding

$$L_{\bar{\alpha}}(M_\omega[\vec{\kappa}] \cap V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}),$$

which is what we wanted. □

The exact same proof with $\vec{\kappa}$ replaced by g shows the following.

Theorem 24. *Suppose that I_0 holds at λ, as witnessed by*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Suppose $g \in V$ is $\mathbb{P}$-generic over M_ω where $\mathbb{P} \in M_\omega$. Also assume that cof$(\lambda)^{M_\omega[g]} = \omega$. Then if $\alpha < \Theta$ is good, for some $\bar{\alpha} < \lambda$ there is an elementary embedding

$$L_{\bar{\alpha}}(M_\omega[g] \cap V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}).$$

Note that this theorem gives a slightly stronger version of generic absoluteness than that found in [10]. In particular we do not require that $\mathbb{P} \in j_{0,\omega}(V_\lambda)$. This is potentially a significant strengthening as for instance we cannot have $j \in M_\omega[g]$ where g is M_ω-generic for a forcing $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, since $\mathbb{P}$ is a small forcing in M_ω relative to $j_{0,\omega}(\lambda)$. This reasoning however does not apply to forcings in M_ω.

This theorem can also be generalized beyond $L(V_{\lambda+1})$ by increasing the large cardinal assumption, using Theorem 22 again.

Theorem 25. *Suppose that $j : L(V^{\#}_{\lambda+1}) \to L(V^{\#}_{\lambda+1})$ is an $I_0^{\#}$ embedding, and $g \in V$ is $\mathbb{P}$-generic over M_ω, the ωth iterate of j, where $\mathbb{P} \in M_\omega$. Also assume that cof$(\lambda)^{M_\omega[g]} = \omega$. Then for some $\bar{\alpha} < \lambda$ there is an elementary embedding*

$$L_{\bar{\alpha}}((M_\omega[g] \cap V_{\lambda+1})^{\#}, M_\omega[g] \cap V_{\lambda+1}) \to L_\alpha(V^{\#}_{\lambda+1}, V_{\lambda+1}).$$

We close by mentioning a couple consequences of the theorems in this section. First by results independently shown by Dimonte-Friedman [5] and Woodin, we have the following theorem on the failure of the singular cardinal hypothesis at λ. Note that although [5] deals principally with the case of I_1, the same methods carry over to I_0 once the above generic absoluteness result is shown.

Theorem 26. *Assume that there is an elementary embedding $j : L(V^{\#}_{\lambda+1}) \to L(V^{\#}_{\lambda+1})$. Then it is consistent that I_0 holds at λ and the singular cardinal hypothesis fails at λ.*

In addition, by a theorem of Shi-Woodin [9] the above results give a proof (alternative to that found in [1]) of the λ-splitting perfect set property for all subsets of $V_{\lambda+1}$. in $L(V_{\lambda+1})$ We refer the reader [10] Section 9 for more applications of the above generic absoluteness theorem.

References Cited

[1] Scott Cramer. UC Berkeley PhD thesis: Inverse limit reflection and the structure of $L(V_{\lambda+1})$. 2012.

[2] Scott Cramer. Existence of tree representations from very large cardinals. *submitted*, 2015.

[3] Scott Cramer. Strong inverse limit reflection. *submitted*, 2015.

[4] Scott S. Cramer. Inverse limit reflection and the structure of $L(V_{\lambda+1})$. *J. Math. Log.*, 15(1):1550001 (38 pages), 2015.

[5] Vincenzo Dimonte and Sy-David Friedman. Rank-into-rank hypotheses and the failure of GCH. *Arch. Math. Logic*, 53(3-4):351–366, 2014.

[6] Paul B. Larson. *The stationary tower*, volume 32 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004. Notes on a course by W. Hugh Woodin.

[7] Richard Laver. Implications between strong large cardinal axioms. *Ann. Pure Appl. Logic*, 90(1-3):79–90, 1997.

[8] Richard Laver. Reflection of elementary embedding axioms on the $L[V_{\lambda+1}]$ hierarchy. *Ann. Pure Appl. Logic*, 107(1-3):227–238, 2001.

[9] Xianghui Shi. Axiom I_0 and higher degree theory. *Journal of Symbolic Logic*, (to appear), 2014.

[10] W. Hugh Woodin. Suitable extender models II: beyond ω-huge. *J. Math. Log.*, 11(2):115–436, 2011.