

Implications of very large cardinals

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Abstract

We trace some of the history of results which follow from the existence of very large cardinals. In particular we concentrate on the axiom I_0 and its implications for the structure of the inner model $L(V_{\lambda+1})$.

1 Introduction

The large cardinal hierarchy is a bedrock of modern set theory as the key measure against which seemingly every other axiom in mathematics can be compared in terms of consistency strength.¹ In this article we explore some of the large cardinals axioms which exist at the edge of this hierarchy: what we call ‘very large cardinals’. Our goals in this study are both to identify consequences of the existence of very large cardinals and to understand how these consequences help us understand the structure and extent of the large cardinal hierarchy itself.

The central focus of our study is the axiom I_0 and the structure $L(V_{\lambda+1})$. This structure was first defined and studied by H. Woodin in the 1980’s in unpublished work in order to more fully understand how the structure of $L(\mathbb{R})$ is influenced by large cardinals. Surprisingly, similarities between the structure of $L(V_{\lambda+1})$ and the structure of $L(\mathbb{R})$ abound under large cardinal assumptions, and much of this article will be devoted to recounting what is presently understood about similarities and differences in their structure.

Beyond understanding the structure of very large cardinals by themselves, recent work has also returned to Woodin’s original motivation and found increasingly deep structural connections between $L(V_{\lambda+1})$ and models of determinacy. These results give a new method for obtaining strong models of determinacy axioms, and could potentially help motivate and analyze very strong determinacy axioms. They also give a picture of the interaction between large cardinals and determinacy axioms on a global scale.

We hope to paint a picture of intricate structure at the level of $L(V_{\lambda+1})$ which gives some evidence of the consistency of the axiom I_0 . Optimistically, the analysis of $L(V_{\lambda+1})$ extends well beyond this level, and hopefully, if our analysis becomes detailed enough, where

¹See [12] for a general history of large cardinals

this structure breaks down and inconsistency occurs will become more clear. At this point, however, this occurrence is a hope rather than a reality.

The outline of this article is as follows. In Section 2 we discuss Kunen’s Inconsistency Theorem and the various large cardinals which exist just below this barrier. In Section 3 we look at reflection properties of these cardinals. In Section 4 we look more closely at the axiom I_0 and the structural aspects of $L(V_{\lambda+1})$ under I_0 . In Section 5 we consider the notion of an AD-like axiom for $L(V_{\lambda+1})$ and various attempts at defining such an axiom. Finally in Section 6 we briefly consider axioms beyond I_0 .

Our proofs are throughout rather sketchy as we focus on key points while trying to avoid a tidal wave of calculations. Hopefully our brevity will be appreciated by the reader, and, moreover, the exercise of filling in the appropriate details, in some cases, would perhaps serve as a fruitful introduction to the subject.

1.1 Set theory conventions

We denote by V_α for α an ordinal the stratification of V according to rank. So $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ and $V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$ for λ a limit. For a transitive set A , $L(A)$ is the constructible hierarchy built on top of A . So $L_0(A) = A$, $L_{\alpha+1}(A) = \text{Def}(L_\alpha(A))$ and $L_\lambda(A) = \bigcup_{\alpha < \lambda} L_\alpha(A)$ for λ a limit. By $L(X, A)$ we mean the constructible hierarchy built on top of A , with X as a predicate. So $L_0(X, A) = A$, $L_{\alpha+1}(X, A) = \text{Def}(L_\alpha(A), X \cap L_\alpha(A))$ and $L_\lambda(X, A) = \bigcup_{\alpha < \lambda} L_\alpha(X, A)$ for λ a limit.

Suppose that M and N are models of a fragment of set theory. Then $j : M \rightarrow N$ is an elementary embedding if for all $\phi[x_1, \dots, x_n]$ and $a_1, \dots, a_n \in M$ we have that

$$M \models \phi(a_1, \dots, a_n) \Rightarrow N \models \phi(j(a_1), \dots, j(a_n)).$$

We use the convention that all elementary embeddings are nontrivial, so $j \neq \text{id}$. And $\text{crit}(j)$ is the least ordinal α such that $j(\alpha) > \alpha$.

2 Kunen’s Theorem and Rank-into-Rank Embeddings

By far the most common type of large cardinal axiom is the assertion that there is a non-trivial elementary embedding j of the universe V into an inner model M . In general, the more M agrees with V the stronger the large cardinal. This observation led W. Reinhardt in 1967 to propose the following axiom, called a Reinhardt cardinal: there exists a non-trivial elementary embedding $j : V \rightarrow V$. The existence of a Reinhardt cardinal is a natural upper bound of this type of elementary embedding existence axiom, but K. Kunen[15] subsequently showed that Reinhardt cardinals are inconsistent with the Axiom of Choice.

Theorem 1 (Kunen[15]). *(KM²) Suppose that $j : V \rightarrow M$ is a non-trivial elementary embedding. Then $M \neq V$.*

²Note that Kunen’s theorem in this form is a statement in Kelley-Morse set theory. See [11] for a discussion of this fact.

In fact he showed the following stronger fact.

Theorem 2 (Kunen). *(ZFC) There is no λ such that there is a non-trivial elementary embedding*

$$j : V_{\lambda+2} \rightarrow V_{\lambda+2}.$$

This theorem follows immediately from the proof of Theorem 23 below and Solovay's Theorem on splitting stationary sets under ZFC. Several additional proofs can be found in [12].

A key point, and the starting point for the analysis of axioms just below this inconsistency, is that all known proofs of Kunen's Theorem require the Axiom of Choice (AC). So, on the one hand, there is the natural question of whether there can be a (non-trivial) elementary embedding $V_{\lambda+2} \rightarrow V_{\lambda+2}$ under ZF, and on the other hand, in analyzing this possibility, we must necessarily consider models in which AC fails. This is perhaps the first indication that such an analysis would have similarities to other non-choice settings, such as the context of the Axiom of Determinacy (AD).

We now define some of the large cardinals which lie just below this inconsistency. These elementary embeddings are collectively referred to as 'rank-into-rank embeddings'. Note that from now on all elementary embeddings are assumed to be non-trivial.

Definition 3. Let λ be an ordinal. We make the following definitions.

1. I_3 holds at λ if there exists an elementary embedding $j : V_\lambda \rightarrow V_\lambda$ and λ is a limit.
2. I_2 holds at λ if there exists an elementary embedding $j : V \rightarrow M$ such that $V_\lambda \subseteq M$ and $\text{crit}(j) < \lambda$.
3. I_1 holds at λ if there exists an elementary embedding $j : V_{\lambda+1} \rightarrow V_{\lambda+1}$.
4. $I_1(X)$ holds at λ if there exists an X -elementary embedding $j : V_{\lambda+1} \rightarrow V_{\lambda+1}$.³

W. H. Woodin defined an axiom called I_0 which is even stronger than I_1 but still below the level of Reinhardt cardinals. Before introducing this axiom, we mention as motivation that the existence of an elementary embedding $V_{\lambda+2} \rightarrow V_{\lambda+2}$ can be weakened by restricting to certain sets in $V_{\lambda+2} = P(V_{\lambda+1})$. This restriction succeeds if we avoid the subsets of $V_{\lambda+1}$ which the axiom of choice gives us in the proof of Theorem 2. This situation is very reminiscent of considering regularity properties, such as Lebesgue measurability, of sets of real numbers in classical descriptive set theory. In that situation the axiom of choice implies that there are sets of reals which do not satisfy, for instance, Lebesgue measurability. However, by restricting to only certain sets of reals (such as Borel sets or projective sets) we have a chance of showing that all such sets have the desired regularity properties. We will see in Section 4 that this analogy holds to a remarkable extent, and that we can actually achieve remarkably similar descriptive set-theoretic results in this context.

We now define I_0 and some variations.

³For instance $I_1(\Sigma_1)$ states that there is such a Σ_1 -elementary embedding. We use this notation with a fair amount of liberty but our meaning should always be clear.

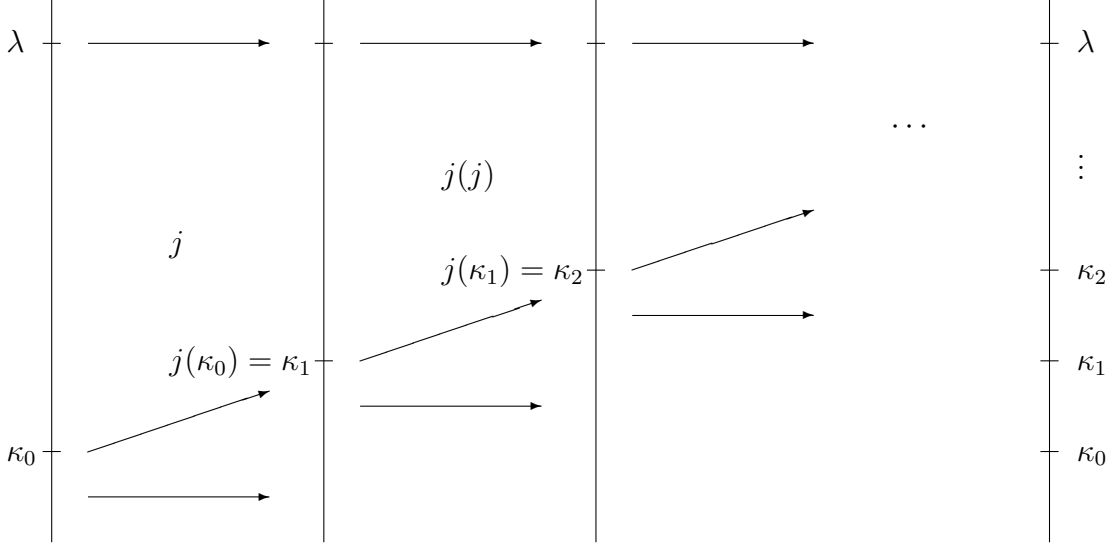


Figure 1: Rank-into-rank embeddings.

Definition 4. ⁴ Let λ and α be ordinals. We make the following definitions.

1. $I_0(\alpha)$ holds at λ if there exists an elementary embedding $j : L_\alpha(V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1})$ such that $\text{crit}(j) < \lambda$ and λ is a limit.
2. $I_0(< \alpha)$ holds at λ if for all $\beta < \alpha$, $I_0(\beta)$ holds at λ .
3. I_0 holds at λ if there exists an elementary embedding $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ such that $\text{crit}(j) < \lambda$.
4. $I_0^\#$ holds at λ if there exists an elementary embedding⁵

$$j : L(V_{\lambda+1}^\#, V_{\lambda+1}) \rightarrow L(V_{\lambda+1}^\#, V_{\lambda+1})$$

such that $\text{crit}(j) < \lambda$.

5. Suppose $X \subseteq V_{\lambda+1}$. We say that X is *Icarus* or $I_0(X)$ holds if there exists a non-trivial elementary embedding $j : L(X, V_{\lambda+1}) \rightarrow L(X, V_{\lambda+1})$ such that $\text{crit}(j) < \lambda$.

We now collect some basic facts about rank-into-rank embeddings and the structure $L(V_{\lambda+1})$.

Proposition 5. *Let λ be an ordinal. Then the following hold.*

1. $(\lambda^+)^V = (\lambda^+)^{L(V_{\lambda+1})}$, so in $L(V_{\lambda+1})$, λ^+ is regular.

⁴Our notation above in (1) and (2) is not standard in the literature, although we shall use it for convenience in this paper. The terminology in (5) first appeared in [8].

⁵See [23] for the definition of $\mathbb{R}^\#$ and [8] for the corresponding definition of $V_{\lambda+1}^\#$, which is very similar.

2. Suppose that $j : V_\lambda \rightarrow V_\lambda$ is an elementary embedding. For $\langle \kappa_i \mid i < \omega \rangle$ defined by⁶ induction by $\kappa_0 = \text{crit}(j)$ and $\kappa_{i+1} = j(\kappa_i)$ for $i < \omega$, we have that

$$\lim_{i < \omega} \kappa_i = \lambda'$$

where either $\lambda' = \lambda$ or $\lambda' + 1 = \lambda$.

3. If I_3 holds at λ then $\text{cof}(\lambda) = \omega$.

4. Suppose I_0 holds at λ . Then $L(V_{\lambda+1})$ does not satisfy the axiom of choice.

5. Suppose that $j : V_\lambda \rightarrow V_\lambda$ is an elementary embedding. Then j has a unique extension to a Σ_0 -elementary embedding

$$\hat{j} : V_{\lambda+1} \rightarrow V_{\lambda+1}.$$

6. If $n < \omega$ is odd and $j : V_{\lambda+1} \rightarrow V_{\lambda+1}$ is Σ_n -elementary, then j is Σ_{n+1} -elementary (Martin).

Proof. To see (1), note that for any ordinal $\alpha < \lambda^+$, there is an $x \in V_{\lambda+1}$ which codes an ordering of λ in ordertype α . And hence $(\lambda^+)^V = (\lambda^+)^{L(V_{\lambda+1})}$ since $V_{\lambda+1} \subseteq L(V_{\lambda+1})$.

To see (2), let $\lim_{i < \omega} \kappa_i = \lambda'$, and assume towards a contradiction that $\lambda' + 2 \leq \lambda$. Note that $j(\lambda') = \lambda'$. And hence $j \upharpoonright V_{\lambda'+2} : V_{\lambda'+2} \rightarrow V_{\lambda'+2}$. But $\lambda' > \text{crit}(j)$, and hence $j \upharpoonright V_{\lambda'+2}$ is non-trivial and violates Theorem 2.

(3) immediately follows from (2), and (4) follows by Kunen's Theorem⁷.

To see (5), define \hat{j} by $\hat{j}(a) = \bigcup_{n < \omega} a \cap V_{\lambda_n}$ for $\langle \lambda_n \mid n < \omega \rangle$ some increasing cofinal sequence in λ . Then clearly \hat{j} does not depend on which cofinal sequence was chosen, and it is a Σ_0 -elementary embedding $V_{\lambda+1} \rightarrow V_{\lambda+1}$.

For a proof of (6) see [18]. □

Because of (5), we will use the convention below that \hat{j} and j are identified. So if $j : V_\lambda \rightarrow V_\lambda$ is elementary then we will write $j(j)$ for $\hat{j}(j)$. Note that we have the following theorem.

Theorem 6 (Laver[18]). *If $j, k : V_{\lambda+1} \rightarrow V_{\lambda+1}$ are Σ_n -elementary then $j(k) : V_{\lambda+1} \rightarrow V_{\lambda+1}$ is Σ_n -elementary.*

Hence, conveniently, the set of I_1 -embeddings acts on itself using the extensions \hat{j} for j an I_1 embedding. This is also true for I_3 -embeddings. Such facts allow for algebras of elementary embeddings. We refer the interested reader to [17] and [20] for more on these algebras.

⁶ $\langle \kappa_i \mid i < \omega \rangle$ is referred to as the *critical sequence* of j .

⁷Note that the proof of Kunen's Theorem does not require that j actually be in V .

3 Reflection properties

We will now look at the reflection properties of rank-into-rank embeddings. These reflection theorems are in a sense harder to achieve than is typically the case with large cardinal axioms. The reason is that in reflecting a rank-into-rank embedding we not only wish to decrease the critical point of the embedding, but we wish to decrease λ , the first fixed point of the embedding. The usual arguments for reflecting large cardinals in this case give us what we will call *square roots* of our embedding, whereas to reflect the large cardinal by decreasing λ will generally require more involved arguments, and, in particular, sequences of embeddings. While this might at first seem like a simple nuisance, the techniques needed for these reflection results will end up being useful in a wide range of Theorems below.

We concentrate on the following fairly weak form of reflection: we say a large cardinal axiom LCA1 *reflects* a large cardinal axiom LCA2 if for any λ if LCA1 holds at λ then LCA2 holds at λ and there is $\bar{\lambda} < \lambda$ such that LCA2 holds at $\bar{\lambda}$. We also write $\text{LCA2} < \text{LCA1}$. The following theorem summarizes some results on reflection of rank-into-rank embeddings in the order in which they were proved. These results can be extended to stronger forms of reflection, such as finding an ω -club of $\bar{\lambda}$ below λ at which LCA2 holds. We will discuss other strengthenings of these results at the end of this section.

Theorem 7. 1. $I_3 < I_2$ (Gaifman[10], Solovay).

2. $I_3 < I_1(\text{iterable}) < I_1(\Sigma_2) = I_2$ (Martin).

3. $I_1 < I_0$ (Woodin[26]).

4. $I_1(\Sigma_2) < I_1(\Sigma_4) < I_1(\Sigma_6) < \dots < I_1 < I_0(1)$ (Laver[18]).

5. $I_0(1) < I_0(\omega + 1) < I_0(\lambda^+) < I_0(\lambda^+ + \omega + 1)$. (Laver[19])⁸

6. $I_0(< \lambda^+ + \omega) < I_0(\lambda^+ + \omega)$ and $I_0(< \alpha + \omega) < I_0(\alpha + \omega)$ for α the least admissible ordinal (C. [5]).

7. For α good⁹, $I_0(\alpha + \omega)$ holding at λ implies that there is $\bar{\lambda} < \lambda$ and $\bar{\alpha}$ such that for all $n < \omega$ there is an elementary embedding

$$L_{\bar{\alpha}+n}(V_{\bar{\lambda}+1}) \rightarrow L_{\alpha+n}(V_{\lambda+1})$$

and hence $I_0(< \bar{\alpha} + \omega)$ holds at $\bar{\lambda}$ (C. [5]).¹⁰

8. $I_0 < I_0^\#(\omega)$ (C. [5]).

⁸Note that in (5) and (6) we mean that if $I_0(\lambda^+ + \omega + 1)$ holds at λ then there is a $\bar{\lambda} < \lambda$ such that $I_0(\bar{\lambda}^+)$ holds at $\bar{\lambda}$.

⁹ α is *good* if every element of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from an element of $V_{\lambda+1}$. The good ordinals are cofinal in Θ_λ (see [19])

¹⁰We state (7) in this manner because for larger α there is no simple definition of α as in the case of $\alpha = \lambda^+$ for instance. So we cannot state the reflection as we do above.

The proofs for (1) and (2) involved taking the ω th iterate of an embedding and then considering the tree of attempts to build an I_3 embedding. The proofs of (4)-(7) on the other hand involved inverse limits of rank-into-rank embeddings, a technique introduced by Laver that we will describe below.

As mentioned above, these reflection results were shown using sequences of embeddings (either direct limits or inverse limits) rather than individual embeddings. The usual method for obtaining reflection from a large cardinal in the case of rank-into-rank embeddings gives us what we call *square roots*. The primary method for obtaining such square roots is the following fact, which we generally refer to as the Square Root Lemma:

Lemma 8. (Martin, Laver) *Let α be good. If $j : L_{\alpha+1}(V_{\lambda+1}) \rightarrow L_{\alpha+1}(V_{\lambda+1})$ is an elementary embedding with $\text{crit}(j) < \lambda$, $a, b \in V_{\lambda+1}$, then there is an elementary embedding $k : L_\alpha(V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1})$ such that*

1. $k(k \upharpoonright V_\lambda) = j \upharpoonright V_\lambda$
2. $a \in \text{rng } k$
3. $j(b) = k(b)$.

Proof sketch. Apply j to the statement of the lemma and then show that $j \upharpoonright L_\alpha(V_{\lambda+1})$ witnesses the result. The fact that α is good implies that $j \upharpoonright L_\alpha(V_{\lambda+1}) \in L_{\alpha+1}(V_{\lambda+1})$. \square

Of course a could be replaced by a finite tuple a_1, \dots, a_n and similarly for b . In fact these can be replaced by tuples of length $< \text{crit}(j)$.

If $k(k \upharpoonright V_\lambda) = j \upharpoonright V_\lambda$ then we say that k is a *square root of j* . The Square Root Lemma has many variations, as for instance Laver included the fact that for a given $\alpha < \text{crit}(j)$, k has the property that $\alpha < \text{crit}(k) < \text{crit}(j)$. Notice however, that we can require that $\alpha \in \text{rng } k$. And this implies that $\alpha < \text{crit}(k) < \text{crit}(j)$ because $k(\text{crit}(k)) = \text{crit}(j)$ implies $\text{crit}(k) < \text{crit}(j)$, and $\alpha < \text{crit}(k)$ since for any $\beta \in [\text{crit}(k), \text{crit}(j))$, $\beta \notin \text{rng } k$. In fact, property (3) in the statement of the Square Root Lemma is in fact redundant, by property (2) by the following proposition.

Proposition 9. *Suppose that $j, k : V_\lambda \rightarrow V_\lambda$ and k is a square root of j . Then the following hold.*

1. $\text{crit}(k) < \text{crit}(j)$ and $k(\text{crit}(k)) = \text{crit}(j)$.
2. If $a \in V_{\lambda+1}$ and $a \in \text{rng } k$, then $k(a) = j(a)$.
3. For all $\alpha < \lambda$, $k(\alpha) \geq j(\alpha)$.

Proof. (1) follows by computing¹¹

$$k(\text{crit}(k)) = \text{crit}(k(k)) = \text{crit}(j).$$

¹¹Note that we are being careful only to use elementarity below λ . That is, we are using properties of k which are properties of $k \upharpoonright V_\alpha$ for large enough $\alpha < \lambda$.

Similarly, for (2), if $k(b) = a$, then, informally,¹²

$$k(a) = k(k(b)) = k(k)(k(b)) = j(a).$$

For (3), suppose not, and let α be least such that $k(\alpha) < j(\alpha)$. Then using the fact that α is least such, $j(\alpha)$ must be in the range of k , since it is the least ordinal above $k(\alpha)$ in the range of j . Hence, $\alpha \in \text{rng } k$. But then $k(\alpha) = j(\alpha)$ by (2), a contradiction. \square

We now come to a key application of the Square Root Lemma: the construction of inverse limits of rank-into-rank embeddings. These structures are key to proving reflection results for rank-into-rank embeddings, and they naturally arise from the Square Root Lemma. In fact, an inverse limit can be constructed simply as the result of ω -many applications of the Square Root Lemma for a fixed j , where at each stage we choose k_i such that everything in the construction so far is required to be in the range of k_i .

Definition 10 (Laver). We say that $(J, \langle j_i \mid i < \omega \rangle) = (J, \vec{j})$ is an *inverse limit* if the following hold.

1. For all $i < \omega$, $j_i : V_\lambda \rightarrow V_\lambda$ is elementary.
2. $\text{crit}(j_0) < \text{crit}(j_1) < \dots < \lambda$ and $\lim_{i < \omega} \text{crit}(j_i) = \bar{\lambda}_J < \lambda$.
3. $J : V_{\bar{\lambda}_J} \rightarrow V_\lambda$ is given by

$$J(a) = \lim_{i < \omega} (j_0 \circ j_1 \circ \dots \circ j_i)(a)$$

for any $a \in V_{\bar{\lambda}_J}$.

We write $J = j_0 \circ j_1 \circ \dots$ to denote an inverse limit. For $n < \omega$, we write $J_n = j_n \circ j_{n+1} \circ \dots$.

Notice in this definition that for any $a \in V_{\bar{\lambda}_J}$ there is an $i < \omega$ such that for all $n \geq i$, $j_n(a) = a$, so the limit in the definition of J makes sense. An inverse limit can also be written as a direct limit as follows (see Figure 2):

$$J = j_0 \circ j_1 \circ \dots = \dots (j_0 \circ j_1)(j_2) \circ j_0(j_1) \circ j_0.$$

Proposition 11 (Laver[18]). *Suppose that α is good and $j : L_{\alpha+1}(V_{\lambda+1}) \rightarrow L_{\alpha+1}(V_{\lambda+1})$ is elementary. Then there is an inverse limit (K, \vec{k}) such that for all $i < \omega$, k_i is a square root of j and k_i extends to an elementary embedding $L_\alpha(V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1})$.*

*Proof.*¹³ Choose k_i for $i < \omega$ by induction using the Square Root Lemma, requiring that for all $i < \omega$, $k_0, \dots, k_i \in \text{rng } k_{i+1}$. Then we have by the above remarks

$$\text{crit}(k_0) < \text{crit}(k_1) < \dots < \text{crit}(j) < \lambda.$$

And hence $K = k_0 \circ k_1 \circ \dots$ is the desired inverse limit. \square

¹²As before, we should restate this in terms of $k \upharpoonright V_{\alpha+1}(a \cap V_\alpha)$ for arbitrarily large $\alpha < \lambda$, and then use continuity of this property.

¹³The following proof is a slight simplification from Laver's original as we use the remarks made above about the Square Root Lemma. Also, Laver did not include the fact that k_i is a square root of j .

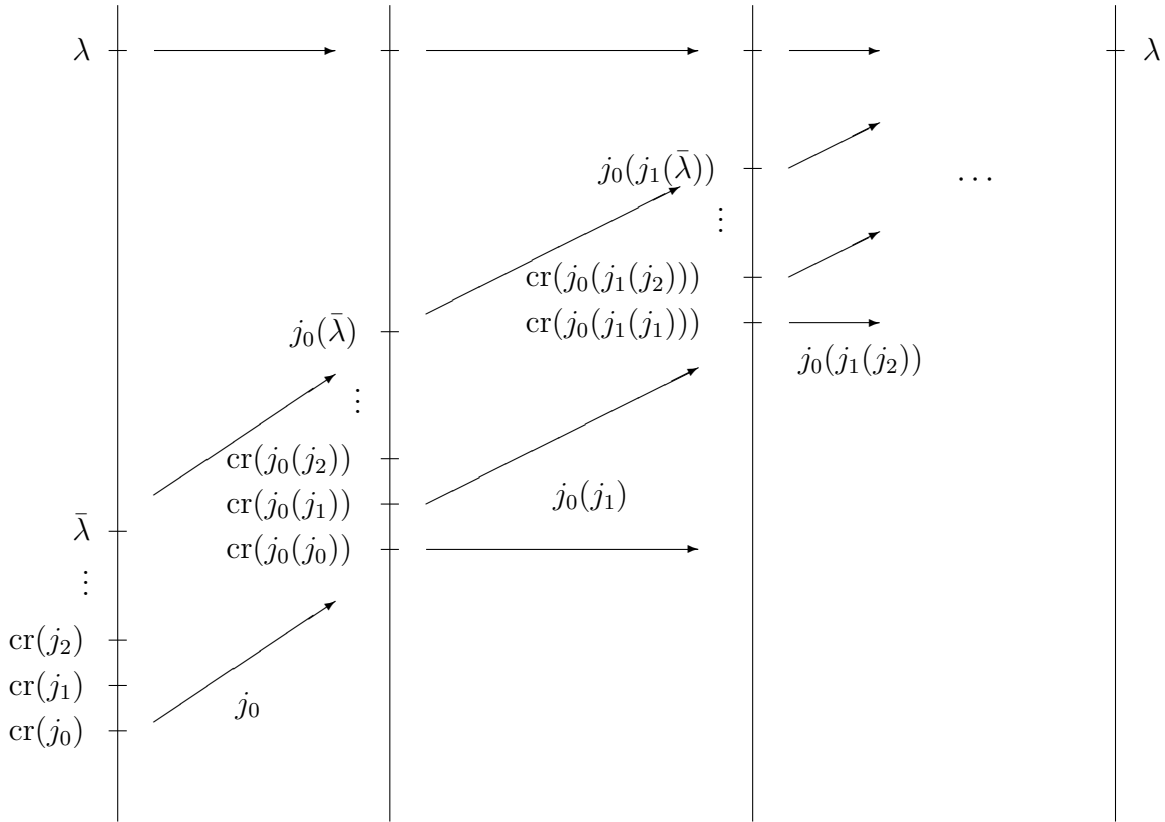


Figure 2: Direct limit decomposition of an inverse limit.

As is to some extent clear from the definition, inverse limits are especially useful for proving reflection results on rank-into-rank embeddings. The key question in performing this reflection is to what extent an inverse limit (J, \vec{j}) , which only gives an embedding $V_{\bar{\lambda}_J} \rightarrow V_\lambda$ can be extended to $V_{\bar{\lambda}_{J+1}}$ or beyond to $L(V_{\bar{\lambda}_{J+1}})$. One key fact in finding such an extension is that inverse limits also satisfy a certain Square Root Lemma. In fact, both the extension of inverse limits and the corresponding Square Root Lemma can be viewed as instances of the same phenomenon: a property of the embeddings comprising an inverse limit *transfers* to the inverse limit itself. Although some instances of this transference can be viewed as immediate consequences of the continuity of the given property, in other cases the proof is much more involved.

We now state the Square Root Lemma for inverse limits.

Proposition 12 (Laver). *Suppose that α is good and (J, \vec{j}) is an inverse limit such that for all $i < \omega$, j_i extends to an elementary embedding $L_{\alpha+1}(V_{\lambda+1}) \rightarrow L_{\alpha+1}(V_{\lambda+1})$. Then for any $a \in V_{\bar{\lambda}_{J+1}}$ and $b \in V_{\lambda+1}$ there is an inverse limit (K, \vec{k}) such that the following hold:*

1. for all $i < \omega$ k_i is a square root of j_i ,
2. k_i extends to an elementary embedding $L_\alpha(V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1})$,
3. $\bar{\lambda}_J = \bar{\lambda}_K$,
4. $K(a) = J(a)$ and $b \in \text{rng } K$.

Proof sketch. Choose k_i by induction on $i < \omega$ using the Square Root Lemma, ensuring that the following hold:

1. for all $n < \omega$, $j_n \in \text{rng } k_i$,
2. for all $n < i$, $k_n \in \text{rng } k_i$,
3. $a, b \in \text{rng } k_i$.

A computation then shows that (K, \vec{k}) is as desired. □

The square root lemma for inverse limits, Proposition 12, is the key technique used in Laver's proof that $I_0(1) < I_0(\omega + 1)$. The proof proceeds by using the square root lemma to derive elementarity of the inverse limit. This type of argument generally proceeds by induction on the elementarity of the embeddings making up the inverse limit, and the square root lemma is used to 'capture' witnesses to a given statement. Notice that the strength of the inverse limit (K, \vec{k}) given by the square root lemma decreases from that of (J, \vec{j}) , a fact which is necessary for this specific statement of the lemma.¹⁴ However, this decrease in strength is what necessitates induction in this method.

On the other hand, the proof that $I_0 < I_0^\#(\omega)$ uses an alternative form of Proposition 12 and avoids the need for induction. This alternative square root lemma is also important

¹⁴To see this, take (J, \vec{j}) with critical point as small as possible. Then notice that $\text{crit}(K) < \text{crit}(J)$.

in defining the axiom we will call Inverse Limit Reflection in Section 5. First we define the notion of a *limit root*.

Definition 13 (C.). Let (J, \vec{j}) and (K, \vec{k}) be inverse limits and $n < \omega$. Then we say that (K, \vec{k}) is an (*n-close limit root of (J, \vec{j})*) if the following hold

1. for all $i < n$, $k_i = j_i$,
2. for all $i \geq n$, k_i is a square root of j_i ,
3. $\bar{\lambda}_J = \bar{\lambda}_K$.

A key difference between limit roots and square roots is the following: if k is a square root of j then $\text{crit}(k) < \text{crit}(j)$ (since $k(\text{crit}(k)) = \text{crit}(j)$), but if (K, \vec{k}) is a limit root of (J, \vec{j}) then it is possible to have $\text{crit}(K) = \text{crit}(J)$ (in particular, iff K is n -close to J for $n \geq 1$). Hence this opens up the possibility of a sequence of inverse limits $\langle K^i \mid i < \omega \rangle$ such that for all $i < \omega$, K^{i+1} is a limit root of K^i . We can in fact achieve this by considering an inverse limit (J, \vec{j}) such that as $i \rightarrow \omega$, the strength of j_i increases. We illustrate this phenomenon in the following lemma.

Lemma 14 (C.). Assume there is an elementary embedding $j : L_\omega(V_{\lambda+1}) \rightarrow L_\omega(V_{\lambda+1})$. Let E be the set of inverse limits (J, \vec{j}) such that for some sequence $\langle i_n \mid n < \omega \rangle$ below ω the following hold:

1. $\lim_{n \rightarrow \omega} i_n = \omega$,
2. for all $n < \omega$, j_n extends to an elementary embedding $L_{i_n}(V_{\lambda+1}) \rightarrow L_{i_n}(V_{\lambda+1})$.

Then E satisfies the following: for any $(J, \vec{j}) \in E$ there is an $n < \omega$ such that for all $a \in V_{\bar{\lambda}_{J+1}}$ and $b \in V_{\lambda+1}$, there is $(K, \vec{k}) \in E$ such that the following hold:

1. K is an n -close limit root of J ,
2. $K(a) = J(a)$ and $b \in \text{rng } K_n$.

Notice that we would not be able to have such a fact be true about square roots of embeddings. The key point is that we only require that $b \in \text{rng } K_n$ rather than $b \in \text{rng } K$. It is also important in practice that the given n work for any a and b (although this is not much more difficult to achieve).

Proof. For $(J, \vec{j}) \in E$, let $\langle i_n \mid n < \omega \rangle$ witness this fact. Then let n be such that for all $n' \geq n$, $i_{n'} > 0$. We can use the proof of Proposition 12 on J_n , a and b to get K_n . Let K be the n -close limit root of J defined this way. We then have that

$$\begin{aligned} K(a) &= (k_0 \circ k_1 \circ \cdots \circ k_{n-1})(K_n(a)) \\ &= (j_0 \circ j_1 \circ \cdots \circ j_{n-1})(K_n(a)) \\ &= (j_0 \circ j_1 \circ \cdots \circ j_{n-1})(J_n(a)) = J(a) \end{aligned}$$

and $b \in \text{rng } K_n$. □

We say that E is *saturated* if it satisfies the conclusion of Lemma 14. The following then illustrates the type of reflection result which can be obtained for inverse limits.

Theorem 15 (C.[5]). *Suppose that α is good and there is an elementary embedding*

$$L_{\alpha+\omega}(V_{\lambda+1}) \rightarrow L_{\alpha+\omega}(V_{\lambda+1}).$$

Then there is $\bar{\alpha}$, $\bar{\lambda}$ and a saturated set of inverse limits E such that for all (J, \vec{j}) , J extends to an elementary embedding

$$J : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \rightarrow L_{\alpha}(V_{\lambda+1}).$$

The importance of this stronger form of reflection¹⁵ comes from its structural implications for $L(V_{\lambda+1})$. This fact allows us to isolate a stronger property of subsets $X \subseteq V_{\lambda+1}$ which appears to be more closer to an AD-like axiom for $L(X, V_{\lambda+1})$ than the property that X is Icarus. We will investigate this idea in Section 5.

4 Structure of $L(V_{\lambda+1})$

We now look at the structure of $L(V_{\lambda+1})$ under the assumption that I_0 holds at λ . The main theme in this study is that the structure of $L(V_{\lambda+1})$ under I_0 is very similar to the structure of $L(\mathbb{R})$ assuming $AD^{L(\mathbb{R})}$. However, there are several important differences between the two situations which we highlight:

Perhaps the most apparent distinction, but the most easily remedied, is that I_0 is not a first-order statement in $L(V_{\lambda+1})$, whereas AD is a first-order statement in $L(\mathbb{R})$. The statement $I_0(< \Theta_\lambda)$ is however a first-order statement in $L(V_{\lambda+1})$, and most of the structural implications of I_0 for $L(V_{\lambda+1})$ follow from this weaker axiom.

Woodin showed that assuming a proper class of Woodin cardinals, the theory of $L(\mathbb{R})$ is generically absolute. Such a fact is not the case for $L(V_{\lambda+1})$ however, as the theory of V_λ is not generically absolute for small forcings (such as changing the size of the continuum). Therefore in deciding whether a given property holds in $L(V_{\lambda+1})$ assuming I_0 , one must also allow for the possibility that it is dependent on V . One solution to this problem, suggested by Woodin, is to consider the structure of $L(V_{\lambda+1})$ in a canonical inner model satisfying I_0 . However, as such an ‘ultimate L ’ does not yet exist, there have been no such results. Nevertheless there are many important properties of $L(V_{\lambda+1})$ which do not depend on V , and we will describe these below.

Arguably the most important distinction is the following. The structural properties of $L(V_{\lambda+1})$ under I_0 do not necessarily translate into properties of $L(X, V_{\lambda+1})$ assuming $X \subseteq V_{\lambda+1}$ is Icarus. The axiom I_0 therefore does not appear to be an appropriate analog of AD in the sense of being a fundamental regularity property, for instance. This fact naturally leads to the question of what would be an appropriate AD-like axiom, and this question is the subject of Section 5.

¹⁵In fact you need a slightly stronger form of reflection, Strong Inverse Limit Reflection, which we will define below.

Some of the results below require that the embedding j be *proper*. As we are focusing on the case of $L(V_{\lambda+1})$, we refer the reader to [26] Section 3 for the definition of proper elementary embeddings. In the case of $L(V_{\lambda+1})$, the word proper can simply be removed from everything we say (although not all I_0 embeddings are proper).

4.1 Measurable cardinals below Θ_λ

We begin by looking at measurable cardinals in $L(V_{\lambda+1})$. We first recall the situation of $L(\mathbb{R})$, where it has been shown that $\Theta^{L(\mathbb{R})}$ is a limit of large cardinals.

Theorem 16 (Solovay(see [14]), Steel[25]). *Assume $AD^{L(\mathbb{R})}$. Then in $L(\mathbb{R})$, ω_1 is measurable, and in fact the club filter is an ultrafilter. Also every regular cardinal below Θ is measurable.*

Woodin was able to show a number of these facts hold for $L(X, V_{\lambda+1})$ assuming X is Icarus.

Theorem 17 (Woodin[26]). *Assume that $j : L(X, V_{\lambda+1}) \rightarrow L(X, V_{\lambda+1})$ is a proper elementary embedding. Then λ^+ is measurable in $L(X, V_{\lambda+1})$, and in fact $\Theta = \Theta^{L(X, V_{\lambda+1})}$ is a limit of γ such that the following hold.*

1. γ is weakly inaccessible.
2. $\gamma = \Theta^{L_\gamma(X, V_{\lambda+1})}$ and $j(\gamma) = \gamma$.
3. for all $\beta < \gamma$, $P(\beta) \cap L(X, V_{\lambda+1}) \subseteq L_\beta(X, V_{\lambda+1})$.
4. for cofinally many $\kappa < \gamma$, κ is measurable in $L(X, V_{\lambda+1})$ as witnessed by the club filter on a stationary set,
5. $L_\gamma(X, V_{\lambda+1}) \prec L_\Theta(X, V_{\lambda+1})$.

Below we will sketch a proof that there is a stationary subset S of λ^+ such that the club filter restricted to S is an ultrafilter in $L(V_{\lambda+1})$ (Theorem 22). This fact shows that λ^+ is indeed measurable in $L(V_{\lambda+1})$. The main point is that the existence of the I_0 embedding shows that λ^+ cannot be partitioned into λ -many disjoint stationary sets. We mention a few questions left open by this analysis.

Question 18. *If I_0 holds at λ , in $L(V_{\lambda+1})$ are λ^{++} , λ^{+++} , etc. regular or singular?*

Question 19. *If I_0 holds at λ , in $L(V_{\lambda+1})$ is every regular cardinal below Θ_λ and above λ measurable?*

In the course of proving this theorem, Woodin also proved the following analog of Moschovakis' Coding Lemma.

Theorem 20 (Woodin). *Assume that $X \subseteq V_{\lambda+1}$ is Icarus. Suppose $\rho : V_{\lambda+1} \rightarrow \eta$ is a surjection with $\rho \in L(X, V_{\lambda+1})$. Then there exists $\gamma_\rho < \Theta^{L(X, V_{\lambda+1})}$ such that for all sets $A \subseteq V_{\lambda+1} \times V_{\lambda+1}$ with $A \in L(X, V_{\lambda+1})$, there exists $B \subseteq V_{\lambda+1} \times V_{\lambda+1}$ such that*

1. $B \subseteq A$,
2. for all $\alpha < \eta$, if there exists $(a, b) \in A$ with $\rho(a) = \alpha$, then there exists $(a, b) \in B$ with $\rho(a) = \alpha$,
3. $B \in L_{\gamma_\rho}(X, V_{\lambda+1})$.

This theorem however gives a rather coarse version of the Coding Lemma, and therefore leaves open such questions as the following.

Question 21 (Woodin). *If I_0 holds at λ , is*

$$P(\lambda^+) \cap L(V_{\lambda+1}) = P(\lambda^+) \cap L_\lambda(H(\lambda^+))?$$

As mentioned above, a key fact in the proof of Theorem 17 is a restriction on partitioning regular cardinals into stationary sets. In particular we have the following.

Theorem 22 (Woodin). *Assume $X \subseteq V_{\lambda+1}$ is Icarus and $\gamma < \Theta$ is such that $\text{cof}(\gamma) > \lambda$. Then for*

$$S = \{\xi < \gamma \mid \text{cof}(\xi) = \omega\},$$

there is a partition

$$\langle S_\alpha \mid \alpha < \eta \rangle \in L(X, V_{\lambda+1}),$$

of S such that $\eta < \lambda$ and such that in $L(X, V_{\lambda+1})$, for all $\alpha < \eta$, $\mathcal{F} \upharpoonright S_\alpha$ is an ultrafilter, where \mathcal{F} is the ω -club filter on γ .

Proof. Suppose towards a contradiction that there is such a γ , and let γ^* be least such. Hence for j an $I_0(X)$ -embedding, we have $j(\gamma^*) = \gamma^*$. Let $\langle S_\alpha \mid \alpha < \text{crit}(j) \rangle$ be a partition of S into $\text{crit}(j)$ -many disjoint stationary sets. Let

$$\langle T_\alpha \mid \alpha < j(\text{crit}(j)) \rangle = j(\langle S_\alpha \mid \alpha < \text{crit}(j) \rangle).$$

Then by elementarity $\langle T_\alpha \mid \alpha < j(\text{crit}(j)) \rangle$ is a partition of S into disjoint stationary sets (here we are using that $j(S) = S$). Note that

$$C = \{\delta < \gamma^* \mid j(\delta) = \delta\}$$

is ω -club below γ^* . Hence there is $\delta \in C \cap T_{\text{crit}(j)}$. But there is some $\alpha < \text{crit}(j)$ such that $\delta \in S_\alpha$. And hence

$$j(\delta) = \delta \in T_\alpha = j(S_\alpha).$$

And this is a contradiction since then T_α and $T_{\text{crit}(j)}$ are not disjoint. □

Note that in Theorem 22 (if j is finitely iterable) we can replace S by $S^\beta = \{\xi < \gamma \mid \text{cof}(\xi) = \beta\}$ where $\beta < \lambda$ is a regular cardinal. We then modify the proof by considering a large enough finite iterate of j so that C is indeed β -club below γ^* .

We now look at the specific case of λ^+ in $L(V_{\lambda+1})$. In that case we have the following.

Theorem 23 (Woodin). *Assume I_0 holds at λ . Then in $L(V_{\lambda+1})$, λ^+ can be partitioned into stationary sets S_α for $\alpha < \lambda$ such that for all $\alpha < \lambda$, $\mathcal{F} \upharpoonright S_\alpha$ is an ultrafilter, where \mathcal{F} is the club filter on λ^+ .*

One question is to what extent this theorem can be improved so that it more closely resembles the corresponding result under AD that ω_1 is measurable as witnessed by the club filter. Since there are λ many regular cardinals below λ^+ , this result cannot be improved as stated, but one might ask whether the club filter restricted to $S^\kappa = \{\eta < \lambda^+ \mid \text{cof}(\eta) = \kappa\}$ is an ultrafilter. The next theorem shows that for κ uncountable this cannot be proven from I_0 .

Theorem 24 (Woodin). *Suppose I_0 holds at λ . Let $\kappa < \lambda$ be an uncountable cardinal and $g \subseteq \text{Coll}(\kappa, \kappa^+)$ be V -generic. Then in $V[g]$, I_0 holds at λ and in $L(V[g]_{\lambda+1})$, S^κ can be partitioned into two stationary sets.*

Proof. Let $S_1 = \{\alpha < \lambda^+ \mid \text{cof}(\alpha)^V = \kappa\}$ and $S_2 = \{\alpha < \lambda^+ \mid \text{cof}(\alpha)^V = \kappa^+\}$. Since g does not add any ω -sequences we have that $V_{\lambda+1} \in L(V[g]_{\lambda+1})$ since $V_\lambda \in V[g]_{\lambda+1}$. So clearly $S_1, S_2 \in L(V[g]_{\lambda+1})$, they are stationary, and they partition S^κ . \square

In fact Woodin has shown a stronger fact using Radin forcing.

Theorem 25 (Woodin). *Suppose I_0 holds at λ as witnessed by j . Let $\kappa < \text{crit}(j)$. Then there is $\mathbb{P} \in V_{\text{crit}(j)}$ such that for $g \subseteq \mathbb{P}$ V -generic, we have in $V[g]$ that I_0 holds at λ , $V_\kappa = V[g]_\kappa$, and in $L(V[g]_{\lambda+1})$, S^{ω_1} can be partitioned into κ -many stationary sets.*

This theorem shows that I_0 alone gives virtually no information as far as to what extent, for instance, S^{ω_1} can be partitioned. Similar results hold for cofinalities larger than ω_1 . Importantly, however, it leaves open the case of $\kappa = \omega$, and in fact Woodin proved the following theorem which shows that the above proof fails in the $\kappa = \omega$ case. This theorem is a corollary of Theorems 36 and 35.

Theorem 26 (Woodin). *Suppose that I_0 holds at λ . Suppose that $\mathbb{P} \in V_\lambda$, g is \mathbb{P} -generic over V and*

$$(\lambda^\omega)^V \neq (\lambda^\omega)^{V[g]}.$$

Then in $V[g]$,

$$V_{\lambda+1} \notin L(V[g]_{\lambda+1}).$$

Now, concentrating on the case of S^ω , the following theorem gives evidence that perhaps the club filter is an ultrafilter when restricted to S^ω .

Theorem 27 (C., Woodin (for $L_\lambda(V_{\lambda+1})$)). *Suppose that I_0 holds at λ . Then there are no disjoint stationary (in V) sets $S_1, S_2 \in L(V_{\lambda+1})$ such that $S_1, S_2 \subseteq S^\omega$.*

However this theorem does not quite show that in $L(V_{\lambda+1})$ there are no disjoint stationary subsets of S^ω , since such sets might not be stationary in V . Hence we have the following question.

Question 28 (Woodin). *Suppose I_0 holds at λ . In $L(V_{\lambda+1})$ is the club filter restricted to S^ω an ultrafilter?*

As for partition properties at λ^+ , we have since ω_1 -DC holds in $L(V_{\lambda+1})$ that

$$L(V_{\lambda+1}) \not\models \lambda^+ \rightarrow (\lambda^+)_\lambda^{\omega_1}.$$

This leads to the following question.

Question 29 (Woodin). *If I_0 holds at λ , then for all $\alpha < \omega_1$ do we have*

$$L(V_{\lambda+1}) \models \lambda^+ \rightarrow (\lambda^+)_\lambda^\alpha?$$

Woodin showed that for α such that $\omega \cdot \alpha = \alpha$, if this properties fails in V then it must fail in all generic extensions by a forcing $\mathbb{P} \in V_\lambda$. In the positive direction the best known partial result is the following, whose proof uses Radin forcing together with generic absoluteness properties, Theorems 62 and 82 below.

Theorem 30 (Woodin). *Suppose I_0 holds at λ . Then for all $\alpha < \beta < \omega_1$,*

$$L(V_{\lambda+1}) \models \lambda^+ \rightarrow (\beta)_\lambda^\alpha.$$

4.2 Perfect set property

The perfect set property for a set of reals X is a strong version of the continuum hypothesis. It states that either X is countable or it contains a perfect set¹⁶, in which case $|X| = |\mathbb{R}| = 2^{\aleph_0}$. While the Axiom of Choice implies that there is a set of reals which does not satisfy the perfect set property, in fact AD implies that every set of reals has the perfect set property.

Theorem 31 (M. Davis[7]). *Assume AD holds. Then if $X \subseteq \mathbb{R}$, either X is countable or X contains a perfect set.*

Using $U(j)$ -representations (see Section 5), Woodin was able to show that a version of the perfect set theorem in the context of AD holds for $L_\lambda(V_{\lambda+1})$. To do this, we regard $V_{\lambda+1}$ as a topological space with basic open sets $O_{(a,\alpha)}$, where $\alpha < \lambda$, $a \subseteq V_\alpha$ and

$$O_{(a,\alpha)} = \{b \in V_{\lambda+1} \mid b \cap V_\alpha = a\}.$$

¹⁶A set is *perfect* if it is closed and has no isolated points.

Since $\text{cof}(\lambda) = \omega$, this is a metric topology, and it is complete.

Perfect sets in the usual context of the reals are somewhat simpler than perfect sets in the context of $V_{\lambda+1}$, since perfect subsets of $V_{\lambda+1}$ can have varying sizes. We are most interested in perfect sets which have the largest possible size, so that our perfect set property continues to be a strong form of the generalized continuum hypothesis. With this in mind we make the following definition.

Definition 32. Suppose $P \subseteq V_{\lambda+1}$ is a perfect set. For $\kappa \leq \lambda$, we say that P is κ -*splitting*¹⁷ if for all $a \in P$, $\gamma < \lambda$, and $\alpha < \kappa$, there is a $\beta < \lambda$ such that

$$|\{b \cap V_\beta \mid b \in P \text{ and } b \cap V_\gamma = a \cap V_\gamma\}| \geq \alpha.$$

We say that $X \subseteq V_{\lambda+1}$ has the λ -*splitting perfect set property* if either $|X| \leq \lambda$ or X contains a λ -splitting perfect set.

Using $U(j)$ -representations (see Section 5.1), Woodin was able to prove the following.

Theorem 33 (Woodin). *Suppose I_0 holds at λ . Then if $Z \in L_\lambda(V_{\lambda+1}) \cap V_{\lambda+2}$, then either $|Z| \leq \lambda$ or Z contains a perfect set.*

Shi-Woodin later improved this fact using Theorem 62 with diagonal supercompact Prikry forcing.

Theorem 34 (Shi-Woodin[22]). *Suppose I_0 holds at λ . Then for all $Z \in L_\lambda(V_{\lambda+1}) \cap V_{\lambda+2}$, either $|Z| \leq \lambda$ or Z contains a λ -splitting perfect set, and therefore $|Z| = 2^\lambda$, so Z has the λ -splitting perfect set property.*

Using the tool of inverse limit reflection, we were able to improve this result to all of $L(V_{\lambda+1})$.

Theorem 35 (C.[5]). *Suppose I_0 holds at λ . Then if $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$, then either $|Z| \leq \lambda$ or Z contains a λ -splitting perfect set, and therefore $|Z| = 2^\lambda$.*

There are several interesting consequences of this perfect set theorem. First we obtain some information on $L(V_{\lambda+1})$ in forcing extensions. In particular we have the following.

Theorem 36 (Woodin[26] Theorem 175). *Suppose that I_0 holds at λ . Suppose that $\mathbb{P} \in V_\lambda$, g is \mathbb{P} -generic over V and*

$$(\lambda^\omega)^V \neq (\lambda^\omega)^{V[g]}.$$

Then $L(V_{\lambda+1}, V[g]_{\lambda+1})$ does not satisfy the λ -splitting perfect set theorem.

Proof sketch. The point is that in $V[g]$, since $|V_{\lambda+1}| > \lambda$, there must be a λ -splitting perfect set $P \subseteq V_{\lambda+1}$ such that $P \in L(V_{\lambda+1}, V[g]_{\lambda+1})$. But since \mathbb{P} is a small forcing, this shows¹⁸ that in fact any ω sequence of ordinals below λ can be coded into an element of P . And hence $(\lambda^\omega)^{V[g]} \subseteq V$, which is a contradiction. \square

¹⁷One can also define κ -splitting in terms of the splitting number of the natural tree of initial segments of elements of P .

¹⁸See [26] Theorem 175, Page 395.

Theorem 36 gives an interesting example of an $X \subseteq V_{\lambda+1}$ such that X is Icarus, but subsets of $V_{\lambda+1}$ in $L(X, V_{\lambda+1})$ do not all satisfy a certain regularity property, namely the λ -splitting perfect set property. This example has a number of interesting consequences for the AD-like axioms which we will consider in Section 5. In particular it gives a contradiction to the assertion that I_0 is in fact the appropriate analog of AD in the context of $V_{\lambda+1}$, since $I_0(X)$ holds for this X , and it at the same time displays the difficulty in propagating the AD-like axioms of Section 5, as these axioms do not propagate generally throughout all $L(X, V_{\lambda+1})$ structures. On the other hand, results such as Theorem 36 potentially give criteria for evaluating extensions of $L(V_{\lambda+1})$, and they highlight the importance of propagating the potential AD-like axioms beyond I_0 (see Section 6).

4.3 Uniformization

The notion of a uniformization of a relation R on $\mathbb{R} \times \mathbb{R}$ is easily generalized to $V_{\lambda+1}$.

Definition 37. Suppose $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$ is a relation. Then $U \subseteq R$ is a *uniformization* of R if for all $x \in \text{dom}(R)$, there is exactly one $y \in V_{\lambda+1}$ such that $(x, y) \in U$.

However, the question in $L(V_{\lambda+1})$ of which relations have a uniformization is not very well understood at this moment. As in the case of $L(\mathbb{R})$, uniformization cannot hold in $L(V_{\lambda+1})$.

Fact 38. *In $L(V_{\lambda+1})$ there is a set $Z \subseteq V_{\lambda+1} \times V_{\lambda+1}$ such that Z has no uniformization.*

Proof. Work in $L(V_{\lambda+1})$. Let Z be defined by

$$Z = \{(x, y) \mid y \text{ is not OD from } x\}.$$

Suppose $U \subseteq Z$ is a uniformization. Then U is OD from some $x \in V_{\lambda+1}$. But then $U(x)$ is OD from x , which is a contradiction. \square

In the case of $L(\mathbb{R})$, however, uniformizations can be obtained from scales, and every $\Sigma_1(L(\mathbb{R}), \{\mathbb{R}\} \cup \mathbb{R})$ -set has a scale (see [21]).

In the $L(V_{\lambda+1})$ case on the other hand, where the first failure of uniformization occurs is far from clear.

Question 39 (Woodin). *Suppose I_0 holds at λ . Let $R_{\text{sqr}} \subseteq V_{\lambda+1} \times V_{\lambda+1}$ be defined by*

$$R_{\text{sqr}} = \{(j, k) \mid j, k : V_{\lambda+1} \rightarrow V_{\lambda+1} \text{ are elementary and } k \text{ is a square root of } j\}.$$

Does R_{sqr} have a uniformization in $L(V_{\lambda+1})$?

Most of the results on uniformization for $L(V_{\lambda+1})$ are negative results stemming from Fact 38 and the positive results on AD-like properties for subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$. In particular we have Corollaries 68 and 76 which state that uniformization does not follow from either $U(j)$ -representability or strong inverse limit reflection.

We now give some background surrounding Question 39 and show that we can get somewhat close to such a uniformization. In particular we will restrict the domain of the relation R_{sqr} , and then show that the restriction has a uniformization. Of course, this process involves a restriction of the domain, and therefore does not succeed in uniformizing R_{sqr} . However, we can also show that if a uniformization of R_{sqr} exists, it is an extension of the type that we define below. This fact gives perhaps a small indication that such a uniformization does not exist.

We first make the following definition for $\alpha < \Theta_\lambda$, $j : V_\lambda \rightarrow V_\lambda$ elementary, and $b \in L_\alpha(V_{\lambda+1})$:

$$E_\alpha(b) = \{k : V_\lambda \rightarrow V_\lambda \mid k \text{ extends to } \hat{k}, \hat{k} : L_\alpha(V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1}) \text{ with } b \in \text{rng } \hat{k}\},$$

$$E_\alpha = E_\alpha(\emptyset) \quad \text{and} \quad E_\alpha^j(b) = \{k \in E_\alpha(b) \mid k \text{ is a square root of } j\}.$$

Proposition 40. *Assume I_0 holds at λ . Suppose that $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$ and $R \in L(V_{\lambda+1})$. Let $\alpha < \Theta$ be good and such that $R \in L_\alpha(V_{\lambda+1})$. Let $j \in E_{\alpha+1}$ and suppose there is $k \in E_{\alpha+1}^j(R)$ and $a \in V_{\lambda+1}$ such that $(k, a) \in R$. Then $R \upharpoonright E_{\alpha+1}^j(R, k(a))$ is uniformizable in $L(V_{\lambda+1})$.*

Proof. Let j, k and a be as in the hypothesis. Then by applying \hat{k} to the fact that $(k, a) \in R$ we have that $(j, k(a)) \in \hat{k}(R)$. But $\hat{k}(R) = \hat{j}(R)$ since k is a square root of j , α is good, and $R \in \text{rng } k$. So $(j, k(a)) \in \hat{j}(R)$.

We define

$$U = \{(\ell, \ell^{-1}(k(a))) \mid \ell \in E_{\alpha+1}^j(R, k(a))\}.$$

Note that we have

$$(j, k(a)) \in \hat{j}(R) \Rightarrow (\ell, \ell^{-1}(k(a))) \in \hat{\ell}^{-1}(\hat{j}(R)) = R$$

since we can pull back by $\hat{\ell}$, and $\ell \in E_{\alpha+1}^j(R, k(a))$. Hence, U is a uniformization of $R \upharpoonright E_{\alpha+1}^j(R, k(a))$, and clearly $U \in L(V_{\lambda+1})$. \square

Corollary 41. *Assume I_0 holds at λ . Let $j \in E_1$ and suppose $(j, k) \in R_{sqr}$. Then $R_{sqr} \upharpoonright E_1^j(k)$ is uniformizable in $L(V_{\lambda+1})$.*

We can also prove the following partial converse of the above proposition.

Proposition 42. *Assume I_0 holds at λ . Suppose that $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$ and $R \in L(V_{\lambda+1})$. Also assume that U is a uniformization of R and $U \in L_\alpha(V_{\lambda+1})$ where α is good. Then for any $j \in E_{\alpha+1}$ if $b \in V_{\lambda+1}$ is such that $(j, b) \in \hat{j}(U)$, we have that*

$$U \upharpoonright E_{\alpha+1}^j(U, b) = \{(\ell, \ell^{-1}(b)) \mid \ell \in E_{\alpha+1}^j(U, b)\}.$$

Proof. To see this, suppose that $(j, b) \in \hat{j}(U)$ is as in the hypothesis. Let $\ell \in E_{\alpha+1}^j(U, b)$. Then we have that

$$(j, b) \in \hat{j}(U) \Rightarrow (\ell, \ell^{-1}(b)) \in \hat{\ell}^{-1}(\hat{j}(U)) = U.$$

\square

For the next definition we use the notation that $j_{(n)}$ is the n th iterate of j . That is, $j_{(0)} = j$ and $j_{(n+1)} = j(j_{(n)})$ for $n < \omega$.

Definition 43. For $j \in E_\kappa$ and $a \in L_\kappa(V_{\lambda+1})$, we let $\tilde{E}_\kappa^j(a)$ be the set of embeddings $k \in E_\kappa$ such that the following hold:

1. For some m and $n < \omega$, $j_{(m)} = k_{(n)}$.
2. For some (all) m and n as in (1), $j_{0,m}(a) \in \text{rng } k_{0,n}$.

If $k \in \tilde{E}_\kappa^j(a)$, we set $\tilde{E}_\kappa^j(k)[a] = k_{0,n}^{-1}(j_{0,m}(a))$ where n and m are as in (2).

We can then improve Proposition 40 to these larger sets in the case R is fixed by j .

Proposition 44. *Suppose that $j \in E_\kappa$ and $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$, $R \in L_\kappa(V_{\lambda+1})$ is such that $j(R) = R$. Suppose that $(j, a) \in R$. Then for $U_{j,a}$ the set of (k, b) such that $k \in \tilde{E}_\kappa^j(\{a, R\})$ and $b = \tilde{E}_\kappa^j(k)[a]$, we have that the following:*

1. $U_{j,a}$ is a uniformization of $R \upharpoonright \tilde{E}_\kappa^j(\{a, R\})$.
2. If κ is good and j extends to an elementary embedding $j^* : L_{\kappa+1}(V_{\lambda+1}) \rightarrow L_{\kappa+1}(V_{\lambda+1})$ then $j^*(U_{j,a}) = U_{j,a}$.

The proof of this proposition is very similar to the proof of Proposition 40. And we also have the corresponding result to Proposition 42, though we leave the statement to the reader. These results lead us to the following conjecture.

Conjecture 45. *Assuming I_0 holds at λ as witnessed by j , the relations $R_{\text{sqr}} \upharpoonright E_0^j$ and $R_{\text{sqr}} \upharpoonright \tilde{E}_0^j(\emptyset)$ have no uniformizations in $L(V_{\lambda+1})$.*

We make this conjecture as a result of the above converse propositions, which seem to indicate that the only method for creating uniformizations of these types of relations is through the above ‘pullback’ method, which is necessarily insufficient for a full uniformization.

5 AD-like axioms for $L(V_{\lambda+1})$

We have seen in the previous section that many structural results can be obtained for $L(V_{\lambda+1})$ by assuming I_0 holds at λ . These facts point to I_0 being an ‘AD-like axiom’ in the sense that it gives a structure theory of $L(V_{\lambda+1})$. As we noted above, however, there are various objections which could be raised against this assertion. In this section we introduce several alternative axioms¹⁹ which have been proposed as AD-like axioms. First we will introduce $U(j)$ -representations which were defined by Woodin, then we will introduce inverse limit reflection which was defined by the author, and lastly we will define some more recent axioms motivated by $U(j)$ -representations.

¹⁹Note that, importantly, we will see that these axioms do follow from I_0 for $L(V_{\lambda+1})$. It would therefore make sense to think of I_0 as being the large cardinal axiom which is implying the AD-like axiom for $L(V_{\lambda+1})$ much in the same way as large cardinals imply $AD^{L(\mathbb{R})}$.

5.1 $U(j)$ -representations

By way of motivation we first define Suslin and weakly homogeneously Suslin representations. In general, in the context of \mathbb{R} , for a set A we say that $X \subseteq \mathbb{R}$ has a *tree representation* T on $\omega \times A$ if T is a tree²⁰ on²¹ $(\omega \times A)^{<\omega}$ and for any $x \in \mathbb{R}$,

$$x \in X \iff T_x \text{ is illfounded.}$$

Here

$$T_x = \{b \mid \exists n < \omega ((x \upharpoonright n, b) \in T)\}$$

is the tree on $A^{<\omega}$ associated to x . Saying that T_x is illfounded is equivalent to the existence of some $b \in A^\omega$ such that for all $n < \omega$, $(x \upharpoonright n, b \upharpoonright n) \in T$.

Tree representations in general are usually not of interest, as any $X \subseteq \mathbb{R}$ has a trivial tree representation on $\omega \times \mathbb{R}$, for instance, defined by

$$(x, b) \in T \iff x \in \omega^{<\omega} \text{ and } \exists x^* \in X (b = (x^*, x^*, \dots, x^*) \wedge |b| = |x| \wedge x = x^* \upharpoonright |x|).$$

If however we restrict the set A , we arrive at a much more interesting notion. For κ an ordinal, we say that $X \subseteq \mathbb{R}$ is κ -*Suslin* if X has a tree representation on $\omega \times \kappa$. Of course, if AC holds and $\kappa \geq |X|$, this again is a rather uninteresting notion. However, in the context of determinacy, these Suslin representations are very important(see [13]).

In the context of AC however, the notion of a Suslin representation can be augmented with the use of measures. We then arrive at what are called homogeneously Suslin and weakly homogeneously Suslin representations (for a complete introduction to these representations see [16]).

For δ a cardinal and Z a set, we say that T is a δ -*weakly homogeneous* tree if there exists a countable set σ of δ -complete measures on $Z^{<\omega}$ such that for all $x \in p[T]$, there exists a countably complete tower of measures²² $\langle \mu_n \mid n < \omega \rangle$ such that for all $n < \omega$, $\mu_n(T_{x \upharpoonright n}) = 1$. And $X \subseteq \mathbb{R}$ is δ -weakly homogeneously Suslin if there is a δ -weakly homogeneous tree T such that $A = p[T]$.

We now define $U(j)$ -representations, which are analogous to weakly homogeneously Suslin representations, but in the context of $L(V_{\lambda+1})$. The main difference is that we restrict the measures which can be used to witness homogeneity to a collection of measures called $U(j)$ -measures. In addition we have to work with the particularities of working with λ instead of ω ; in particular there is no natural cofinal sequence in λ .

We first introduce some terminology. We fix $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ an iterable elementary embedding with $\text{crit}(j) < \lambda$. We say that a sequence $\vec{a} = \langle a_n \mid n < \omega \rangle$ is *weakly fixed by* j if for all $n < \omega$, $|a_n| < \lambda$, $a_n \subseteq a_{n+1}$ and there exists an m such that $j_{(m)}(a_n) = a_n$.

²⁰A tree is a set closed under initial segments.

²¹We will confuse $(\omega \times A)^{<\omega}$ with $\omega^{<\omega} \times A^{<\omega}$, as is customary.

²²A tower of measures is such that each measure projects down to the previous measures, and n th measure concentrates on n -sequences. So for instance $A \in \mu_1 \iff \{t \frown \langle s \rangle \mid t \in A, s \in Z\} \in \mu_2$. To be countable complete means that each sequence of measure one sets can be threaded. Equivalently, the direct limit of the ultrapowers given by the measures is wellfounded. See [16] for more details.

For S a set of embeddings we let

$$\text{Fix}(S) = \{a \mid \forall k \in S (k(a) = a)\},$$

and we set $\text{Fix}(k) = \text{Fix}(\{k\})$. We then set $\mathcal{F}^k(\kappa, a)$ to be the filter generated by the sets $\text{Fix}(S)$ where $S \in [E^k(\kappa, a)]^\lambda$.

For $\vec{a} \in [L_\kappa(V_{\lambda+1})]^\omega$ weakly fixed by j , we let $\mathcal{A}^j(\kappa, \vec{a})$ be the set of sequences $\langle A_n \mid n < \omega \rangle$ such that for all $n < \omega$, $A_n \in \mathcal{F}^j(\kappa, a_n)$. We also set for $\vec{A} \in \mathcal{A}^j(\kappa, \vec{a})$, $T^F(\vec{A})$ to be the largest tree T such that any node s of T is such that for all large enough n , $s \in A_n$.

We now proceed to define the set of $U(j)$ -measures and $U(j)$ -representations.

Definition 46 (Woodin). Let $U(j)$ be the set of $U \in L(V_{\lambda+1})$ such that in $L(V_{\lambda+1})$ the following hold:

1. U is a λ^+ -complete ultrafilter.
2. For some $\gamma < \Theta$, $U \in L_\gamma(V_{\lambda+1})$.
3. For all sufficiently large $n < \omega$, $j_{(n)}(U) = U$ and for some $A \in U$,

$$\{a \in A \mid j_{(n)}(a) = a\} \in U.$$

The next lemma gives us a way of generating lots of $U(j)$ measures. The proof is very much along the lines of the proof of Theorem 22. For each ordinal κ , let $\Theta^{L_\kappa(V_{\lambda+1})}$ denote the supremum of the ordinals α such that there is a surjection $\rho : V_{\lambda+1} \rightarrow \alpha$ such that $\{(a, b) \mid \rho(a) < \rho(b)\} \in L_\kappa(V_{\lambda+1})$.

Lemma 47 (Woodin). Suppose $\kappa < \Theta$, $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$, $a \in L_\kappa(V_{\lambda+1})$ and that $j(\kappa, a) = (\kappa, a)$. Then there is $\delta < \text{crit}(j)$ and a partition $\{S_\alpha \mid \alpha < \delta\} \in L(V_{\lambda+1})$ of $L_\kappa(V_{\lambda+1})$ into $\mathcal{F}^j(\kappa, a)$ -positive sets such that for each $\alpha < \delta$,

$$\mathcal{F}^j(\kappa, a) \upharpoonright S_\alpha \in U(j).$$

Suppose that $\kappa < \Theta$ and $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$ and $\langle a_i \mid i < \omega \rangle$ is weakly fixed by j . Let $U(j, \kappa, \langle a_i \mid i < \omega \rangle)$ denote the set of $U \in U(j)$ such that there exists $n < \omega$ such that for all $k \in E^j(\kappa, \langle a_i \mid i \leq n \rangle)$,

$$\text{Fix}(k) \in U.$$

We can now define $U(j)$ -representations for subsets of $V_{\lambda+1}$.

Definition 48 (Woodin). Suppose $\kappa < \Theta$, κ is weakly inaccessible in $L(V_{\lambda+1})$, and $\langle a_i \mid i < \omega \rangle$ is an ω -sequence of elements of $L_\kappa(V_{\lambda+1})$ such that for all $i < \omega$ there is an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.

Suppose that $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$. Then Z is $U(j, \kappa, \langle a_i \mid i < \omega \rangle)$ -representable if there exists an increasing sequence $\langle \lambda_i \mid i < \omega \rangle$, cofinal in λ and a function

$$\pi : \bigcup \{V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} \mid i < \omega\} \rightarrow U(j, \kappa, \langle a_i \mid i < \omega \rangle)$$

such that the following hold:

1. For all $i < \omega$ and $(a, b, i) \in \text{dom}(\pi)$ there exists $A \subseteq (L(V_{\lambda+1}))^i$ such that $A \in \pi(a, b, i)$.
2. For all $i < \omega$ and $(a, b, i) \in \text{dom}(\pi)$, $\pi(a, b, i) \in U(j, \kappa, a_i)$.²³
3. For all $i < \omega$ and $(a, b, i) \in \text{dom}(\pi)$, if $m < i$ then

$$(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m) \in \text{dom}(\pi)$$

and $\pi(a, b, i)$ projects to $\pi(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m)$.

4. For all $x \subseteq V_\lambda$, $x \in Z$ if and only if there exists $y \subseteq V_\lambda$ such that

(a) for all $m < \omega$, $(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) \in \text{dom}(\pi)$,

(b) the tower

$$\langle \pi(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) \mid m < \omega \rangle$$

is well founded.

For $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$ we say that Z is $U(j)$ -representable if there exists $(\kappa, \langle a_i \mid i < \omega \rangle)$ such that Z is $U(j, \kappa, \langle a_i \mid i < \omega \rangle)$ -representable.

5.1.1 Propagation of $U(j)$ -representations

First we give an outline of how $U(j)$ -representations have been propagated throughout $L(V_{\lambda+1})$. Then we give details of some of the methods involved in this propagation.

It is relatively easy to see that $U(j)$ -representations are closed under unions of size λ and existential quantification (see [26], Lemmas 114, 115). However, the case of complements is much more complicated. In spite of this, Woodin was able to first propagate $U(j)$ -representations up to λ .

Theorem 49 (Woodin[26], Theorem 134). *Suppose j witnesses I_0 holds at λ . Then in $L(V_{\lambda+1})$ every set $X \in L_\lambda(V_{\lambda+1}) \cap V_{\lambda+2}$ is $U(j)$ -representable.*

He also showed that a certain continuous ill-foundedness condition called the Tower Condition (see Definition 50) implies that $U(j)$ -representations are closed under complements. We then showed [5] that the Tower Condition holds in general for $L(X, V_{\lambda+1})$ assuming that X is Icarus. This result by a Theorem of Woodin, pushed the $U(j)$ -representations in $L(V_{\lambda+1})$ beyond λ^+ . Beyond this point, Woodin also defined a certain game (see Definition 54) on fixed points of embeddings, which if one could prove had a high enough rank, then the set of $U(j)$ -representations could be propagated even further. We gave such an analysis[3], which effectively pushed the $U(j)$ -representations beyond the first Σ_1 -gap (Theorem 60). Finally, in order to show that every subset of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ has a $U(j)$ -representation, these representations were propagated along with j -Suslin representations (see [1] and Section 5.3).

²³This condition is slightly strengthened from the corresponding condition in [26]. The set of $U(j)$ -representable sets is unchanged however. We make this restriction because of useful properties exploited in [1].

We now look at some of the details of these theorems and some of the methods used in their arguments. First we start with the Tower Condition, which basically says that given a collection of $U(j)$ -measures there is a continuous choice of measure one sets, such that if there is an illfounded tower, this is witnessed by the chosen measure one sets.

Definition 50 (Woodin). Suppose $A \subseteq U(j)$, $A \in L(V_{\lambda+1})$, and $|A| \leq \lambda$. The *Tower Condition for A* is the following statement: There is a function $F : A \rightarrow L(V_{\lambda+1})$ such that the following hold:

1. For all $U \in A$, $F(U) \in U$.
2. Suppose $\langle U_i \mid i < \omega \rangle \in L(V_{\lambda+1})$ and for all $i < \omega$, there exists $Z \in U_i$ such that $Z \subseteq L(V_{\lambda+1})^i$, $U_i \in A$, and U_{i+1} projects to U_i . Then the tower $\langle U_i \mid i < \omega \rangle$ is wellfounded in $L(V_{\lambda+1})$ if and only if there exists a function $f : \omega \rightarrow L(V_{\lambda+1})$ such that for all $i < \omega$, $f \upharpoonright i \in F(U_i)$.

The *Tower Condition for $U(j)$* is the statement that for all $A \subseteq U(j)$ if $A \in L(V_{\lambda+1})$ and $|A| \leq \lambda$ then the Tower Condition holds for A .

Theorem 51. *Let $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ be elementary such that $\text{crit}(j) < \lambda$. Then the following hold:*

1. *Tower Condition for $U(j)$ holds in $L(V_{\lambda+1})$ (C. [5]).*
2. *The set of $U(j)$ -representable sets is closed under complements (Woodin [26]).*

Proof sketch. In order to prove (1), for a λ -sized set A of $U(j)$ -measures, there is no inverse limit J such that for all $U \in A$, U is in the range of the extension of J . However, we can almost achieve this in that any $U \in A$ is in the range of the extension of $(j_0 \circ j_1 \circ \dots \circ j_{n-1})(J_n)$ for some n . Having such a J , we pull back²⁴ to an \bar{A} which has size $\bar{\lambda}_j$. Using that each measure in $U(j)$ is λ^+ complete, we can show that \bar{A} has the Tower Condition. A tower function for \bar{A} can then be pushed forward to obtain a tower function for A .

To prove (2), we look at the tree of attempts to continuously illfounded the tower given by a $U(j)$ -representation π . □

In fact the proof holds more generally for X Icarus.

Theorem 52. *Let $j : L(X, V_{\lambda+1}) \rightarrow L(X, V_{\lambda+1})$ be a proper elementary embedding such that $\text{crit}(j) < \lambda$. Then the following hold:*

1. *Tower Condition for $U(j)$ holds in $L(X, V_{\lambda+1})$ (C. [5]).*
2. *The set of $U(j)$ -representable sets in $L(X, V_{\lambda+1})$ is closed under complements (Woodin [26]).*

²⁴We do not actually need to reflect, which is the reason why we can prove this result for $L(X, V_{\lambda+1})$.

As a consequence of the Tower Condition, Woodin showed that $U(j)$ -representations propagate beyond λ^+ .

Theorem 53 (Woodin[26]). *Suppose there exists an elementary embedding*

$$j : L(X, V_{\lambda+1}) \rightarrow L(X, V_{\lambda+1}).$$

Let Y be $U(j)$ -representable in $L(X, V_{\lambda+1})$. Let $\kappa = \lambda^+$ and set

$$\eta = \sup\{(\kappa^+)^{L[A]} \mid A \subseteq \lambda\}.$$

Then every set

$$Z \in L_\eta(Y, V_{\lambda+1}) \cap V_{\lambda+2}$$

is $U(j)$ -representable in $L(X, V_{\lambda+1})$.

To propagate $U(j)$ -representations beyond this point we now look at the following game on fixed points of embeddings, introduced by Woodin.

Definition 54. Suppose $\gamma < \Theta^{L(V_{\lambda+1})}$,

$$\text{Emb}(j, \gamma) := \{k \mid k : L_\gamma(V_{\lambda+1}) \rightarrow L_\gamma(V_{\lambda+1}) \text{ is elementary}\},$$

and

$$\langle a_i \mid i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$$

and we have:

1. $\gamma \leq \Theta^{L_\gamma(V_{\lambda+1})}$,
2. for all $i < \omega$, $a_i \subseteq a_{i+1} \subseteq \gamma$ and $|a_i| < \lambda$,
3. for all $i < \omega$, there exists an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.

Then let $G(j, \gamma, \langle a_i \mid i < \omega \rangle)$ denote the following game. Player I plays a sequence

$$\langle (\gamma_i, \langle b_m^i : m < \omega \rangle) : i < \omega \rangle$$

and player II plays a sequence $\langle \mathcal{E}_i : i < \omega \rangle$ such that the following hold:

1. $\mathcal{E}_i \subseteq \text{Emb}(j, \gamma_i)$, $|\mathcal{E}_i| \leq \lambda$, and for each $k \in \mathcal{E}_i$ there exists $m < \omega$ such that $k(b_m^i) = b_m^i$.
2. $\gamma_0 = \gamma$, $\gamma_{i+1} < \gamma_i$ and there exists $m < \omega$ such that

$$k(b_m^i) = b_m^i \Rightarrow k(\gamma_{i+1}) = \gamma_{i+1}$$

for all $k \in \mathcal{E}_i$.

3. for all $i < \omega$, $\gamma_i \leq \Theta^{L_{\gamma_i}(V_{\lambda+1})}$,

4. $\langle b_m^0 : m < \omega \rangle = \langle a_m : m < \omega \rangle$.
5. for all $m < \omega$, $b_m^i \subseteq b_{m+1}^i \subseteq \gamma_i$ and $|b_m^i| < \lambda$
6. for all $m < \omega$ there exists $m^* < \omega$ such that

$$k(b_{m^*}^i) = b_{m^*}^i \Rightarrow k(b_m^{i+1}) = b_m^{i+1}$$

for all $k \in \mathcal{E}_i$.

Of course II always wins this game, but we are interested in the rank of this game, which we define as follows.

Definition 55. Let $G_\delta(j, \gamma, \langle a_i | i < \omega \rangle)$ have the same definition as $G(j, \gamma, \langle a_i | i < \omega \rangle)$ except that II must also play ordinals $\delta_0 > \delta_1 > \dots$ such that $\delta_0 < \delta$. Then if δ is least such that II has a quasi-winning strategy in $G_\delta(j, \gamma, \langle a_i | i < \omega \rangle)$, then we set $\delta = \text{rank}(j, \gamma, \langle a_i | i < \omega \rangle)$.

Theorem 58 shows that for any $\delta < \Theta$ we can find γ and $\langle a_i | i < \omega \rangle$ such that

$$\text{rank}(j, \gamma, \langle a_i | i < \omega \rangle) \geq \delta.$$

That is, the rank of this game can be made arbitrarily large by an appropriate choice of parameters. We give some definitions now so that we can give sufficient criteria for the rank to be large.

Definition 56. Suppose $\gamma < \Theta^{L(V_{\lambda+1})}$, $S \subseteq L_\gamma(V_{\lambda+1})$, and $\langle a_i | i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$ and we have:

1. $\gamma \leq \Theta^{L_\gamma(V_{\lambda+1})}$,
2. for all $i < \omega$, $a_i \subseteq a_{i+1} \subseteq \gamma$ and $|a_i| < \lambda$,
3. for all $i < \omega$, $a_i \in \mathcal{F}_{\gamma+1}^\omega(j)$.
4. $S = \bigcup_{i < \omega} a_i$.

Then we say that $\langle a_i | i < \omega \rangle$ is a *j-stratification* of S .

Suppose that $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$. Note that for all $S \subseteq \mathcal{F}_\Theta^\omega(j)$ such that $|S| \leq \lambda$, there is a $\gamma < \Theta$ and a $\langle a_i | i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$ such that $\langle a_i | i < \omega \rangle$ is a *j-stratification* of S . Hence for any $\gamma < \lambda^+$, if $\langle a_i | i < \omega \rangle$ is a *j-stratification* of γ , then $\text{rank}(j, \gamma, \langle a_i | i < \omega \rangle) = \gamma$. An instructive example then is to show that $\text{rank}(j, \lambda^+, \emptyset) = \lambda^+$, which we leave to the reader.

Definition 57. Fix $\kappa < \Theta$ good with $\text{cof}(\kappa) > \lambda$. Let $S \subseteq \kappa$ such that $|S| \leq \lambda$. Then we say that S is *λ -threaded* if the following hold:

1. Suppose $\alpha < \sup S$ is such that there exists $\vec{\beta} \in S^{<\omega}$ and $a \in V_\lambda$ such that α is definable over $L_\kappa(V_{\lambda+1})$ from $\vec{\beta}$ and a . Then $\alpha \in S$.

2. Suppose $\alpha \in S$ is a limit and $\text{cof}(\alpha) < \lambda$. Then $S \cap \alpha$ is cofinal in α .

Theorem 58. *Let $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ be elementary. Fix $\kappa < \Theta$ good in $L(V_{\lambda+1})$. Suppose that S has a largest element α_0 , S is λ -threaded, and $\langle a_i \mid i < \omega \rangle$ is a j -stratification of S . Then $\text{rank}(j, \kappa + \alpha_0, \vec{a}) \geq \alpha_0$.*

Proof sketch. We guide Player I by using reflection. Roughly speaking, if a certain rank β is to be achieved, then this β is reflected to a $\bar{\beta}$, and then the least image of $\bar{\beta}$ above some γ by an inverse limit, say β^* is considered. If γ was chosen appropriately then this β^* is definable from parameters in V_λ (in particular $\bar{\beta}$), and hence is a fixed point of embeddings in \mathcal{E}_i for i large enough. The details of the proof involve a certain amount of bookkeeping to ensure that this strategy works. \square

We immediately have the following corollary, which gives the following result on $U(j)$ -representations by Theorem 148 of [26].

Corollary 59. *Suppose there exists an elementary embedding $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$. Then the supremum of $\text{rank}(j, \kappa, \vec{a})$ for all possible κ and \vec{a} is Θ .*

Theorem 60. *Assume there exists an elementary embedding $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$. Let κ be least such that*

$$L_\kappa(V_{\lambda+1}) \not\stackrel{V_{\lambda+1} \cup \{V_{\lambda+1}\}}{\mathcal{K}_1} L_{\kappa+1}(V_{\lambda+1}).$$

Then for all sets $X \subseteq V_{\lambda+1}$ such that $X \in L_\kappa(V_{\lambda+1})$, X is $U(j)$ -representable in $L(V_{\lambda+1})$.

Finally, by propagating $U(j)$ -representations along with j -Suslin representations (Section 5.3), we showed the following in [1].

Theorem 61. *Assume j witnesses that I_0 holds at λ . Then in $L(V_{\lambda+1})$ every set $X \in V_{\lambda+2}$ is $U(j)$ -representable.*

We will discuss some of the details of this proof in Section 5.3.

5.1.2 Consequences of $U(j)$ -representations

Two of the most important consequences of $U(j)$ -representations are (1) a certain generic absoluteness theorem for certain forcings between M_ω and V , and (2) obtaining weakly homogeneously Suslin representations after collapsing λ to ω . The generic absoluteness result allows us to obtain structural consequences for $L(V_{\lambda+1})$, and the fact that we can obtain weakly homogeneously Suslin sets allows us to connect the study of $L(V_{\lambda+1})$ to models of determinacy.

We start with a generic absoluteness result due to Woodin, and then we show some of its consequences.

Theorem 62 (Woodin).²⁵ *Suppose j witnesses that I_0 holds at λ and j is iterable. Let $j_{0,\omega} : L(V_{\lambda+1}) \rightarrow M_\omega$ be the embedding into the ω th iterate of $L(V_{\lambda+1})$ by j . Suppose $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, $g \in V$ is \mathbb{P} -generic over M_ω , and $(\text{cof}(\lambda))^{M_\omega[g]} = \omega$. Then we have for any $\alpha < \lambda$ an elementary embedding*

$$L_\alpha(M_\omega[g] \cap V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1})$$

which is the identity below λ .

We now isolate the generic absoluteness property present in this theorem.

Definition 63. Suppose j witnesses that I_0 holds at λ and j is iterable. Let

$$j_{0,\omega} : L(V_{\lambda+1}) \rightarrow M_\omega$$

be the embedding into the ω th iterate of $L(V_{\lambda+1})$ by j . We say that *generic absoluteness holds between M_ω and $L_\alpha(V_{\lambda+1})$* if for some $\bar{\alpha}$ we have the following. Suppose $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, $g \in V$ is \mathbb{P} -generic over M_ω , and $(\text{cof}(\lambda))^{M_\omega[g]} = \omega$. Then there is an elementary embedding

$$L_{\bar{\alpha}}(M_\omega[g] \cap V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1})$$

which is the identity below λ .

We could then rephrase Theorem 62 as stating that if I_0 holds at λ as witnessed by j , then for any $\alpha < \lambda$, generic absoluteness holds between M_ω and $L_\alpha(V_{\lambda+1})$. We will see below that there are in fact other representations which imply this generic absoluteness property.

We mention a couple of consequences of this theorem. Theorem 34 follows from Theorem 62 and we also obtain the following theorem on the failure of SCH at λ .

Theorem 64 (Dimonte-Friedman[9], Woodin independently). *Suppose I_0 holds at λ . Then it is consistent that I_1 holds at λ and the singular cardinal hypothesis fails at λ .*

Proof sketch. Let j be an I_0 embedding. Force so that GCH fails at $\text{crit}(j)$ and so that the generic g is mapped coherently by j , so that we may extend j to the extension. Let j^* be the natural extension of j . In the extension let M_ω be the ω th iterate of $L(V[g]_{\lambda+1})$ by j^* . Then M_ω satisfies that GCH fails at λ . Let $\vec{\kappa}$ be the critical sequence of j^* . Then $M_\omega[\vec{\kappa}]$ satisfies that SCH fails at λ , and by Theorem 62, I_1 holds at λ . \square

Note that in defining an AD-like axiom from $U(j)$ -representations it is the generic absoluteness property which seems to imply regularity properties for subsets of $V_{\lambda+1}$. And this generic absoluteness property does not obviously follow in the same way in the context of $L(X, V_{\lambda+1})$, assuming the existence of uniform $U(j)$ -representations. Therefore the AD-like axiom would seem to be the generic absoluteness result itself, unless a general generic absoluteness result could be proven from the existence of uniform $U(j)$ -representations.

²⁵Notice here that $V_\lambda^{M_\omega} = V_\lambda$. Also, this theorem does not actually follow directly from the existence of $U(j)$ -representations, but rather it follows from the existence of a uniform version of a $U(j)$ -representation.

We now show that $U(j)$ -representations can be used to connect to models of determinacy when collapsing λ to ω . To see this, let Γ^∞ be the set of universally Baire sets of reals. The determinacy axiom LSA states that the largest Suslin cardinal exist and is a Θ_α , that is a member of the Solovay sequence. Corollary 168 of [26] together with Theorem 61 gives us the following theorem.

Theorem 65. *Suppose that λ is a limit of supercompact cardinals and there is a proper class of Woodin cardinals. Suppose that I_0 holds at λ . Let $G \subseteq \text{Coll}(\omega, \lambda)$ be V -generic. Then for*

$$\Gamma_G^\infty = (\Gamma^\infty)^{V[G]} \cap L(V_{\lambda+1})[G]$$

we have that $L(\Gamma_G^\infty)$ satisfies LSA .

Although G. Sargsyan recently showed that $Con(LSA)$ follows from a Woodin limit of Woodins, the above theorem gives an alternative proof of $Con(LSA)$ from large cardinals. We also obtain the following theorem, which shows a strong relationship between $L(V_{\lambda+1})$ and models of determinacy after collapsing λ to ω .

Theorem 66. *Suppose that λ is a limit of supercompact cardinals and there is a proper class of Woodin cardinals. Suppose that I_0 holds at λ . Let $G \subseteq \text{Coll}(\omega, \lambda)$ be V -generic. Then for*

$$\Gamma_G^\infty = (\Gamma^\infty)^{V[G]} \cap L(V_{\lambda+1})[G]$$

we have that

$$\Theta^{L(V_{\lambda+1})} = \Theta^{L(\Gamma_G^\infty)}.$$

Proof. To see that

$$\Theta^{L(V_{\lambda+1})} \leq \Theta^{L(\Gamma_G^\infty)}$$

let $\alpha < \Theta$ and let $X \subseteq V_{\lambda+1}$ code a prewellordering of $V_{\lambda+1}$ of ordertype at least α . We have that X is $U(j)$ -representable in $L(V_{\lambda+1})$, which implies that in $L(V_{\lambda+1})[G]$, X can be coded as a subset $Y \subseteq \mathbb{R}^{L(V_{\lambda+1})[G]}$ such that Y is weakly homogeneously Suslin. Hence since there is a proper class of Woodin cardinals, Y is universally Baire. And hence $\alpha < \Theta^{L(\Gamma_G^\infty)}$.

To see that

$$\Theta^{L(V_{\lambda+1})} \geq \Theta^{L(\Gamma_G^\infty)}$$

we show that in $L(V_{\lambda+1})[G]$ there is no surjection $f : V_{\lambda+1} \rightarrow \Theta^{L(V_{\lambda+1})}$. Suppose this is not the case, and let $\tau \in L(V_{\lambda+1})$ be a term for f . Then we have that $g : V_{\lambda+1} \times \text{Coll}(\omega, \lambda) \rightarrow \Theta^{L(V_{\lambda+1})}$ defined by $g(x, p) = \alpha$ iff $p \Vdash \tau(x) = \alpha$ is clearly a surjection onto $\Theta^{L(V_{\lambda+1})}$. But $g \in L(V_{\lambda+1})$, which is a contradiction. Hence the theorem follows. \square

This theorem gives some evidence that the connection between large cardinals and determinacy axioms holds even up to the level of very large cardinals. However, the nature of this connection is far from understood at this point. For instance we have the following open question of Woodin.

Question 67. Suppose that λ is a limit of supercompact cardinals and there is a proper class of Woodin cardinals. Suppose that I_0 holds at λ . Let $G \subseteq \text{Coll}(\omega, \lambda)$ be V -generic. Then for

$$\Gamma_G^\infty = (\Gamma^\infty)^{V[G]} \cap L(V_{\lambda+1})[G]$$

what is the largest Suslin cardinal of $L(\Gamma_G^\infty)$?

Finally, we mention one negative result which is that uniformization does not follow from $U(j)$ -representability. This is a direct corollary of Theorem 61 and Fact 38.

Theorem 68 (C.). Assume I_0 holds at λ . Then in $L(V_{\lambda+1})$ there is a relation $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$ which is $U(j)$ -representable, but has no uniformization.

5.2 Inverse limit reflection

In this section we introduce the axioms *inverse limit reflection* and *strong inverse limit reflection*. These axioms arose out of the reflection results in Section 3 and the structural results in Section 4. In order to state these axioms we will need to use the terminology introduced in Section 3. In addition we make the definition for E saturated that

$$CL(E) = \{(J, \vec{j}) \in \mathcal{E} \mid \forall n < \omega \exists (K, \vec{k}) \in E (\vec{k} \upharpoonright n = \vec{j} \upharpoonright n)\}.$$

Also let \mathcal{E} be the set of inverse limits (J, \vec{j}) .

Definition 69. We define *inverse limit reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in E$, J extends to $\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \rightarrow L_\alpha(V_{\lambda+1})$ which is elementary.

We define *strong inverse limit reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to $\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \rightarrow L_\alpha(V_{\lambda+1})$ which is elementary.

We can also make this definition relativized to some $X \subseteq V_{\lambda+1}$. To do this we let

$$\begin{aligned} \mathcal{E}(X) = \{ & (J, \langle j_i \mid i < \omega \rangle) \mid \forall i (j_i : (V_{\lambda+1}, X) \rightarrow (V_{\lambda+1}, X)) \text{ and} \\ & J = j_0 \circ j_1 \circ \dots : (V_{\bar{\lambda}+1}, \bar{X}) \rightarrow (V_{\lambda+1}, X) \text{ is } \Sigma_0 \}. \end{aligned}$$

Here we let $\bar{X} = J^{-1}[X]$. We modify the definition of saturated to X -saturated, requiring in addition that $J^{-1}[X] = K^{-1}[X]$.

Definition 70. Suppose $X \subseteq V_{\lambda+1}$. We define *inverse limit X -reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X -saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in E$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \rightarrow L_\alpha(X, V_{\lambda+1})$ which is elementary.

We define *strong inverse limit X -reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X -saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \rightarrow L_\alpha(X, V_{\lambda+1})$ which is elementary.

We will say *strong inverse limit X -reflection* for strong inverse limit X -reflection at ω , and similarly for the other terminology.

The following result was shown in [5].

Theorem 71. *Suppose that there exists an elementary embedding*

$$j : L_{\Theta}(V_{\lambda+1}) \rightarrow L_{\Theta}(V_{\lambda+1}).$$

Then inverse limit reflection holds at α for all $\alpha < \Theta$.

It was subsequently shown in [4] that:

Theorem 72. *For any good $\alpha < \Theta^{L(V_{\lambda+1})}$, if there exists an elementary embedding*

$$L_{\alpha+\omega}(V_{\lambda+1}) \rightarrow L_{\alpha+\omega}(V_{\lambda+1})$$

then strong inverse limit reflection holds at α .

These results in fact extend slightly beyond I_0 by the same proof to get the following.

Theorem 73. *For any good $\alpha < \Theta^{L(V_{\lambda+1}^{\#}, V_{\lambda+1})}$, if there exists an elementary embedding*

$$L_{\alpha+\omega}(V_{\lambda+1}^{\#}, V_{\lambda+1}) \rightarrow L_{\alpha+\omega}(V_{\lambda+1}^{\#}, V_{\lambda+1})$$

then strong inverse limit $V_{\lambda+1}^{\#}$ -reflection holds at α .

Some of the methods needed for proving these theorems were discussed in Section 3. Theorem 72, however, requires a more detailed analysis of how inverse limits behave when passing to limit roots. We refer the reader to [6] for a detailed analysis of inverse limit reflection results throughout $L(V_{\lambda+1})$. A key question is whether these results can be extended way beyond I_0 through the E_{α}^0 hierarchy (see Section 6).

5.2.1 Consequences of inverse limit reflection

The reason for defining strong inverse limit X -reflection is its consequences for the properties of X .

Theorem 74 (C. [5]). *Let $X \subseteq V_{\lambda+1}$ and suppose that strong inverse limit X -reflection holds. Then the following hold:*

1. *there are no disjoint sets $S_1, S_2 \in L_{\omega}(X, H(\lambda^+))$ such that $S_1, S_2 \subseteq \lambda^+$ and both S_1 and S_2 are stationary (in V);*
2. *every set $X \in L_{\omega}(X, V_{\lambda+1}) \cap V_{\lambda+2}$ either has size λ or contains a λ -splitting perfect set.*

Theorem 74 gives very strong consequences of strong inverse limit X -reflection for sets $X \subseteq V_{\lambda+1}$, and it is this theorem which seems to give the most compelling AD-like properties of any axiom defined for the context of $V_{\lambda+1}$ so far.

We also have the following important corollary.

Theorem 75. *Suppose I_0 is consistent. Then it is consistent that for some $X \subseteq V_{\lambda+1}$, X is Icarus, but strong inverse limit X -reflection does not hold.*

Proof. This follows immediately from Theorem 36. □

By Fact 38 we also have the following.

Corollary 76. *Suppose I_0 holds at λ . Then in $L(V_{\lambda+1})$ there is $R \subseteq V_{\lambda+1} \times V_{\lambda+1}$ such that strong inverse limit R -reflection holds, but R has no uniformization.*

On the other hand, it is not clear that any generic absoluteness results follow directly from strong inverse limit X -reflection, which is one of the most compelling consequences of the existence of $U(j)$ -representations. Hence we are left with the following question.

Question 77. *Does generic absoluteness between M_ω and $L_\alpha(V_{\lambda+1})$ follow directly from strong inverse limit reflection at α ?*

5.3 j -Suslin representations

In [2] we defined tree representations for subsets of $V_{\lambda+1}$ which also give generic absoluteness between M_ω and $L(V_{\lambda+1})$. These tree representations seem rather similar to Suslin representations in the sense that they are not augmented by measures. On the other hand, they are weaker than Suslin representations in the sense that instead of a tree on ordinals, they are given by a tree on fixed points of iterates of a fixed embedding j . This definition is very much motivated by $U(j)$ -representations, which, through the Tower Condition, also give such trees. However, we must add an additional requirement on the tree structure itself in order to obtain a non-trivial representation and our generic absoluteness result. These representations, which we call j -Suslin representations, are therefore ostensibly not directly related to $U(j)$ -representations. At the end of this section we will define what is very naturally a combination of j -Suslin and $U(j)$ -representations, which we call a weakly homogeneously j -Suslin representation. This stronger representation in fact does immediately give the two weaker representations.

The main point of defining these representations was in order to help in the propagation of $U(j)$ -representations. However, we will see that j -Suslin representations (and therefore weakly homogeneously j -Suslin representations) give a slightly stronger generic absoluteness result.

For this section we fix $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ an elementary embedding with $\text{crit}(j) < \lambda$. For k an elementary embedding we denote by $k_{(n)}$ the n th iterate of k , and we let

$$\mathcal{F}_\kappa(k) = \{a \in L_\kappa(V_{\lambda+1}) \mid k(a) = a\}, \quad \mathcal{F}_\kappa^\omega(k) = \bigcup_{n < \omega} \mathcal{F}_\kappa^\omega(k_{(n)}),$$

and let

$$E^k(\kappa) = \{k' : L_\kappa(V_{\lambda+1}) \rightarrow L_\kappa(V_{\lambda+1}) \mid \exists n, m (k'_{(n)} = k_{(m)})\}$$

if $k : L_\kappa(V_{\lambda+1}) \rightarrow L_\kappa(V_{\lambda+1})$ is elementary and iterable. Also for $a \in L_\kappa(V_{\lambda+1})$ let

$$E^k(\kappa, a) = \{k \in E^k(\kappa) \mid k(a) = a\}.$$

Note that if $j(\kappa) = \kappa$ then $j(\mathcal{F}_\kappa^\omega(j)) = \mathcal{F}_\kappa^\omega(j)$.

Definition 78. For $\vec{\kappa} = \langle \kappa_i \mid i < \omega \rangle$ increasing and cofinal in λ , we let $\mathcal{W}^{\vec{\kappa}}$ be the set of sequences $s \in V_\lambda^\omega$ such that

1. for some $n < \omega$, $|s| = n$ and for all $i < n$, $s(i) \subseteq V_{\kappa_i}$,
2. if $i \leq m < |s|$ then $s(i) = s(m) \cap V_{\kappa_i}$.

Also let $\mathcal{W}_n^{\vec{\kappa}} = \{s \in \mathcal{W}^{\vec{\kappa}} \mid |s| = n\}$. In this context if $x \in V_{\lambda+1}$, we set

$$\hat{x} = \hat{x}_{\vec{\kappa}} = \langle x \cap V_{\kappa_n} \mid n < \omega \rangle \in \mathcal{W}^{\vec{\kappa}},$$

where we use the first notation if the sequence $\vec{\kappa}$ is understood.

Suppose that $\kappa < \Theta$. Let $X \subseteq V_{\lambda+1}$. We say that T is a (j, κ) -Suslin representation for X if for some sequence $\langle \kappa_i \mid i < \omega \rangle$ increasing and cofinal in λ the following hold.

1. T is a (height ω) tree on $V_\lambda \times \mathcal{F}_\kappa^\omega(j)$ such that for all $(s, a) \in T$, $s \in \mathcal{W}_{|s|}^{\vec{\kappa}}$.
2. For all $s \in \mathcal{W}^{\vec{\kappa}}$, $T_s \in \mathcal{F}_\Theta^\omega(j)$.
3. For all $x \in V_{\lambda+1}$, $x \in X$ iff $T_{\hat{x}}$ is illfounded.

We say that X is j -Suslin if for some κ , X has a (j, κ) -Suslin representation. If T satisfies conditions 1 and 3 then we say that T is a *weak* (j, κ) -Suslin representation for X .

By definition of the Tower Condition, if π is a $U(j)$ -representation for X and F is a tower function for $\text{rng } \pi$, then we immediately obtain a weak j -Suslin representation for X from F . However, obtaining weak j -Suslin representations is not particularly difficult (and apparently not useful). To see this consider the pointwise image of a set $X \subseteq V_{\lambda+1}$ under the map $j_{0,\omega}$ of $L(V_{\lambda+1})$ to M_ω . Every element of $M_\omega \cap L_\Theta(V_{\lambda+1})$ is fixed by an iterate of j , for j an I_0 embedding, and hence we can obtain from this pointwise image a weak j -Suslin representation for X . We will have to work considerably harder to obtain j -Suslin representations.

Similarly we say that T is a *uniform* (j, κ) -Suslin representation for X if the following hold.

1. T is a function on $[\lambda]^{<\omega}$ such that for all $s \in [\lambda]^\omega$, if $T(s)$ is the tree whose n th level is given by $T(s \upharpoonright n)$, then $T(s)$ is a (height ω) tree on $V_\lambda \times \mathcal{F}_\kappa^\omega(j)$.
2. For all $s \in [\lambda]^\omega$ such that s is cofinal in λ , $T(s)$ is a (j, κ) -Suslin representation for X .

We now state the generic absoluteness results which follow from the existence of uniform j -Suslin representations. In order to do this, we make the following definition about the existence of uniform j -Suslin representations.

Definition 79. Suppose that $j : L(Y, V_{\lambda+1}) \rightarrow L(Y, V_{\lambda+1})$ is an $I_0(Y)$ elementary embedding. We say that the *uniform j -Suslin conjecture* holds in $L(Y, V_{\lambda+1})$ if the following holds. Suppose $X \subseteq V_{\lambda+1}$, $X \in L(Y, V_{\lambda+1})$ is such that there is an $a \in \mathcal{F}_0^\omega(j)$ and a δ such that X is definable over $L_\delta(V_{\lambda+1})$ from a (so $X \in \mathcal{F}_\Theta^\omega(j)$). Then for some $\kappa < \Theta$ there exists a uniform (j, κ) -Suslin representation T for X such that $T \in \mathcal{F}_\Theta^\omega(j)$.

The following is proved in [1], which shows that uniform j -Suslin representations do exist in $L(V_{\lambda+1})$ under I_0 .

Theorem 80. *Suppose that I_0 holds at λ . Then the uniform j -Suslin conjecture holds in $L(V_{\lambda+1})$. In fact if $I_0^\#$ holds at λ then the uniform j -Suslin conjecture holds in $L(V_{\lambda+1}^\#)$.*

Using this result, the following generic absoluteness results are shown in [2].

Theorem 81. *Suppose that I_0 holds at λ as witnessed by j . Then for $\vec{\kappa}$ the critical sequence of j , if $\alpha < \Theta$ is good then for some $\bar{\alpha} < \lambda$ there is an elementary embedding*

$$L_{\bar{\alpha}}(M_\omega[\vec{\kappa}] \cap V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1}).$$

Theorem 82. *Suppose that I_0 holds at λ as witnessed by j . Suppose $g \in V$ is \mathbb{P} -generic over M_ω where $\mathbb{P} \in M_\omega$. Also assume that $\text{cof}(\lambda)^{M_\omega[g]} = \omega$. Then if $\alpha < \Theta$ is good, for some $\bar{\alpha} < \lambda$ there is an elementary embedding*

$$L_{\bar{\alpha}}(M_\omega[g] \cap V_{\lambda+1}) \rightarrow L_\alpha(V_{\lambda+1}).$$

Note that this theorem gives a slightly stronger version of generic absoluteness than the definition of generic absoluteness between M_ω and $L(V_{\lambda+1})$ which we made above. In particular we do not require that $\mathbb{P} \in j_{0,\omega}(V_\lambda)$. This is potentially a significant strengthening as for instance we cannot have $j \in M_\omega[g]$ where g is M_ω -generic for a forcing $\mathbb{P} \in j_{0,\omega}(V_\lambda)$. This reasoning however does not apply to forcings in M_ω .

Now we wish to define the notion of a weakly homogeneously j -Suslin representation. This representation combines properties of a j -Suslin representation and a $U(j)$ -representation, and is important for [1] when propagating these representations throughout $L(V_{\lambda+1})$. We use the same terminology as in Section 5.1.

Definition 83. Let $X \subseteq V_{\lambda+1}$. We say that T is a *weakly (δ, \vec{a}) -homogeneously (j, κ) -Suslin representation* for X if T is a (j, κ) -Suslin representation for X , and the following hold.

1. $\vec{a} \in [L_\kappa(V_{\lambda+1})]^\omega$ is weakly fixed by j .
2. For all $x \in V_{\lambda+1}$ and $\vec{A} \in \mathcal{A}^j(\delta, \vec{a})$, if $[T_{\hat{x}}] \neq \emptyset$ then

$$[T_{\hat{x}}] \cap [T^F(\vec{A})] \neq \emptyset.$$

We say that X is *weakly homogeneously j -Suslin* if for some δ, \vec{a} and κ , X is weakly (δ, \vec{a}) -homogeneously (j, κ) -Suslin.

Theorem 84 (C. [1]). *Assume that $j : L(V_{\lambda+1}) \rightarrow L(V_{\lambda+1})$ is an I_0 embedding. Then every subset $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$ satisfies the following in $L(V_{\lambda+1})$.*

1. X is $U(j)$ -representable.
2. X is j -Suslin.
3. X is weakly homogeneously j -Suslin.

The proof of this theorem has many similarities to the propagation of scales in $L(\mathbb{R})$. In particular there is a version of a closed game representation (see [21] and [24]) which is sufficiently weaker so that every subset of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ has such a representation. This fact together with the results of [3] are the key elements in the proof.

One unfortunate aspect of Theorem 84 is that the result is not local, especially not as we saw of strong inverse limit reflection above (Theorem 72). While obtaining $U(j)$ -representations locally is basically impossible because the measures do not occur locally, this is not obviously the case for j -Suslin representations. On the other hand, the definition of $U(j)$ -representations could be weakened in order to allow for such a local result. We state these ideas in the following question.

Question 85. 1. *Is there a local version of Theorem 84?*

2. *Do $U(j)$, j -Suslin, and weakly homogeneously j -Suslin representations follow directly from strong inverse limit reflection?*

An affirmative answer to (2) would presumably simplify the propagation of these representations dramatically, and would perhaps answer (1) at the same time.

6 Beyond I_0

In the previous sections we for the most part concentrated on consequences of I_0 and $I_0^\#$, and so a natural question is to what extent these results extend beyond this point²⁶ A natural way of framing this question is through the AD-like axioms which we considered in Section 5. So we could ask, for which $X \subseteq V_{\lambda+1}$ is it consistent that $L(X, V_{\lambda+1})$ satisfies these AD-like axioms? As we saw, however, we can find examples of $X \subseteq V_{\lambda+1}$ for which these AD-like axioms fail in $L(X, V_{\lambda+1})$, and so the answer to this question will most likely be rather complicated.

On the other hand, there is a natural hierarchy of axioms beyond I_0 , introduced by Woodin, which do seem like good candidates for the propagation of these AD-like axioms. This hierarchy, called the E_α^0 -hierarchy, was inspired by the hierarchy of models above $L(\mathbb{R})$

²⁶Of course another question is how to define such large cardinals.

which give the minimal model of $AD_{\mathbb{R}}$. Very roughly speaking, the sequence E_{α}^0 is defined by progressively increasing the size of $\Theta^{L(E_{\alpha}^0)}$, and E_{α}^0 is such that

$$E_{\alpha}^0 = V_{\lambda+2} \cap L(E_{\alpha}^0).$$

This is achieved by either adding in the sharp of the previous model, adding in λ -sequences of the previous model, or adding in a predicate for the ω -club filter on Θ of the previous model. Alternatively the next model can be defined using the set of elementary embeddings on the previous model. A key feature of this hierarchy is that it satisfies absoluteness properties similar to $L(V_{\lambda+1})$.

The analysis of the E_{α}^0 -hierarchy is beyond the scope of this article, and we refer the interested reader to [26], [8], and [9], and state the following question mentioned above.

Question 86. *Do the AD-like axioms of Section 5 ($U(j)$ -representability, (strong) inverse limit reflection, j -Suslin representability, weakly homogeneously j -Suslin representability, generic absoluteness) propagate throughout the E_{α}^0 -hierarchy?*

We close with one additional question, first posed by Woodin, which seems to be at the heart of the dynamics of passing to larger cardinals beyond I_0 .

Question 87. *Suppose that j is an I_0 embedding. Then does the following necessarily hold*

$$L(j \upharpoonright L_{\Theta_{\lambda}}(V_{\lambda+1}), V_{\lambda+1}) \cap V_{\lambda+2} \neq L(V_{\lambda+1}) \cap V_{\lambda+2}?$$

If the answer is yes, and this property holds more generally, there would perhaps be a straightforward way of building a hierarchy beyond I_0 , by building models which include the predicate for j up to some point. Understanding this question is therefore seemingly essential to understanding how (and to what extent) the hierarchy of large cardinals proceeds beyond I_0 .

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