Inverse Limit Reflection and the Structure of $L(V_{\lambda+1})$

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Abstract

We extend the results of Laver on using inverse limits to reflect large cardinals of the form, there exists an elementary embedding $L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$. Using these inverse limit reflection embeddings directly and by broadening the collection of $U(j)$-representable sets, we prove structural results of $L(V_{\lambda+1})$ under the assumption that there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. As a consequence we show the impossibility of a generalized inverse limit X-reflection result for $X \subseteq V_{\lambda+1}$, thus focusing the study of $L(\mathbb{R})$ generalizations on $L(V_{\lambda+1})$.

1 Introduction

The study of $L(V_{\lambda+1})$ is motivated primarily by the two goals of uncovering the structure of large cardinal axioms just below the limitation of Kunen's Theorem and understanding the relationship between $L(V_{\lambda+1})$ and $L(\mathbb{R})$. In terms of the structure of large cardinals, one of the most basic questions, which we consider here, is whether apparently stronger large cardinals reflect weaker large cardinals. As for the relationship between $L(V_{\lambda+1})$ and $L(\mathbb{R})$, the basic impression at present is that the structure of $L(V_{\lambda+1})$ assuming an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point less than λ (we will always assume $\text{crit}\,(j) < \lambda$ often without mention) is similar to the structure of $L(\mathbb{R})$ assuming $AD^{L(\mathbb{R})}$. Here we show the somewhat surprising fact that a tool, inverse limits, originally used for reflecting large cardinals, is useful in proving structural properties of $L(V_{\lambda+1})$ as well.

Laver ([7], [8]) first introduced inverse limits in the study of rank into rank embeddings. An inverse limit is an embedding $V_{\bar\lambda+1} \to V_{\lambda+1}$ for some $\bar\lambda < \lambda$ which is built out of an ω-sequence of embeddings $V_{\lambda+1} \to V_{\lambda+1}$ (we give the precise definition below). The basic question of inverse limits is to what extent they extend to embeddings $L_{\bar\alpha}(V_{\bar\lambda+1}) \to L_\alpha(V_{\lambda+1})$, and inverse limit reflection is the statement that an inverse limit does have such an extension as long as the embeddings that make up the inverse limit are sufficiently strong. In Section 3 we show that inverse limit reflection holds assuming there is an elementary embedding $L(V_{\lambda+1}) \to L(V_{\lambda+1})$ (as well as more local results). This result is enough to show that the existence of an elementary embedding $L(V^\#_{\lambda+1}) \to L(V^\#_{\lambda+1})$ implies that there is some $\bar\lambda < \lambda$ such that there is an elementary embedding $L(V_{\bar\lambda+1}) \to L(V_{\bar\lambda+1})$.

From inverse limit reflection we show a number of structural properties of $L(V_{\lambda+1})$. In $L(V_{\lambda+1})$, let $\kappa < \Theta$ be a cardinal with cofinality bigger than λ, and let $\alpha < \lambda$ be an infinite cardinal and $S_\alpha = \{\beta < \kappa \mid \mathrm{cof}(\beta) = \alpha\}$. Woodin showed that, assuming there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$, in $L(V_{\lambda+1})$ κ is measurable, as witnessed by the club filter restricted to a stationary set. He also showed however that under the same assumptions, if $\alpha > \omega$ then it is consistent that the club filter restricted to S_α is not an ultrafilter (this problem is part of a larger issue in studying $L(V_{\lambda+1})$ which is the 'right V' problem; see the remarks before Theorem 2.5). This leaves open the case of $\alpha = \omega$. We show, assuming there is an elementary embedding

$$L_{\omega+1}(V^\#_{\lambda+1}) \to L_{\omega+1}(V^\#_{\lambda+1})$$

with critical point less than λ, that S_ω cannot be partitioned into two stationary sets which are in $L(V_{\lambda+1})$. Woodin showed a similar result follows from $U(j)$-representations (see Section 6), but it is unclear at present if all subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ have $U(j)$-representations.

This relationship between inverse limit reflection and the structure of $L(V_{\lambda+1})$ has an interesting consequence. Suppose that $X \subseteq V_{\lambda+1}$ and there exists an elementary embedding $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$. Then one might expect the analysis of $L(V_{\lambda+1})$ to carry over to $L(X, V_{\lambda+1})$, and that this more general situation is really the appropriate area to study. We show, however, that inverse limit X-reflection cannot hold in general, and that the set of $X \subseteq V_{\lambda+1}$ such that inverse limit X-reflection holds is very restricted. As inverse limit reflection is a very natural property one would expect of these structures, this fact highlights $L(V_{\lambda+1})$ and its extensions satisfying inverse limit reflection as the most natural objects to study at this level.

Xianghui Shi and Woodin showed that the Perfect Set Property in $L(V_{\lambda+1})$ follows from a forcing argument and the generic absoluteness of Theorem 6.4 (which follows from $U(j)$-representations). In Section 5 we prove an analogous result using inverse limit reflection.

Some of the above structural results which we obtain from inverse limit reflection were shown by Woodin to follow from $U(j)$-representations. In fact he showed that even stronger reflection properties follow from these representations. The extent of $U(j)$-representable sets is however rather minimal at present. The similarity in the structural consequences of inverse limit reflection and $U(j)$-representations suggests that there might be some connection between the two. We make an initial step towards exploring this connection in Section 6 by proving the Tower Condition using inverse limit techniques.

The above results evidence the potentially wide-ranging usefulness of inverse limits in the study of $L(V_{\lambda+1})$. These results also hint at a connection between $U(j)$-representations and inverse limits which is currently unclear.

2 Basic Properties of $L(V_{\lambda+1})$

We first give some background on $L(V_{\lambda+1})$ (for a more thorough introduction and an alternative exposition of some of the results in Section 3, see [4]).

Lemma 2.1 (Kunen (see [5])). *(ZFC) Suppose that α is such that there exists an elementary embedding $j : V_\alpha \to V_\alpha$. Then for $\lambda = \sup_{i<\omega} \kappa_i$ where $\kappa_0 = \mathrm{crit}\, j$ and for $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$, we have*

1. *Either $\lambda = \alpha$ or $\lambda + 1 = \alpha$.*

2. *For all β such that $\mathrm{crit}\, j \leq \beta < \lambda$, $j(\beta) > \beta$.*

Definition 2.2. *Fix λ. Call an ordinal α* good *if every member of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from a member of $V_{\lambda+1}$. Define*

$$\Theta = \Theta_\lambda := \sup\{\alpha |\, (\exists\sigma(\sigma : V_{\lambda+1} \to \alpha \textit{ is a surjection}))^{L(V_{\lambda+1})}\}.$$

Lemma 2.3. *Fix λ a strong limit such that $\mathrm{cof}(\lambda) = \omega$. Then the following hold:*

1. *$L(V_{\lambda+1}) \models ZF + \lambda$-$DC$.*

2. *The good ordinals are cofinal in* Θ_λ.

3. Θ_λ *is regular in* $L(V_{\lambda+1})$.

4. $L_{\Theta_\lambda}(V_{\lambda+1}) \models ZF^-$.

5. *Suppose that* $j : L_\alpha(V_{\lambda+1}) \to L_\beta(V_{\lambda+1})$ *is elementary for* α *good. Then* j *is induced by* $j \restriction V_\lambda$.

Proof. 1, 3, and 4 are as in the $L(\mathbb{R})$ case. For 2 and 5, see [8]. □

Woodin has shown that the structure of $L(V_{\lambda+1})$ under the assumption that there is an elementary embedding $L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is similar in many respects to the structure of $L(\mathbb{R})$ assuming $AD^{L(\mathbb{R})}$. The following is a selection of results to that effect.

Theorem 2.4 (Woodin [10])**.** *Fix* λ *such that there exists an elementary embedding* $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. *Then the following hold in* $L(V_{\lambda+1})$:

1. *For cofinally many* $\kappa < \Theta_\lambda$, κ *is measurable, and this is witnessed by the club filter restricted to a stationary subset of* κ.

2. *If* $\alpha < \Theta_\lambda$ *then* $P(\alpha) \in L_{\Theta_\lambda}(V_{\lambda+1})$.

While this Theorem suggests the two situations are similar, there are important differences. One of the most important is the 'right V' problem: the theory of V_λ can be changed by small forcing, but the theory of V_ω cannot. Hence a property of $L(V_{\lambda+1})$ might depend on the theory of V_λ, and thus not be provable from the existence of the elementary embedding alone. The example of whether the club filter restricted to a certain cofinality is an ultrafilter is an example of such a phenomenon.

Theorem 2.5 (Woodin)**.** *Fix* λ *and let* $\kappa < \lambda$ *be an uncountable regular cardinal. Let* $S_\kappa = \{\alpha < \lambda^+ \mid cof(\alpha) = \kappa\}$ *and let* $\mathcal{F}$ *be the club filter on* λ^+. *If* $G \subseteq Coll(\kappa, \kappa^+)$ *is* V*-generic then*

$$L(V[G]_{\lambda+1}) \models \mathcal{F} \textit{ restricted to } S_\kappa \textit{ is not an ultrafilter.}$$

In fact for any $\beta < \lambda$, *there exists a poset* $\mathbb{P}$ *such that if* $G \subseteq \mathbb{P}$ *is* V*-generic then in* $L(V[G]_{\lambda+1})$ *there is a partition* $\langle T_\alpha \mid \alpha < \gamma \rangle$ *of* S_κ *into stationary sets for some* $\gamma \geq \beta$, *such that for all* $\alpha < \gamma$, $\mathcal{F}$ *restricted to* T_α *is an ultrafilter.*

Note however that $\kappa > \omega$ is required, which leaves open the case of $\kappa = \omega$. Theorems 4.4 and 4.9 give partial evidence that perhaps $\mathcal{F}$ restricted to S_ω is an ultrafilter in $L(V_{\lambda+1})$, assuming there is an elementary embedding $L(V_{\lambda+1}) \to L(V_{\lambda+1})$,

2.1 Inverse Limits

In this section we introduce the theory of inverse limits. These structures are most readily used for reflecting large cardinal hypotheses of the form: there exists an elementary embedding $L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$. The use of inverse limits in reflecting such large cardinals is originally due to Laver [7]. For a more thorough introduction see [7], [8], [1].

Suppose there exists an elementary embedding $j : V_\lambda \to V_\lambda$. Then if j extends to an elementary embedding $j^* : V_{\lambda+1} \to V_{\lambda+1}$ we have $j^*(A) = \bigcup_i j(A \cap V_{\lambda_i})$ for $\langle \lambda_i |\, i < \omega \rangle$ any cofinal sequence in λ, as λ is a continuity point. Hence any elementary embedding $V_{\lambda+1} \to V_{\lambda+1}$ can be coded as an element of $V_{\lambda+1}$.

Suppose that $\langle j_i |\, i < \omega \rangle$ is a sequence of elementary embeddings such that the following hold:

1. For all i, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.
2. There exists $\bar{\lambda} < \lambda$ such that
$$\operatorname{crit} j_0 < \operatorname{crit} j_1 < \cdots < \bar{\lambda}$$
and $\lim_{i<\omega} \operatorname{crit} j_i = \bar{\lambda}$.

Then we can form the inverse limit
$$J = j_0 \circ j_1 \circ \cdots : V_{\bar{\lambda}} \to V_\lambda$$
by setting
$$J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a)$$
for any $a \in V_{\bar{\lambda}}$. $J : V_{\bar{\lambda}} \to V_\lambda$ is elementary, and can be extended to a Σ_0-embedding $J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ by $J(A) = \bigcup_i J(A \cap V_{\bar{\lambda}_i})$ for $\langle \bar{\lambda}_i |\, i < \omega \rangle$ any cofinal sequence in $\bar{\lambda}$. Furthermore by a theorem of Laver [7], if for all i, j_i extends to an elementary embedding $V_{\lambda+1} \to V_{\lambda+1}$, then J^* is elementary. We will always assume that $J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ is elementary and define the inverse

limit of $\langle j_i | \, i < \omega \rangle$ to be $J = J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$. But we will sometimes treat J as if it were an element of $V_{\lambda+1}$. We write $\bar{\lambda}_J$ for the unique $\bar{\lambda}$ such that $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$. We will often drop the sequence $\langle j_i | \, i < \omega \rangle$ in our notation when talking about the inverse limit J, though the sequence is not unique for a given inverse limit J (for instance, by simply grouping the embeddings as, say, $J = (j_0 \circ j_1) \circ j_2 \circ \cdots$); it will always be clear from context which embeddings we mean when referring to $\langle j_i | \, i < \omega \rangle$.

Suppose $J = j_0 \circ j_1 \circ \cdots$ is an inverse limit. Then for $i < \omega$ we write $J_i := j_i \circ j_{i+1} \circ \cdots$, the inverse limit obtained by 'chopping off' the first i embeddings. For $i, n < \omega$ we write

$$J^{(i)} := (j_0 \circ \cdots \circ j_i)(J), \qquad J_n^{(i)} = (j_0 \circ \cdots \circ j_i)(J_n),$$

and

$$j_n^{(i)} := (j_0 \circ \cdots \circ j_i)(j_n).$$

Then we can rewrite J in the following useful ways:

$$\begin{aligned} J = j_0 \circ j_1 \circ \cdots &= \cdots (j_0 \circ j_1)(j_2) \circ j_0(j_1) \circ j_0 \\ &= \cdots j_2^{(1)} \circ j_1^{(0)} \circ j_0 \end{aligned}$$

and

$$\begin{aligned} J &= j_0 \circ J_1 = j_0(J_1) \circ j_0 = J_1^{(0)} \circ j_0 \\ &= (j_0 \circ \cdots \circ j_{i-1})(J_i) \circ j_0 \circ \cdots \circ j_{i-1} = J_i^{(i-1)} \circ j_0 \circ \cdots \circ j_{i-1} \end{aligned}$$

for any $i > 0$. Hence we can view an inverse limit J as a direct limit (see Figure 2.1), though both perspectives are useful in different situations. We let $\mathcal{E}$ be the set of inverse limits. So

$$\mathcal{E} = \{(J, \langle j_i | \, i < \omega \rangle) | \, J = j_0 \circ j_1 \circ \cdots : V_{\bar{\lambda}_J+1} \to V_{\lambda+1}\}.$$

Lemma 2.6. *If $(K, \vec{k}) \in \mathcal{E}$ and $A \in V_{\lambda+1}$ are such that $A \in rng\, K$, then for all i, $A \in rng\, (k_0 \circ \cdots \circ k_i)$.*

Proof. It is enough to see this for any $A \in V_\lambda$. But then there is an $\bar{A}$ and an n such that

$$K(\bar{A}) = (k_0 \circ \cdots \circ k_n)(\bar{A}) = A,$$

and for all $i > n$, $k_i(\bar{A}) = \bar{A}$. Hence for all i we have that $A \in \operatorname{rng}(k_0 \circ \cdots \circ k_i)$. □

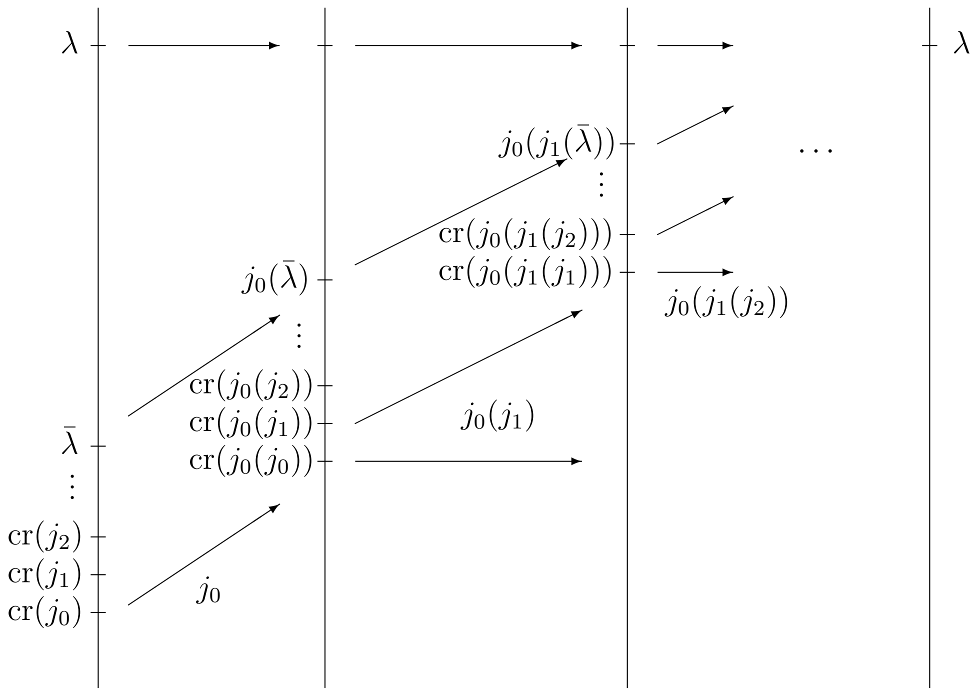

Figure 1: Direct limit decomposition of an inverse limit.

Suppose $j, k : V_{\lambda+1} \to V_{\lambda+1}$. Then we say k is a *square root of* j if $k(k) = j$ (thinking of k and j as elements of $V_{\lambda+1}$, so actually $k(k \restriction V_\lambda) = j \restriction V_\lambda$). We use the same terminology for $j, k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ where α is good. We have the following 'square root lemma' which says that strength of the embedding gives a large number of square roots. This is the key lemma which takes advantage of the strength of our embeddings, and we will use many variations of it below.

Lemma 2.7 (Martin)**.** *Suppose α is good. If $j : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$ is elementary then for all $A, B \in V_{\lambda+1}$ and $\beta < crit(j)$ there exists a $k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that k is a square root of j, $k(A) = j(A)$, $B \in rng\, k$ and $\beta < crit(k) < crit(j)$.*

Proof. Given α, j, A, B, and β as in the hypothesis, we want to show that $L_{\alpha+1}(V_{\lambda+1}) \models \exists k : V_\lambda \to V_\lambda$ which induces $\hat{k} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that

$$\beta < \operatorname{crit} k < \operatorname{crit} j,\ j(A) = k(A) \text{ and } B \in \operatorname{rng}(k).$$

Note that since α is good, an elementary embedding $k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is induced by $k \restriction V_\lambda$. Applying j, this is equivalent to $L_{\alpha+1}(V_{\lambda+1}) \models \exists k : V_\lambda \to V_\lambda$ which induces $\hat{k} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that $j(\beta) < \operatorname{crit} k < \operatorname{crit} j(j)$, $j(j)(j(A)) = \hat{k}(j(A))$ and $j(B) \in \operatorname{rng}(\hat{k})$. But $j \restriction V_\lambda$ satisfies this second statement. So we are done by elementarity of j. ☐

Note that we can replace A and B with any sequence of length less than $\operatorname{crit} j$ by coding. We will do so below without any comment.

Define

$$\mathcal{E}_\alpha = \{(J, \vec{j}) \in \mathcal{E} \,|\, \forall i < \omega\, (j_i \text{ extends to an elementary embedding } \hat{j}_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

Lemma 2.8 (Laver)**.** *Suppose there exists an elementary embedding*

$$j : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

where α is good. Then $\mathcal{E}_\alpha \neq \emptyset$.

Proof. Inductively define j_i as follows, repeatedly using Lemma 2.7. Let j_0 be such that $\operatorname{crit} j_0 < \operatorname{crit} j$ and $j_0 : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is elementary. Having chosen

$$j_0, \ldots, j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$$

such that

$$\operatorname{crit} j_0 < \operatorname{crit} j_1 < \cdots < \operatorname{crit} j_i < \operatorname{crit} j,$$

let j_{i+1} be such that

$$\operatorname{crit} j_i < \operatorname{crit} j_{i+1} < \operatorname{crit} j$$

and j_{i+1} extends to $j_{i+1} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$.

Then clearly we have that

$$\operatorname{crit} j_0 < \operatorname{crit} j_1 < \cdots < \operatorname{crit} j$$

and hence $\lim_{i\to\omega} \operatorname{crit} j_i = \bar{\lambda} < \lambda$ for some $\bar{\lambda}$. Let $J = j_0 \circ j_1 \circ \cdots$. □

There is a corresponding square root lemma for inverse limits. Suppose

$$(J, \langle j_i \rangle), (K, \langle k_i \rangle) \in \mathcal{E}.$$

Then we say that K *is a limit root of* J if there is $n < \omega$ such that $\bar{\lambda}_J = \bar{\lambda}_K$ and

$$\forall i < n\, (k_i = j_i) \text{ and } \forall i \geq n\, (k_i(k_i) = j_i).$$

We say K *is an n-close limit root of* J if n witnesses that K is a limit root of J. We also say that K and J *agree up to* n if for all $i < n$, $j_i = k_i$.

Lemma 2.9 (Laver [7])**.** *Suppose α is good. If $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$ then for all $\bar{A} \in V_{\bar{\lambda}+1}$ and $B \in V_{\lambda+1}$ there exists a $(K, \vec{k}) \in \mathcal{E}_\alpha$ such that K is a limit root of J, $K(\bar{A}) = J(\bar{A})$ and $B \in rng\, K$.*

While Laver's original statement did not include the notion of being a limit root, the proof is identical.

Proof. We use Lemma 2.7 ω-many times to $j_0, j_1, \ldots$ in succession. Define $k_0, k_1, \ldots$ by induction as follows. Let $k_0 : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ be given by Lemma 2.7 such that $B \in \operatorname{rng} k_0$ and for all i

$$j_0((j_1 \circ \cdots \circ j_i)(\bar{A})) = k_0((j_1 \circ \cdots \circ j_i)(\bar{A})).$$

After defining $k_0, \ldots, k_n$ let $k_{n+1} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ be given by Lemma 2.7 such that

$$(k_0 \circ \cdots \circ k_n)^{-1}(B) \in \operatorname{rng} k_{n+1},$$

$\operatorname{crit} j_n < \operatorname{crit} k_{n+1} < \operatorname{crit} j_{n+1}$ and for all i

$$j_n((j_{n+1} \circ \cdots \circ j_{n+i})(\bar{A})) = k_n((j_{n+1} \circ \cdots \circ j_{n+i})(\bar{A})).$$

A calculation shows that $\operatorname{crit} k_0 < \operatorname{crit} k_1 < \cdots < \bar{\lambda}$, $\lim_{i \to \omega} \operatorname{crit}(k_i) = \bar{\lambda}$, and for

$$K := k_0 \circ k_1 \circ \cdots$$

we have $K(\bar{A}) = J(\bar{A})$ and $B \in \operatorname{rng} K$:

To see that $K(\bar{A}) = J(\bar{A})$, note that it is enough to see that for all $\beta < \bar{\lambda}$, if $\bar{A}' = \bar{A} \cap V_\beta$, then $K(\bar{A}') = J(\bar{A}')$. Let n be large enough so that $\operatorname{crit}(k_n) > \beta$. Then we have that

$$\begin{aligned} J(\bar{A}') &= (j_0 \circ \cdots \circ j_{n-1})(\bar{A}') \\ &= (j_0 \circ \cdots \circ j_{n-2})(k_{n-1}(\bar{A}')) \\ &= (j_0 \circ \cdots \circ j_{n-3})((k_{n-2} \circ k_{n-1})(\bar{A}')) = \cdots \\ &= (k_0 \circ \cdots \circ k_{n-1})(\bar{A}') = K(\bar{A}') \end{aligned}$$

To see that $B \in \operatorname{rng} K$, let $\bar{\kappa}_i = \operatorname{crit} k_i$ and set $\kappa_i = K(\bar{\kappa}_i)$. It is enough to see that for all $i < \omega$, if $B' = B \cap V_{\kappa_i}$, then $B' \in \operatorname{rng} K$. Let $i < \omega$. Then we have that $(k_0 \circ \cdots \circ k_i)^{-1}(B')$ is defined since

$$K(\bar{\kappa}_i) = (k_0 \circ \cdots \circ k_i)(\bar{\kappa}_i).$$

But then we have that

$$K((k_0 \circ \cdots \circ k_i)^{-1}(B')) = B',$$

which is what we wanted. □

A key difference between embeddings for square roots and being a limit root for inverse limits is that if $k(k) = j$ then $\operatorname{crit} k < \operatorname{crit} j$ whereas if K is a limit root of J then $\operatorname{crit} K \le \operatorname{crit} J$. So while there is no sequence $k_0, k_1, \ldots$ such that for all $i < \omega$, $k_{i+1}(k_{i+1}) = k_i$, we have the following lemma for limit roots.

Lemma 2.10. *Suppose that α is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Then there exists a sequence $\left\langle (K^i, \vec{k}^i) \middle| \, i < \omega \right\rangle$ such that the following hold:*

1. *$K^0 = J$.*
2. *For all i, $(K^i, \vec{k}^i) \in \mathcal{E}_\alpha$.*
3. *For all i, K^{i+1} is a limit root of K^i.*

Proof. Let $(J,\vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Set $K^0 = J$, and choose $(K^{m+1}, \vec{k}^{m+1})$ by induction as follows. Suppose that $(K^0, \vec{k}^0), \ldots, (K^m, \vec{k}^m)$ have been chosen so that $(K^m, \vec{k}^m) \in \mathcal{E}_\alpha$ and there exists $\langle n_i^m | \, i < \omega \rangle$ such that for all $i < \omega$, $n_i^m < \omega$, k_i^m extends to

$$\hat{k}_i^m : L_{\alpha+n_i^m}(V_{\lambda+1}) \to L_{\alpha+n_i^m}(V_{\lambda+1}),$$

and $\lim_{i\to\omega} n_i^m = \infty$. Let i be large enough so that for all $i' \geq i$, $n_{i'}^m > 0$. Then by the proof of Lemma 2.9, there is K^{m+1} which is an i-close limit root of K^m such that for all $i' \geq i$, $k_{i'}^{m+1}$ extends to

$$\hat{k}_{i'}^{m+1} : L_{\alpha+n_{i'}^m-1}(V_{\lambda+1}) \to L_{\alpha+n_{i'}^m-1}(V_{\lambda+1}).$$

We have that

$$\lim_{i\to\omega}(n_i^m - 1) = \infty,$$

and hence we can continue the induction. The sequence we produce

$$\left\langle (K^i, \vec{k}^i) | \, i < \omega \right\rangle$$

clearly satisfies the lemma. □

Of course, if we considered the more restrictive notion of being a 0-close limit root, then such sequences as in Lemma 2.10 would indeed be impossible. We will see though that the added benefit afforded by Lemma 2.10 will be very useful. As a first example, we obtain sets of inverse limits which are in a sense closed under the square root lemma.

Definition 2.11. *Suppose $E \subseteq \mathcal{E}$. Then we say that E is* saturated *if for all $(J,\vec{j}) \in E$ there exists an $i < \omega$ such that for all $A \in V_{\bar{\lambda}_J+1}$, and $B \in V_{\lambda+1}$, there exists $(K,\vec{k}) \in E$ such that K is an i-close limit root of J, $K_i(A) = J_i(A)$ and $B \in rng\, K_i$. We set $i(E,J) =$ the least such i.*

Note that if K is an i-close limit root of J and $K_i(A) = J_i(A)$ then $K(A) = J(A)$. However, we cannot conclude that $B \in \operatorname{rng} K$ if $B \in \operatorname{rng} K_i$. For instance if $i = 1$ then we always have that $\operatorname{crit}(J) = \operatorname{crit}(K) \notin \operatorname{rng} K$.

We will use the same terminology of being saturated for E such that there is α good such that for all $(J,\vec{j}) \in E$ and $i < \omega$, $j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is elementary.

As a corollary to the proof of Lemma 2.10 we have:

Corollary 2.12. *Suppose that α is good and $(J,\vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Then there exists a saturated set $E \subseteq \mathcal{E}_\alpha$ such that $(J,\vec{j}) \in E$.*

Proof. Let E be the set of all $(K,\vec{k}) \in \mathcal{E}_\alpha$ such that there exists a sequence $\langle n_i | \, i < \omega \rangle$ such that $\lim_{i\to\omega} n_i = \infty$ and for all $i < \omega$, $n_i < \omega$ and k_i extends to

$$\hat{k}_i : L_{\alpha+n_i}(V_{\lambda+1}) \to L_{\alpha+n_i}(V_{\lambda+1}).$$

Since $(J,\vec{j}) \in \mathcal{E}_{\alpha+\omega}$ we must have that $(J,\vec{j}) \in E$. So the lemma follows by the proofs of Lemmas 2.9 and 2.10. □

In fact we actually proved the following stronger result, which allows us to conclude that the existence of a saturated $E \subseteq \mathcal{E}_\alpha$ follows from the existence of an elementary embedding $L_{\alpha+\omega}(V_{\lambda+1}) \to L_{\alpha+\omega}(V_{\lambda+1})$.

Corollary 2.13. *Suppose that α is good and $(J,\vec{j}) \in \mathcal{E}_\alpha$ is such that for all $i < \omega$ there is an n_i such that j_i extends to*

$$\hat{j}_i : L_{\alpha+n_i}(V_{\lambda+1}) \to L_{\alpha+n_i}(V_{\lambda+1})$$

and $\lim_{i\to\omega} n_i = \omega$. Then there exists a saturated set $E \subseteq \mathcal{E}_\alpha$ such that $(J,\vec{j}) \in E$.

Lemma 2.14. *Suppose $E \subseteq \mathcal{E}$ is saturated. Let $(J,\vec{j}) \in E$, $\bar{A} \in V_{\bar{\lambda}_J+1}$, and suppose*

$$J(\bar{A}) = A \in V_{\lambda+1}.$$

Set

$$E(\bar{A},A) = \{(K,\vec{k}) \in E | \, K(\bar{A}) = A\}.$$

Then $E(\bar{A},A)$ is saturated.

Proof. Suppose $(K,\vec{k}) \in E(\bar{A},A)$. Then $(K,\vec{k}) \in E$, so there is $i < \omega$ such that for all $C \in V_{\bar{\lambda}_J+1}$ and $B \in V_{\lambda+1}$ there exists $(K',\vec{k}') \in E$, an i-close limit root of K such that $K_i(C) = K'_i(C)$ and $B \in \operatorname{rng} K'_i$. But then i is such that for all $C \in V_{\bar{\lambda}_J+1}$ and $B \in V_{\lambda+1}$ there exists $K' \in E$, an i-close limit root of K such that $K_i(C) = K'_i(C)$, $K'_i(\bar{A}) = K_i(\bar{A}) = A$ and $B \in \operatorname{rng} K'_i$. So $K'(\bar{A}) = K(\bar{A}) = A$ and hence $(K',\vec{k}') \in E(\bar{A},A)$. Hence $E(\bar{A},A)$ is saturated. □

Finally note that if $k(k) = j$ and $A \in \operatorname{rng} k$, then $k(A) = j(A)$. To see this suppose $k(B) = A$, and notice

$$k(A) = k(k(B)) = k(k)(k(B)) = j(k(B)) = j(A).$$

2.2 Sequences of inverse limits

We will show in this section that sequences of inverse limit roots have a powerful continuity property. We will use this property many times below. As usual, we often write $\langle K^i | \, i < \omega \rangle$ instead of $\left\langle (K^i, \vec{k}^i) | \, i < \omega \right\rangle$ for a sequence of inverse limits, with the underlying embeddings being understood.

Lemma 2.15. *Suppose $\langle K^i | \, i < \omega \rangle$ is such that for all i, K^{i+1} is a limit root of K^i. Then there exists an increasing sequence $\langle i_n | \, n < \omega \rangle$ such that for all $n < \omega$ and $s \geq i_n$, we have that $k^s_n = k^{i_n}_n$.*

Proof. Suppose the lemma does not hold and n is least such that there is no i_n such that for all $s > i_n$, $k^s_n = k^{i_n}_n$. Then there is a sequence $\langle s_i | \, i < \omega \rangle$ such that $\langle k^{s_i}_n | \, i < \omega \rangle$ is such that for all $i > 0$ there exists an m such that

$$(k^{s_i}_n)_{(m)} = k^{s_{i-1}}_n$$

where we write $j_{(m)}$ for the m-th iterate of an embedding j. But there can be no such sequence since for all $i > 0$, $\operatorname{crit}(k^{s_i}_n) < \operatorname{crit}(k^{s_{i-1}}_n)$. So the lemma follows. □

For $\langle K^i | \, i < \omega \rangle$ such that there exists $\langle i_n | \, n < \omega \rangle$, an increasing sequence satisfying that for all $n < \omega$ and $s \geq i_n$, $k^s_n = k^{i_n}_n$, we call

$$K = k^{i_0}_0 \circ k^{i_1}_1 \circ \cdots$$

the *common part of* $\langle K^i | \, i < \omega \rangle$, and

$$\langle i_n | \, n < \omega \rangle$$

a *common part index sequence for* $\langle K^i | \, i < \omega \rangle$.

The following is a key continuity property of inverse limit sequences.

Lemma 2.16. *Suppose that for $i < \omega$, $(J^i, \vec{j}^i) \in \mathcal{E}$. And suppose the common part of $\langle J^i | \, i < \omega \rangle$ is K and $\bar{\lambda}_{J^0} = \bar{\lambda}_K = \bar{\lambda}$. Then for all $\bar{A} \in V_{\bar{\lambda}+1}$ such that for all i, $J^0(\bar{A}) = J^i(\bar{A})$, we have $K(\bar{A}) = J^0(\bar{A})$.*

Proof. Let $J^0(\bar{A}) = A$ and let $\langle i_n | \, n < \omega \rangle$ be a common part index sequence for $\langle J^i | \, i < \omega \rangle$. It is enough to show that for cofinally many $\bar{\kappa} < \bar{\lambda}$, $K(\bar{A} \cap V_{\bar{\kappa}}) = A \cap V_\kappa$, where $\kappa = K(\bar{\kappa})$. Let $\bar{\kappa} < \bar{\lambda}$, and let $n < \omega$ be least such that $\operatorname{crit}(k_n) > \bar{\kappa}$. Then we have that

$$K(\bar{A} \cap V_{\bar{\kappa}}) = (k_0 \circ \cdots \circ k_{n-1})(\bar{A} \cap V_{\bar{\kappa}}).$$

On the other hand, for some $\kappa^* < \lambda$,

$$A \cap V_{\kappa^*} = J^{i_n}(\bar{A} \cap V_{\bar{\kappa}}) = (j_0^{i_n} \circ \cdots \circ j_{n-1}^{i_n})(\bar{A} \cap V_{\bar{\kappa}}) = (k_0 \circ \cdots \circ k_{n-1})(\bar{A} \cap V_{\bar{\kappa}}).$$

And hence $\kappa^* = K(\bar{\kappa})$, and $K(\bar{A} \cap V_{\bar{\kappa}}) = A \cap V_{\kappa^*}$, as desired. □

It is possible that if K is the common part of $\langle J^i | \, i < \omega \rangle$ then $\bar{\lambda}_K < \bar{\lambda}_{J^0}$. To avoid this possibility, we can fix a sequence $\langle \bar{\lambda}_n | \, n < \omega \rangle$ cofinal in $\bar{\lambda}_{J_0}$. Then we add to our requirement on J^{i+1} that for all $m < \omega$ if n is largest such that $\operatorname{crit} j_m^i > \bar{\lambda}_n$, then $\operatorname{crit} j_m^{i+1} > \bar{\lambda}_n$. In this case we say that J^{i+1} is *a limit root of J^i, supported by $\langle \bar{\lambda}_n | \, n < \omega \rangle$*.

Definition 2.17. *Suppose $E \subseteq \mathcal{E}$ is a set of inverse limits. Then we let $CL(E)$ be the set*

$$CL(E) = \{(K, \vec{k}) \in \mathcal{E} | \, \exists \vec{K} \, (K \textit{ is the common part of } \vec{K}, \bar{\lambda}_K = \bar{\lambda}_{K^0}, \textit{ and } \forall i < \omega \, ((K^i, \vec{k}^i) \in E))\}.$$

3 Inverse Limit Reflection

A fundamental question of inverse limits is to what extent they extend to strong reflection embeddings. We introduce some terminology which identifies the various forms of reflection as obtained by inverse limits.

Definition 3.1. *We define* inverse limit reflection at α *to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in E$, J extends to $\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$ which is elementary.*

We define strong inverse limit reflection at α *to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to $\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$ which is elementary.*

We will also need the notion of inverse limit X-reflection where $X \subseteq V_{\lambda+1}$. As before we let

$$\mathcal{E}(X) = \{(J, \langle j_i | \, i < \omega \rangle) | \, \forall i \, (j_i : (V_{\lambda+1}, X) \to (V_{\lambda+1}, X)) \text{ and } J = j_0 \circ j_1 \circ \cdots : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X) \text{ is } \Sigma_0\}.$$

Here we let $\bar{X} = J^{-1}[X]$. We modify the definition of saturated to X-saturated, requiring in addition that $J^{-1}[X] = K^{-1}[X]$.

Definition 3.2. *Suppose $X \subseteq V_{\lambda+1}$. We define* inverse limit X-reflection at α *to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X-saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in E$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \to L_\alpha(X, V_{\lambda+1})$ which is elementary.*

We define strong inverse limit X-reflection at α *to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X-saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \to L_\alpha(X, V_{\lambda+1})$ which is elementary.*

Note that we cannot immediately conclude elementarity of

$$J : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

as $\bar{X}$ depends on J in general. And in fact we will show that inverse limit X-reflection does not hold in general.

Theorem 3.3. *Suppose there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Then there exists $\bar{\lambda} < \lambda$ such that for all $\alpha < \Theta_\lambda$, there exists $\bar{\alpha}$ such that*

$$L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \equiv L_\alpha(V_{\lambda+1}).$$

Proof. Suppose that $\alpha < \Theta_\lambda$ is good, $\rho : V_{\lambda+1} \to L_\alpha(V_{\lambda+1})$ is a surjection definable over $L_{\alpha+1}(V_{\lambda+1})$, with $X \subseteq V_{\lambda+1}$ the preimage. Let $G \subseteq \mathrm{Coll}(\omega, \lambda)$ be V-generic.

Let $E \subseteq \mathcal{E}_{\alpha+1}$ be saturated and $(J, \vec{j}) \in E$. Let $\vec{\lambda}$ be cofinal in $\bar{\lambda}_J = \bar{\lambda}$. In $V[G]$, let $\langle a_i | \, i < \omega \rangle$ be an enumeration of $V_{\bar{\lambda}+1}$, and let $\langle \phi^i | \, i < \omega \rangle$ be an enumeration of all formulas in the language $(\in)$. We define sequences $\langle J^i | \, i < \omega \rangle$, $\langle n_i | \, i < \omega \rangle$ in $V[G]$ with the following properties:

1. $J^0 = J$. For all $i < \omega$, $J^i \in E$ and J^{i+1} is a limit root of J^i, supported by $\vec{\lambda}$.
2. $\langle n_i | \, i < \omega \rangle$ is increasing, and for all $i < \omega$, for all $n \leq n_i$, $J^{i+1}(a_n) = J^i(a_n)$.
3. For all $i_0 < \omega$, suppose that $L_\alpha(V_{\lambda+1}) \models \exists x \, \phi(x, \vec{B})$ where

$$\vec{B} = \left\langle \rho(J^{i_0}(a_{s_1})), \ldots, \rho(J^{i_0}(a_{s_m})) \right\rangle$$

and for all $i < m$, $s_i \leq i_0$ and $\exists x\phi(x, \vec{X})$ is the formula ϕ^i for some $i < i_0$. Then for some b which is a witness to ϕ with parameter $\vec{B}$, we have

$$\rho(J^{i_0+1}(a_{\bar{t}})) = b$$

and $n_{i_0+1} \geq \bar{t}$.

Note that we can arrange (3) as follows. Suppose that $i_0 < \omega$ and

$$L_\alpha(V_{\lambda+1}) \models \exists x\, \phi(x, \vec{B}),$$

where

$$\vec{B} = \left\langle \rho(J^{i_0}(a_{s_1})), \ldots, \rho(J^{i_0}(a_{s_m})) \right\rangle .$$

Let i be such that for all $A \in V_{\bar{\lambda}+1}$ and $B \in V_{\lambda+1}$, there exists $(K, \vec{k}) \in E$, with K an i-close limit root of J^{i_0}, $K_i(A) = J_i^{i_0}(A)$ and $B \in \operatorname{rng} K_i$. Pulling back by $j_0^{i_0} \circ \cdots \circ j_{i-1}^{i_0}$, we have

$$L_\alpha(V_{\lambda+1}) \models \exists x\, \phi(x, \vec{B}_i),$$

where

$$\vec{B}_i = \left\langle \rho(J_i^{i_0}(a_{s_1})), \ldots, \rho(J_i^{i_0}(a_{s_m})) \right\rangle .$$

Let b be a witness to ϕ with parameter $\vec{B}_i$. Then if $(K, \vec{k}) \in E$ is an i-close limit root of J^{i_0}, satisfies (2), and for some $\bar{t}$, $\rho(K_i(a_{\bar{t}})) = b$ then

$$L_\alpha(V_{\lambda+1}) \models \phi(\rho((j_0^{i_0} \circ \cdots \circ j_{i-1}^{i_0})(b)), \vec{B}).$$

To arrange (3), we simply work with the finitely many $\vec{B}$ and ϕ required by (3) simultaneously.

Let J^* be the common part of $\langle J^i |\, i < \omega\rangle$. Then by (2) and Lemma 2.16 we have that $J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ since for all $a \in V_{\bar{\lambda}+1}$, there is an n such that $a_n = a$. And hence for i large enough, we have that $J^*(a_n) = J^i(a_n) \in V_{\lambda+1}$.

Let $M = \rho[J^*[V_{\bar{\lambda}+1}]]$. We claim that $M \prec L_\alpha(V_{\lambda+1})$. But this follows immediately from condition (3). Furthermore, M is wellfounded. Let $\bar{M}$ be the transitive collapse of M and let π be the inverse of the transitive collapse. We have that $V_{\bar{\lambda}+1} = \pi^{-1}[V_{\lambda+1}]$, and hence by condensation, we have that $\bar{M} = L_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ for some $\bar{\alpha}$. So $L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \equiv L_\alpha(V_{\lambda+1})$. But, by absoluteness, in V we have that $L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \equiv L_\alpha(V_{\lambda+1})$, which is what we wanted. □

For the next theorem we use Jensen's J-hierarchy to stratify $L(V_{\lambda+1})$. We will also use the following notation: Suppose that α is least such that $J_\alpha(V_{\lambda+1}) \models \phi[A]$ where ϕ is Σ_1 and $A \in V_{\lambda+1}$. Then we say that (A, ϕ) *tags* α *(over* $V_{\lambda+1}$*)*. If such a tag exists then there is a partial map $\rho : V_{\lambda+1} \to J_\alpha(V_{\lambda+1})$ which is a surjection, Σ_1-definable over $J_\alpha(V_{\lambda+1})$ (see Steel [9]). We similarly define $\bar{\rho}$ over $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$.

Based on the proof of Theorem 3.3, we fix some terminology which will be useful in the following theorems.

Definition 3.4. *Fix $E \subseteq \mathcal{E}$ saturated, α good, and $J \in E$. Set $\bar{\lambda} = \bar{\lambda}_J$ and let $\vec{\lambda}$ be cofinal in $\bar{\lambda}$. Fix $\langle \phi^i | \, i < \omega \rangle$, an enumeration of all formulas in the language $(\in)$. We define a forcing $\mathbb{P}(E, \alpha, J)$. Conditions are elements $(\langle J^i, n_i | \, i < m \rangle, \langle a_i | \, i < n_{m-1} \rangle)$ where $m \geq 1$ and the following hold.*

1. *$J^0 = J$. For all $i < m-1$, J^{i+1} is a limit root of J^i supported by $\vec{\lambda}$, and $J^{i+1} \in E$.*
2. *$\langle n_i | \, i < m \rangle \in \omega^m$ is an increasing sequence.*
3. *For all $i < n_{m-1}$, $a_i \in V_{\bar{\lambda}_J+1}$.*
4. *For all $1 \leq m' < m$, and $i < n_{m'-1}$, $J^{m'-1}(a_i) = J^{m'}(a_i)$.*
5. *For all $m' < m-1$, suppose that $L_\alpha(V_{\lambda+1}) \models \exists x \, \phi(x, \vec{B})$ where*
$$\vec{B} = \left\langle \rho(J^{m'}(a_{s_1})), \ldots, \rho(J^{m'}(a_{s_n})) \right\rangle$$
and for all $i < n$, $s_i \leq m'$ and $\exists x \phi(x, \vec{X})$ is the formula ϕ^i for some $i < m'$. Then for some b which is a witness to ϕ with parameter $\vec{B}$, we have
$$\rho(J^{m'+1}(a_{\bar{t}})) = b$$
for some $\bar{t} < n_{m'+1}$.

For
$$(\left\langle J^i, n_i | \, i < m \right\rangle, \langle a_i | \, i < n_{m-1} \rangle) \text{ and } (\left\langle K^i, n'_i | \, i < m' \right\rangle, \left\langle a'_i | \, i < n'_{m'-1} \right\rangle)$$
in $\mathbb{P}(E, \alpha, J)$ we put
$$(\left\langle J^i, n_i | \, i < m \right\rangle, \langle a_i | \, i < n_{m-1} \rangle) \geq_{\mathbb{P}(E,\alpha,J)} (\left\langle K^i, n'_i | \, i < m' \right\rangle, \left\langle a'_i | \, i < n'_{m'-1} \right\rangle)$$
iff

1. $m \leq m'$,

2. for all $i < m$, $J^i = K^i$, $n_i \leq n'_i$, *and for all* $s < n_{m-1}$, $a_s = a'_s$.

Suppose that

$$g \subseteq \mathbb{P}(E, \alpha, J)$$

is $L(V_{\lambda+1})$*-generic. Then clearly in* $L(V_{\lambda+1})[g]$ *we obtain a unique sequence* $\langle J^i | \, i < \omega \rangle$ *from* g *such that for all* i, J^{i+1} *is a limit root of* J^i. *We set* J^g *to be the common part of* $\langle J^i | \, i < \omega \rangle$.

Lemma 3.5. *Assume we are in the situation of Definition 3.4. Suppose that*

$$g \subseteq \mathbb{P}(E, \alpha, J)$$

is $L(V_{\lambda+1})$*-generic. Then* J^g *maps* $V_{\bar{\lambda}+1} \to V_{\lambda+1}$, *and there exists an* $\bar{\alpha}$ *such that* J^g *extends to an elementary embedding*

$$\hat{J}^g : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}).$$

Proof. This follows exactly as in the proof of Theorem 3.3. □

Lemma 3.6. *Assume we are in the situation of Definition 3.4. Suppose that*

$$(\left\langle J^i, n_i | \, i < m \right\rangle, \langle a_i | \, i < n_{m-1} \rangle) \in \mathbb{P}(E, \alpha, J)$$

and there exists $\bar{\alpha}$ *such that*

$$(\left\langle J^i, n_i | \, i < m \right\rangle, \langle a_i | \, i < n_{m-1} \rangle) \Vdash J^{\dot{g}} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}) \text{ is elementary.}$$

Then J^{m-1} *extends to an elementary embedding* $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1})$.

Proof. We assume for simplicity of notation that $m = 1$. So we have

$$p := (\langle J, n_0 \rangle, \langle a_i | \, i < n_0 \rangle) \in \mathbb{P}(E, \alpha, J)$$

and there exists $\bar{\alpha}$ such that

$$(\langle J, n_0 \rangle, \langle a_i | \, i < n_0 \rangle) \Vdash J^{\dot{g}} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}) \text{ is elementary.}$$

We extend J to a map $\hat{J}$ as follows. Suppose that $\bar{B} \in V_{\bar{\lambda}+1}$, $B \in V_{\lambda+1}$, $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$, $b \in J_{\alpha}(V_{\lambda+1})$, and ϕ are such that $J(\bar{B}) = B$, and b is the unique element of $J_\alpha(V_{\lambda+1})$ such that $J_\alpha(V_{\lambda+1}) \models \phi(b, B)$ and $\bar{b}$ is the unique element of $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ such that $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \phi(\bar{b}, \bar{B})$. Then set $\hat{J}(\bar{b}) = b$.

We need to check that $\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$ is well-defined, total, and elementary. The proofs of each of these facts are very similar. First we check that $\hat{J}$ is well-defined. Suppose that $\bar{B}_1, B_1, \phi_1$ witness that $\hat{J}(\bar{b}) = b_1$ and $\bar{B}_2, B_2, \phi_2$ witness that $\hat{J}(\bar{b}) = b_2$. Let $p' \leq_{\mathbb{P}(E,\alpha,J)} p$ be the condition

$$p' = (\langle J, n_0 + 2\rangle, \langle a_i | \, i < n_0\rangle^\frown \langle \bar{B}_1, \bar{B}_2\rangle).$$

Then

$$p' \Vdash \dot{J^g}(\bar{B}_1) = B_1 \wedge \dot{J^g}(\bar{B}_2) = B_2,$$

and hence

$$p' \Vdash b_1 = \dot{\hat{J}^g}(\bar{b}) = b_2.$$

So $b_1 = b_2$ by absoluteness, which is what we wanted.

Now we check that $\hat{J}$ is total. We first show that $\bar{\alpha}$ is ($\bar{\lambda}$-)good. Let $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. Suppose $p \in g \subseteq \mathbb{P}(E, \alpha, J)$ is $L(V_{\lambda+1})$-generic and $\hat{J}^g(\bar{b}) = b \in J_\alpha(V_{\lambda+1})$. Then since α is good there exists a $B \in V_{\lambda+1}$ such that $J_\alpha(V_{\lambda+1}) \models b$ is the unique element such that $\phi(b, B)$. Hence

$$J_\alpha(V_{\lambda+1}) \models \exists B' \in V_{\lambda+1}(b \text{ is the unique element such that } \phi(b, B')).$$

But then by elementarity of J^g,

$$J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \exists \bar{B}' \in V_{\bar{\lambda}+1}(\bar{b} \text{ is the unique element such that } \phi(\bar{b}, \bar{B}')).$$

So this shows that $\bar{\alpha}$ is good.

To see that $\hat{J}$ is total, let $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ and let $\bar{B}$ and ϕ be such that $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \bar{b}$ is the unique element such that $\phi(\bar{b}, \bar{B})$. Set $B = J(\bar{B})$. Let $p' \leq_{\mathbb{P}(E,\alpha,J)} p$ be the condition

$$p' = (\langle J, n_0 + 1\rangle, \langle a_i | \, i < n_0\rangle^\frown \langle \bar{B}\rangle).$$

Then

$$p' \Vdash \dot{J^g}(\bar{B}) = B,$$

and hence

$$p' \Vdash J_\alpha(V_{\lambda+1}) \models \dot{J^g}(\bar{b}) \text{ is the unique element such that } \phi(\dot{J^g}(\bar{b}), B).$$

Let $p'' \leq_{\mathbb{P}(E,\alpha,J)} p'$ be such that for some $b \in J_\alpha(V_{\lambda+1})$, $p'' \Vdash J^{\dot{g}}(\bar{b}) = b$. But then by absoluteness we have that

$$J_\alpha(V_{\lambda+1}) \models b \text{ is the unique element such that } \phi(b, B).$$

So we must have that $\hat{J}(\bar{b}) = b$.

To see that $\hat{J}$ is elementary, suppose that $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ and ψ is a formula. Let $\bar{B}$ and ϕ be such that $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \bar{b}$ is the unique element such that $\phi(\bar{b}, \bar{B})$. Set $b = \hat{J}(\bar{b})$ and $B = J(\bar{B})$. Let $p' \leq_{\mathbb{P}(E,\alpha,J)} p$ be the condition

$$p' = (\langle J, n_0 + 1\rangle, \langle a_i |\, i < n_0\rangle^\frown \langle \bar{B}\rangle).$$

Then

$$p' \Vdash J^{\dot{g}}(\bar{B}) = B \wedge \hat{J}^{\dot{g}}(\bar{b}) = b,$$

and hence

$$p' \Vdash J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \psi(\bar{b}) \iff J_\alpha(V_{\lambda+1}) \models \psi(b).$$

But by absoluteness $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \psi(\bar{b}) \iff J_\alpha(V_{\lambda+1}) \models \psi(b)$, which is what we wanted.

So $\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$ is an elementary embedding, as desired. □

Theorem 3.7. *Suppose that there exists an elementary embedding*

$$j : L_\Theta(V_{\lambda+1}) \to L_\Theta(V_{\lambda+1}).$$

Then inverse limit reflection holds at α for all $\alpha < \Theta$.

Proof. It is enough to show that for all $\alpha < \Theta$ good, inverse limit reflection holds at α, since if inverse limit reflection holds at α good then it holds at all $\beta \leq \alpha$. So assume that $\alpha < \Theta$ is good. Since there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$, there must exist a saturated set $E \subseteq \mathcal{E}_\alpha$. Fix $J \in E$.

Let

$$p = (\langle J^i, n_i |\, i < m\rangle, \langle a_i |\, i < n_{m-1}\rangle) \in \mathbb{P}(E, \alpha, J)$$

be a condition such that for some $\bar{\alpha}$

$$p \Vdash J^{\dot{g}} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}) \text{ is elementary.}$$

Then we have that J^{m-1} extends to an elementary embedding $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$. Let E_p be the set of inverse limits $K \in E$ such that for some $q \leq_{\mathbb{P}(E,\alpha,J)} p$ if

$$q = (\langle K^i, n'_i |\, i < m'\rangle, \langle a'_i |\, i < n_{m'-1}\rangle)$$

then $K = K^{m'-1}$.

Clearly by definition of $\mathbb{P}(E, \alpha, J)$ we have that E_p is saturated as well. Furthermore by Lemma 3.6 we have that for all $K \in E_p$ that K extends to an elementary embedding

$$\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}).$$

Hence inverse limit reflection holds at α. $\square$

In the next theorem we show that strong inverse limit reflection holds for rather large ordinals. In fact, it can be shown that strong inverse limit reflection holds all the way up to Θ, but this requires arguments which are more involved (see [1] or [2]).

Theorem 3.8. *Suppose that there exists an elementary embedding*

$$j : L_{\Theta}(V_{\lambda+1}) \to L_{\Theta}(V_{\lambda+1}).$$

Let δ be least such that

$$J_{\delta}(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1} \cup \{V_{\lambda+1}\})} L(V_{\lambda+1}).$$

Then strong inverse limit reflection holds at α for all $\alpha < \delta$.

Proof. Suppose $\alpha < \delta$, $A \in V_{\lambda+1}$ and (A, ϕ) is a tag for α (such α are cofinal in δ). Let $E \subseteq \mathcal{E}_{\alpha+1}$ be a saturated set of inverse limits such that for some $\bar{A} \in V_{\bar{\lambda}+1}$, for all $(J, \vec{j}) \in E$, $J(\bar{A}) = A$.

Let $J \in E$. We claim that for some $\bar{\alpha}$,

$$\emptyset \Vdash_{\mathbb{P}(E,\alpha+1,J)} J^{\dot{g}} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to J_{\alpha+1}(V_{\lambda+1}) \text{ is elementary.}$$

But this is clear since

$$\begin{aligned}\emptyset \Vdash_{\mathbb{P}(E,\alpha+1,J)} \exists \bar{\alpha}' \, (J^{\dot{g}} : J_{\bar{\alpha}'+1}(V_{\bar{\lambda}+1}) \to J_{\alpha+1}(V_{\lambda+1}) \text{ is elementary} \\ \wedge J^{\dot{g}}(\bar{A}) = A \wedge (\bar{A}, \phi) \text{ tags } \bar{\alpha}').\end{aligned}$$

And hence by absoluteness there is an $\bar{\alpha}$ which is tagged by $(\bar{A}, \phi)$, and this $\bar{\alpha}$ is as desired.

Hence we have by Lemma 3.6 that for all $K \in E$, that K extends to an elementary embedding $\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1})$.

We also have that for any $K \in \mathcal{E}_{\alpha+\omega}$ such that $K(\bar{A}) = A$ that there exists a saturated set $E_K \subseteq \mathcal{E}_{\alpha+1}$ such that $K \in E_K$ and for all $K' \in E_K$, $K'(\bar{A}) = A$. Hence this shows that for any $K \in \mathcal{E}_{\alpha+\omega}$ such that $K(\bar{A}) = A$ that K extends to an elementary embedding

$$\hat{K} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

To complete the proof we consider a saturated set $E \subseteq \mathcal{E}_{\alpha+\omega}$ such that for all $J \in E$, $J(\bar{A}) = A$ for some (A, ϕ) a tag for α. Such an E must exist since there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Let $\bar{\alpha}$ be as above. Then for all $K \in CL(E)$ we have that $K(\bar{A}) = A$ and $K \in \mathcal{E}_{\alpha+\omega}$. Hence by what we proved above we have that K extends to an elementary embedding $\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$. Hence this E witnesses strong inverse limit reflection at α. □

Theorem 3.9. *Suppose that there exists an elementary embedding*

$$j : L_\omega(V^{\#}_{\lambda+1}, V_{\lambda+1}) \to L_\omega(V^{\#}_{\lambda+1}, V_{\lambda+1}).$$

Then there exists $\bar{\lambda} < \lambda$ and a $V^{\#}_{\lambda+1}$-saturated set $E \subseteq \mathcal{E}(V^{\#}_{\lambda+1})$ of inverse limits such that for all $(J, \vec{j}) \in E$, J is an elementary embedding

$$J : (V^{\#}_{\bar{\lambda}+1}, V_{\bar{\lambda}+1}) \to (V^{\#}_{\lambda+1}, V_{\lambda+1}).$$

And hence there exists an elementary embedding $\bar{j} : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$. Furthermore strong inverse limit $V^{\#}_{\lambda+1}$-reflection holds at 0.

Proof. We describe how to modify the proof of Theorem 3.3. Let $E \subseteq \mathcal{E}(V^{\#}_{\lambda+1})$ be saturated (but not necessarily $V^{\#}_{\lambda+1}$-saturated). Then proceeding exactly as in the proof of Theorem 3.3, replacing $L_\alpha(V_{\lambda+1})$ with $(V^{\#}_{\lambda+1}, V_{\lambda+1})$, the argument is exactly the same until the point that we defined M.

Let $M = J[V_{\bar{\lambda}+1}]$. Then $M \prec V_{\lambda+1}$ and for $\bar{M}$ the transitive collapse of M we have $\bar{M} = V_{\bar{\lambda}+1}$. Let π be the inverse of the transitive collapse. Let $\bar{X} = \pi^{-1}[V^{\#}_{\lambda+1}]$. Then we have that $\pi : (\bar{X}, V_{\bar{\lambda}+1}) \to (V^{\#}_{\lambda+1}, V_{\lambda+1})$ is elementary. But by definability of the sharp, we must have $\bar{X} = V^{\#}_{\bar{\lambda}+1}$. So we have that $(V^{\#}_{\bar{\lambda}+1}, V_{\bar{\lambda}+1}) \equiv (V^{\#}_{\lambda+1}, V_{\lambda+1})$. But by absoluteness this is true in V.

The rest of the proof proceeds exactly as in the proof of Theorem 3.8.

To see that there is an elementary embedding

$$L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1}),$$

we have that $(V_{\lambda+1}, V^{\#}_{\lambda+1})$ satisfies that there is a Σ_1-elementary embedding

$$(V_{\lambda+1}, V^{\#}_{\lambda+1}) \to (V_{\lambda+1}, V^{\#}_{\lambda+1}).$$

And hence $(V_{\bar\lambda+1}, V^{\#}_{\bar\lambda+1})$ satisfies that there is a Σ_1-elementary embedding

$$\bar{j} : (V_{\bar\lambda+1}, V^{\#}_{\bar\lambda+1}) \to (V_{\bar\lambda+1}, V^{\#}_{\bar\lambda+1}).$$

So $\bar{j} \restriction V_{\bar\lambda+1}$ extends to an elementary embedding

$$\bar{j}^* : L_{\bar\Theta}(V_{\bar\lambda+1}) \to L_{\bar\Theta}(V_{\bar\lambda+1}).$$

Here we are using that every subset of $V_{\bar\lambda+1}$ in $L(V_{\bar\lambda+1})$ is Σ_1-definable over $(V_{\bar\lambda+1}, V^{\#}_{\bar\lambda+1})$ with parameters in $V_{\bar\lambda+1}$. But as in [10] we can define the following ultrafilter $U_{\bar{j}}$ from $\bar{j}$,

$$X \in U_{\bar{j}} \iff \bar{j} \restriction V_{\bar\lambda} \in \bar{j}^*(X).$$

Taking the ultrapower by $U_{\bar{j}}$ yields an elementary embedding

$$L(V_{\bar\lambda+1}) \to L(V_{\bar\lambda+1})$$

which extends $\bar{j} \restriction V_{\bar\lambda+1}$ (see [10]). $\square$

Theorem 3.9 gives an example of an $X \subseteq V_{\lambda+1}$ such that inverse limit X-reflection holds. The set of such X is very restricted however, as inverse limit X-reflection gives structural properties of $L(X, V_{\lambda+1})$. Specifically, we will prove the following theorem in Section 4.

Theorem 3.10. *Suppose $X \subseteq V_{\lambda+1}$ and strong inverse limit X-reflection holds at α. Then there are no disjoint stationary subsets S_1 and S_2 of*

$$\{\beta < \lambda^+ \mid cof(\beta) = \omega\}$$

such that

$$S_1, S_2 \in L_\alpha(X, V_{\lambda+1}).$$

Corollary 3.11. *Assume there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Suppose that $G \subseteq Coll(\omega, \omega_1)$ is V-generic. Then in $V[G]$, (strong) inverse limit $V_{\lambda+1}$-reflection at 1 does not hold.

Proof. We work with $(H(\lambda^+), V_{\lambda+1})$ for ease of notation. We have that for

$$S_1 = \{\alpha < \lambda^+ | \, (\mathrm{cof}(\alpha) = \omega)^{L(V_{\lambda+1})}\}$$

and

$$S_2 = \{\alpha < \lambda^+ | \, (\mathrm{cof}(\alpha) = \omega_1)^{L(V_{\lambda+1})}\},$$

that S_1 and S_2 are definable over $(H(\lambda^+)^{V[G]}, V_{\lambda+1})$. Furthermore, S_1 and S_2 are stationary in $V[G]$. And since

$$S_1, S_2 \in L_1(H(\lambda^+)^{V[G]}, V_{\lambda+1}),$$

we have that inverse limit $V_{\lambda+1}$-reflection at 1 does not hold by Theorem 3.10. □

4 Stationary Subsets of λ^+

In this section we use inverse limit reflection to obtain results related to the club filter on λ^+ in $L(V_{\lambda+1})$. We cannot quite show that the ω-club filter restricted to the cofinality ω ordinals is an ultrafilter in $L(V_{\lambda+1})$, but we obtain a couple approximations to this result. Namely, we show that the weak ω-club filter is an ultrafilter in $L(V_{\lambda+1})$, and that any two disjoint stationary (in V) subsets of the cofinality ω ordinals must not be in $L(V_{\lambda+1})$. A weak ω-club of λ^+ (see Definition 4.8 below) is a set of ordinals which can be written as the set of of all sups below λ^+ of countable elementary substructures of some fixed structure in a countable language. While the Axiom of Choice implies that the weak ω-club filter and the ω-club filter are the same, in our situation we cannot come to such a conclusion.

These results extend to higher ordinals of cofinality greater than λ, though for simplicity of notation we prove them for λ^+. We will simply state these extensions below, as the proofs are nearly identical.

We will define a game which is similar to the 'sup game' on ω_1 (see [6]). I will play an increasing sequence of ordinals below λ^+ and II will, in essence, be playing ordinals as well. However II must play her ordinals by playing inverse limits which send certain specified ordinals α_n below some $\bar{\lambda}^+$ to ordinals $K^n(\alpha_n)$ below λ^+. The key point is that we can choose the α_n for $n < \omega$ such that II has a (quasi-)winning strategy and use the reflection given by the sequence of inverse limits to control where the sup of a winning run ends up.

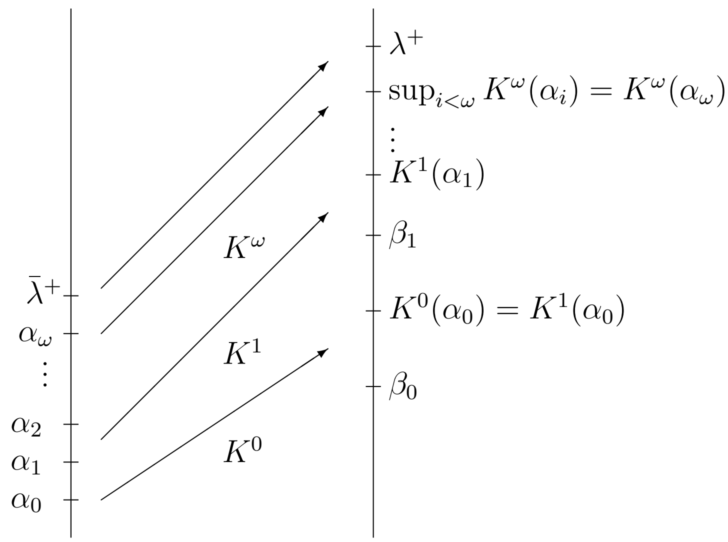

Figure 2: The game $G(\langle \alpha_i |\, i < \omega\rangle, E)$, where K^ω is the common part of $\langle K^i |\, i < \omega\rangle$.

Fix λ, $\bar\lambda < \lambda$ and a surjection $\bar\rho : V_{\bar\lambda+1} \to L_{\bar\lambda^+}(V_{\bar\lambda+1})$ definable over $L_{\bar\lambda^+}(V_{\bar\lambda+1})$. Also let E be a saturated set of inverse limits such that for all $(J, \vec{j}) \in E$, J extends to

$$\hat{J} : L_{\bar\lambda^+ +1}(V_{\bar\lambda+1}) \to L_{\lambda^+ +1}(V_{\lambda+1}).$$

Assume ρ is a surjection $\rho : V_{\lambda+1} \to L_{\lambda^+}(V_{\lambda+1})$ and for all $(J, \vec{j}) \in E$, $\hat{J}(\bar\rho) = \rho$ (see the remark before Theorem 3.7). We will say that A tags b (over $V_{\lambda+1}$) if $\rho(A) = b$, and similarly for $\bar\rho$.

Consider the following game $G(\langle \alpha_i |\, i < \omega\rangle, E)$ (see Figure 4), where

$$\langle \alpha_i |\, i < \omega\rangle$$

is an increasing sequence of ordinals less than $\bar\lambda^+$.

I	β_0		β_1		$\cdots$
II		$(K^0, \vec{k}^0), A_0$		$(K^1, \vec{k}^1), A_1$	$\cdots$

With the following rules:

1. $\beta_0 < \beta_1 < \cdots < \lambda^+$ are limit ordinals.
2. For all i, $(K^i, \vec{k}^i) \in E$, and K^{i+1} is a limit root of K^i.
3. Let $\hat{K}^i$ be the extension of K^i to $L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1})$. Then we have

$$\beta_0 < \hat{K}^0(\alpha_0) < \beta_1 < \hat{K}^1(\alpha_1) < \beta_2 < \cdots < \lambda^+.$$

4. For all i, $\bar{\rho}(A_i) = \alpha_i$.
5. For all i and $n \leq i$, $K^{i+1}(A_n) = K^i(A_n)$.

II wins if the game goes on ω-many steps. This is a closed game for I, and hence determined.

We first show that II can win the analogous one step game.

Lemma 4.1. *Let E be saturated such that for all $(J, \vec{j}) \in E$, J extends to*

$$\hat{J} : L_{\bar{\lambda}^+ +1}(V_{\bar{\lambda}+1}) \to L_{\lambda^+ +1}(V_{\lambda+1}).$$

Then for all $(J, \vec{j}) \in E$, there exists an $\alpha < \bar{\lambda}^+$ such that for all $\beta < \lambda^+$ there is a $(K, \vec{k}) \in E$, a limit root of J, such that $\hat{K}(\alpha) \geq \beta$.

Proof. Let $(J, \vec{j}) \in E$. Then for $i = i(E, J)$ (see Definition 2.11), we have that for all $\gamma < \lambda^+$ there exists a $(K, \vec{k}) \in E$ such that $\hat{K}_i(\bar{\gamma}) = \gamma$ and hence $\hat{K}(\bar{\gamma}) \geq \gamma$ for some $\bar{\gamma} < \bar{\lambda}^+$. So by regularity of λ^+ there is an $\alpha < \bar{\lambda}^+$ such that for cofinally many $\beta < \lambda^+$ there is $(K, \vec{k}) \in E$, a limit root of J, such that $\hat{K}(\alpha) \geq \beta$, which is what we wanted. □

Recall that a quasi-winning strategy σ for II is a function for which, given any position in the game p where it is II's turn to play and p has been played according to σ, $\sigma(p)$ is a set of possible moves (rather than a single move) for II, and any play according to σ is winning for II. We must consider quasi-winning strategies because to obtain an actual winning strategy would require the Axiom of Choice in this situation.

Lemma 4.2. *Let E be saturated such that for all $(J, \vec{j}) \in E$, J extends to*

$$\hat{J} : L_{\bar{\lambda}^+ +1}(V_{\bar{\lambda}+1}) \to L_{\lambda^+ +1}(V_{\lambda+1}).$$

Then there exists an increasing sequence $\langle \alpha_i | \, i < \omega \rangle$ such that II has a quasi-winning strategy in $G(\langle \alpha_i | \, i < \omega \rangle, E)$.

Proof. First note that by Gale-Stewart [3] for all $\langle \alpha_i | \, i < \omega \rangle$ increasing below $\bar{\lambda}^+$, either I has a winning strategy (since he is playing ordinals) or II has a quasi-winning strategy in $G(\langle \alpha_i | \, i < \omega \rangle, E)$.

Suppose towards a contradiction that for all $\vec{\alpha} \in [\bar{\lambda}^+]^\omega$ that II does not have a quasi-winning strategy in $G(\langle \alpha_i | \, i < \omega \rangle, E)$. Then by λ-DC and the fact that λ is a strong limit, we can choose

$$\left\langle \sigma^{\vec{\alpha}} | \, \vec{\alpha} \in [\bar{\lambda}^+]^\omega \right\rangle$$

such that for all $\vec{\alpha} \in [\bar{\lambda}^+]^\omega$, $\sigma^{\vec{\alpha}}$ is a winning strategy for I in $G(\vec{\alpha}, E)$. We use the regularity of λ^+ to play against all of these winning strategies simultaneously.

Choose a sequence $\vec{\alpha}^*$ as follows. Let

$$\beta_0^* = \sup_{\vec{\alpha} \in [\bar{\lambda}^+]^\omega} \sigma^{\vec{\alpha}}(\emptyset) < \lambda^+.$$

Let $K^0 \in E, A_0$ and α_0^* be such that $\hat{K}^0(\alpha_0^*) > \beta_0^*$ and A_0 tags α_0^*. After having chosen $K^0, \ldots, K^n \in E$ and $\alpha_0^*, \ldots, \alpha_n^*$, let

$$\beta_{n+1}^* = \sup\{\sigma^{\vec{\alpha}}(\left\langle K^0, A_0, \ldots, K^n, A_n \right\rangle) | \, \vec{\alpha} \in [\bar{\lambda}^+]^\omega, \forall i \le n \, (\alpha_i = \alpha_i^*)\}.$$

Let $K^{n+1} \in E, A_{n+1}$ and α_{n+1}^* be such that K^{n+1} is a limit root of K^n, for all $i \le n$, $K^{n+1}(A_i) = K^n(A_i)$, $\hat{K}^{n+1}(\alpha_{n+1}^*) > \beta_{n+1}^*$ and A_{n+1} tags α_{n+1}^*.

We then play $\langle K^0, A^0, K^1, A^1, \ldots, \rangle$ in the game $G(\vec{\alpha}^*, E)$, against the winning strategy $\sigma^{\vec{\alpha}^*}$. But by the way we chose K^i, A_i and α_i^*, this must be a winning play by II. Hence $\sigma^{\vec{\alpha}^*}$ is not a winning strategy for I, a contradiction. □

Lemma 4.3. *Let E be saturated such that for all $(J, \vec{j}) \in CL(E)$, J extends to*

$$\hat{J} : L_{\bar{\lambda}^+ + 1}(V_{\bar{\lambda}+1}) \to L_{\lambda^+ + 1}(V_{\lambda+1}).$$

Suppose that $\langle \alpha_i | \, i < \omega \rangle$ is an increasing sequence of ordinals $< \bar{\lambda}^+$ and

$$(\beta_0, K^0, A_0, \beta_1, K^1, A_1, \ldots)$$

is a winning play for II in $G(\langle \alpha_i | \, i < \omega \rangle, E)$. Let K be the common part of $\langle K^i | \, i < \omega \rangle$. Then

$$\hat{K}(\sup_{i<\omega} \alpha_i) = \sup_{i<\omega} \beta_i.$$

Proof. Note that we have for all i and $n \leq i$ that $C_n := K^{i+1}(A_n) = K^i(A_n)$. Hence we have that $K(A_n) = C_n$ and therefore $\hat{K}(\alpha_n) = \gamma_n$, where γ_n is tagged by C_n. And by the rules of the game, we have

$$\beta_0 < \gamma_0 < \beta_1 < \gamma_1 < \cdots .$$

Hence $\hat{K}(\sup_{i<\omega} \alpha_i) = \sup_{i<\omega} \beta_i$ follows by continuity. □

Theorem 4.4. *Assume strong inverse limit reflection at ξ for $\xi > \lambda^+$ good. Let*

$$S_\omega = \{\beta < \lambda^+ |\, cof(\beta) = \omega\}.$$

Then if $S \in L_\xi(V_{\lambda+1})$ and $S \subseteq S_\omega$ is stationary (in V), then $S_\omega \setminus S$ is not stationary.

Proof. Suppose E, $\bar{\xi}$, ξ and $\bar{S}$ are such that ξ is good, $E \in L_\xi(V_{\lambda+1})$ and for all $(K, \vec{k}) \in CL(E)$, K extends to

$$\hat{K} : L_{\bar{\xi}}(V_{\bar{\lambda}+1}) \to L_\xi(V_{\lambda+1})$$

and $\hat{K}(\bar{S}) = S$. Suppose that $\langle \alpha_i |\, i < \omega \rangle$ is such that II has a quasi-winning strategy in $G(\langle \alpha_i |\, i < \omega \rangle, E)$.

Claim 4.5. *If $\sup_{i<\omega} \alpha_i \in \bar{S}$ then S contains an ω-club.*

Proof. Suppose this is not the case, so $\sup \alpha_i \in \bar{S}$ but $S_\omega \setminus S$ is stationary (in V). Let γ be large enough such that in $L_\gamma(V_{\lambda+1})$, II has a quasi-winning strategy in $G(\langle \alpha_i |\, i < \omega \rangle, E)$. Let $M \prec L_\gamma(V_{\lambda+1})$ be such that $|M| = \lambda$, $S, J, E \in M$, $V_\lambda \subseteq M$, and $M \cap \lambda^+ \in S_\omega \setminus S$. Let $\langle \beta_i |\, i < \omega \rangle$ be increasing and cofinal in $M \cap \lambda^+$ such that for all i, $\beta_i \in M$.

The point is that we would like to play the finite initial segments of the sequence $\langle \beta_i |\, i < \omega \rangle$ against II's quasi-winning strategy in M. The sequence $\langle \beta_i |\, i < \omega \rangle$ might not be a legal play for I (since II could play such that $\hat{K}^i(\alpha_i)$ is very large below λ^+), but there is always a legal subsequence of $\langle \beta_i |\, i < \omega \rangle$ that I can play.

So we play a run of the game $G(\langle \alpha_i |\, i < \omega \rangle, E)$ in M such that at each stage I plays an ordinal on the sequence $\langle \beta_i |\, i < \omega \rangle$ and II plays a winning response (in M). Suppose without loss of generality (by passing to a subsequence) that the game is played as $(\beta_0, K^0, A_0, \beta_1, K^1, A_1 \ldots)$ with $(K^i, \vec{k}^i) \in M$ for all i. Let K be the common part of $\langle K^i |\, i < \omega \rangle$ as computed in $L(V_{\lambda+1})$. Then by the previous lemma we have that $\hat{K}(\sup \alpha_i) = \sup M \cap \lambda^+ \in S$ by elementarity. But this is a contradiction. So $S_\omega \setminus S$ is not stationary. □

To obtain the theorem just notice that if $\sup_{i<\omega} \alpha_i \in \bar{\lambda}^+ \setminus \bar{S}$, then $\lambda^+ \setminus S$ contains an ω-club. And hence S is not stationary. Hence we have for any $S \subseteq \lambda^+$ that S is stationary iff there exists $\langle \alpha_i | \, i < \omega \rangle$ such that II has a quasi-winning strategy in $G(\langle \alpha_i | \, i < \omega \rangle, E)$ and there are $E, \bar{\xi}, \xi$, and $\bar{S}$ as above such that $\sup_{i<\omega} \alpha_i \in \bar{S}$. Hence, by the definition of strong inverse limit reflection, the theorem follows. □

Applying Theorem 3.9 we have the following.

Corollary 4.6. *Assume there exists an elementary embedding*

$$j : L_\omega(V_{\lambda+1}^\#, V_{\lambda+1}) \to L_\omega(V_{\lambda+1}^\#, V_{\lambda+1}).$$

Then there are no disjoint stationary subsets S_1 and S_2 of $\{\beta < \lambda^+ | \, cof(\beta) = \omega\}$ such that $S_1, S_2 \in L(V_{\lambda+1})$.

This result can be improved to the following using an improved version of Theorem 3.9.

Theorem 4.7 ([2])**.** *Assume there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Then there are no disjoint stationary subsets S_1 and S_2 of $\{\beta < \lambda^+ | \, cof(\beta) = \omega\}$ such that $S_1, S_2 \in L(V_{\lambda+1})$.

We now restate the above result using the notion of a weak ω-club and a weakly stationary set.

Definition 4.8. *Suppose that $C \subseteq \gamma$ for γ a limit with uncountable cofinality. Then we say that C is* weakly club *if there exists a structure $(M, \ldots)$ in a countable language such that*

$$C = \{\alpha < \gamma | \, \exists (X, \ldots) \prec (M, \ldots), \sup(X \cap \gamma) = \alpha\}.$$

We say that $S \subseteq \gamma$ is weakly stationary *if for all $C \subseteq \gamma$ weakly club, $S \cap C \neq \emptyset$. The* weak club filter on γ *is the filter generated by the set of weakly club subsets of γ. We define weakly ω-club and the weak ω-club filter analogously, restricting to countable elementary substructures.*

As a corollary to Corollary 4.6 we obtain the following result. The hypothesis can be similarly reduced as above by results in [2].

Corollary 4.9. *Suppose there exists an elementary embedding*

$$j : L_\omega(V^\#_{\lambda+1}, V_{\lambda+1}) \to L_\omega(V^\#_{\lambda+1}, V_{\lambda+1}).$$

Let $S_\omega = \{\beta < \lambda^+ | \, cof(\beta) = \omega\}$*. Then in* $L(V_{\lambda+1})$ *the weak club filter restricted to* S_ω *is an ultrafilter.*

Proof. Assume that there exists an α such that α is good and there exists $S \in L_\alpha(V_{\lambda+1})$, $S \subseteq \lambda^+$ such that both S and $S_\omega \setminus S$ are weakly stationary in $L(V_{\lambda+1})$. But by Theorem 3.9 inverse limit reflection holds at α. So by the proof of Theorem 4.4, there is a weakly club $C \in L(V_{\lambda+1})$ such that either $C \subseteq S$ or $C \subseteq S_\omega \setminus S$, a contradiction. □

We can prove similar results in exactly the same way for limit ordinals $\gamma > \lambda^+$ such that $\mathrm{cof}(\gamma) > \lambda$. For instance we have the following.

Theorem 4.10. *Suppose there exists an elementary embedding*

$$j : L_\omega(V^\#_{\lambda+1}, V_{\lambda+1}) \to L_\omega(V^\#_{\lambda+1}, V_{\lambda+1})$$

and that $\gamma < \Theta$ *is such that* $cof(\gamma) > \lambda$*. Let*

$$S_\omega = \{\beta < \gamma | \, cof(\beta) = \omega\}.$$

Then if $S \in L_\alpha(V_{\lambda+1})$ *and* $S \subseteq S_\omega$ *is stationary (in* V*), then* $S_\omega \setminus S$ *is not stationary.*

5 Perfect set property

In this section we prove an approximation to the Perfect Set Property in $L(V_{\lambda+1})$. We regard $V_{\lambda+1}$ as a topological space with basic open sets $O_{(a,\alpha)}$, where $\alpha < \lambda$, $a \subseteq V_\alpha$ and

$$O_{(a,\alpha)} = \{b \in V_{\lambda+1} | \, b \cap V_\alpha = a\}.$$

Since $\mathrm{cof}(\lambda) = \omega$, this is a metric topology, and it is complete. Xianghui Shi and Woodin showed a similar result follows from the conclusion of Theorem 6.4.

Theorem 5.1. *Suppose there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Assume $X \subseteq V_{\lambda+1}$, $X \in L(V_{\lambda+1})$, and $|X| > \lambda$. Then there is a perfect set $Y \subseteq X$ such that $|Y| > \lambda$ and $Y \in L(V_{\lambda+1})$. In fact, for all $a, \alpha \in V_\lambda$ such that $a \subseteq V_\alpha$ and there exists $b \in Y$ such that $a = b \cap V_\alpha$, we have

$$|Y \cap O_{(a,\alpha)}| > \lambda.$$

We will need a version of the pigeonhole principle in our proof of the Perfect Set Property. The next several lemmas demonstrate these kind of principles for saturated sets.

The simplest example of this pigeonhole principle is given in the following lemma.

Lemma 5.2. *Assume $X \subseteq V_{\lambda+1}$, $X \in L(V_{\lambda+1})$, and $|X| > \lambda$. Let $\alpha < \Theta$ be good such that $X \in L_\alpha(V_{\lambda+1})$. Suppose that $E \subseteq \mathcal{E}_{\alpha+1}$ is saturated and $\bar{\alpha}$ are such that for all $(J, \vec{j}) \in E$, J extends to*

$$\hat{J} : L_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

and $X \in rng\, \hat{J}$. Let $(J, \vec{j}) \in E$, let $\bar{X}$ be such that $\hat{J}(\bar{X}) = X$, and define $E' \subseteq E$ by

$$E' = \{(K, \vec{k}) \in E |\, \hat{K}(\bar{X}) = X\}.$$

Then there is an $\bar{A} \in V_{\bar{\lambda}+1}$ such that for

$$Y_{\bar{A}} = \{A \in V_{\lambda+1} |\, \exists (K, \vec{k}) \in E' \text{ a limit root of } J \text{ such that } K(\bar{A}) = A\},$$

we have $Y_{\bar{A}} \subseteq X$ and $|Y_{\bar{A}}| > \lambda$.

Proof. For $(J, \vec{j}) \in E$, let $\bar{X}$ be such that $\hat{J}(\bar{X}) = X$, and define $E' \subseteq E$ by

$$E' = \{(K, \vec{k}) \in E |\, \hat{K}(\bar{X}) = X\}.$$

Clearly E' is also saturated. Let n be such that for any $a \in V_{\lambda+1}$ there is $(K, \vec{k}) \in E'$ an n-close limit root of $(J, \vec{j})$ such that $a \in \operatorname{rng} K_n$. Let $X_n = (\hat{j}_0 \circ \hat{j}_1 \circ \cdots \circ \hat{j}_{n-1})^{-1}(X)$, where $\hat{j}_i$ is the natural extension of j_i to an elementary embedding $j_i : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$. By elementarily we have that $|X_n| > \lambda$. Hence using the fact that $|\bar{X}| < \lambda$ and $|X_n| > \lambda$, there is an $\bar{A} \in \bar{X}$ such that for

$$Y^n_{\bar{A}} = \{A \in V_{\lambda+1} |\, \exists (K, \vec{k}) \in E' \text{ a limit root of } J \text{ such that } K_n(\bar{A}) = A\},$$

$|Y^n_{\bar{A}}| > \lambda$ and $Y^n_{\bar{A}} \subseteq X_n$. Hence since $j_0 \circ j_1 \circ \cdots \circ j_{n-1}$ is injective we have that $|Y_{\bar{A}}| > \lambda$ and $Y_{\bar{A}} \subseteq X$. □

We will be considering saturated sets with a kind of 'homogeneity' property. The next lemma demonstrates the use of this property. We will use this property below to obtain a homogeneous version of our pigeonhole principle.

Lemma 5.3. *Suppose that $E \subseteq \mathcal{E}$ has the property that for all $J, K \in E$ and $n < \omega$, $\bar{\lambda}_J = \bar{\lambda}_K$ and*

$$j_0 \circ \cdots \circ j_{n-1} \circ k_n \circ k_{n+1} \circ \cdots \in E.$$

Fix $J, K \in E$, $a \in V_{\bar{\lambda}_K}$ and $n < \omega$. Suppose that for $\beta < \lambda$, we have that

$$|\{b \in V_{\lambda+1} |\, \exists S \in E (\forall i < n(s_i = k_i) \wedge S(a) = b)\}| \geq \beta.$$

Then

$$|\{b \in V_{\lambda+1} |\, \exists S \in E (\forall i < n(s_i = j_i) \wedge S(a) = b)\}| \geq \beta.$$

Proof. Let $\langle (S^\alpha, \vec{s}^\alpha) |\, \alpha < \beta \rangle$ and $\langle b^\alpha |\, \alpha < \beta \rangle$ witness the above hypothesis. So the b^α for $\alpha < \beta$ are distinct, $S^\alpha(a) = b^\alpha$ for all $\alpha < \beta$, and for $i < n$, $k_i = s^\alpha_i$. Then we have that for $b^\alpha_0 = (k_0 \circ \cdots k_{n-1})^{-1}(b^\alpha)$, that $S^\alpha_n(a)$ are distinct elements for $\alpha < \beta$. But then

$$(j_0 \circ \cdots \circ j_{n-1} \circ S^\alpha_n)(a)$$

are distinct for $\alpha < \beta$. And by the assumed property of E we have that

$$j_0 \circ \cdots \circ j_{n-1} \circ S^\alpha_n \in E$$

for all $\alpha < \beta$. So the lemma follows. □

The following lemma gives us a much more powerful version of the Lemma 5.2.

Lemma 5.4. *Suppose E is a saturated set of inverse limits and $(J, \vec{j}) \in E$. Let Z be the set of $A \in V_{\bar{\lambda}+1}$ such that*

$$|\{K(A) | (K, \vec{k}) \in E \text{ is a limit root of } J\}| < \lambda.$$

Then $|Z| \leq \bar{\lambda}$.

Proof. Let $\kappa < \lambda$ and let Z_κ be the set of $A \in V_{\bar{\lambda}+1}$ such that

$$|\{K(A) | (K, \vec{k}) \in E \text{ is a limit root of } J\}| < \kappa.$$

Suppose $|Z_\kappa| > \bar{\lambda}$. Let $\bar{T}$ be the tree of initial segments of elements of Z_κ. We have $|[\bar{T}]| > \bar{\lambda}$. Let $J(\bar{T}) = T$. Then by elementarity, $|[T]| > \lambda$. But by definition of Z_κ we have that

$$|\bigcup\{K''\bar{T} | (K, \vec{k}) \in E \text{ is a limit root of } J\}| \leq \bar{\lambda} \cdot \kappa < \lambda.$$

We claim this is a contradiction. To see this, let i be such that for all $b \in V_{\lambda+1}$ there exists $(K, \vec{k}) \in E$ a limit root of J such that $b \in \operatorname{rng} K_i$ and $K(\bar{T}) = T$. Let $T_i = (j_0 \circ \cdots j_{i-1})^{-1}(T)$. Then $|T_i| = \lambda$ and for all $b \in T_i$, there exists $(K, \vec{k}) \in E$ a limit root of J such that $b \in \operatorname{rng} K_i$. But then $(j_0 \circ \cdots \circ j_{i-1})(b) \in T$. And hence, since $j_0 \circ \cdots \circ j_{i-1}$ is injective,

$$|\bigcup\{K''\bar{T} | (K, \vec{k}) \in E \text{ is a limit root of } J\}| = \lambda,$$

a contradiction.

The lemma follows by noting that $\operatorname{cof}(\lambda) = \omega$, so $|Z| \leq \bar{\lambda}$. □

Finally, with our 'homogeneity' property we are able to further strengthen the previous lemma. This is the final version of our pigeonhole principle which we will use to prove our theorem.

Lemma 5.5. *Suppose that $E \subseteq \mathcal{E}$ is saturated and has the property that for all $J, K \in E$ and $n < \omega$, $\bar{\lambda}_J = \bar{\lambda}_K$ and*

$$j_0 \circ \cdots \circ j_{n-1} \circ k_n \circ k_{n+1} \circ \cdots \in E.$$

Let Z be the set of $A \in V_{\bar{\lambda}+1}$ such that there exists a $(J, \vec{j}) \in E$ and $n < \omega$ with

$$|\{K(A) | (K, \vec{k}) \in E \text{ agrees up to } n \text{ with } J\}| < \lambda.$$

Then $|Z| \leq \bar{\lambda}$.

Proof. Note that by the assumed property of E, as in the proof of Lemma 5.3, if $A \in V_{\bar{\lambda}+1}$ is such that there exists $(J, \vec{j}) \in E$ and $n < \omega$ with

$$|\{K(A) | (K, \vec{k}) \in E \text{ agrees up to } n \text{ with } J\}| < \lambda$$

then in fact for all $(J, \vec{j}) \in E$

$$|\{K(A) | (K, \vec{k}) \in E \text{ agrees up to } n \text{ with } J\}| < \lambda.$$

Hence the lemma follows by Lemma 5.4. □

We can now prove the perfect set property using the previous lemmas.

Proof of Theorem 5.1. By Σ_1-reflection, if there a counterexample to the Theorem, then there is one below the least stable δ of $L(V_{\lambda+1})$. So we prove the Theorem for subsets of $V_{\lambda+1}$ in $L_\delta(V_{\lambda+1})$.

Let $\alpha < \delta$ be good and let $X \in L_\alpha(V_{\lambda+1})$ be such that $X \subsetneq V_{\lambda+1}$. By strong inverse limit reflection, there is $E \subseteq \mathcal{E}$ saturated, $\bar{\alpha}$, and $\bar{X}$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to

$$\hat{J} : L_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

and $\hat{J}(\bar{X}) = X$. Let $\langle \lambda_i | i < \omega \rangle$ be increasing and cofinal in λ, and let $\langle \kappa_i | i < \omega \rangle$ be increasing and cofinal in $\bar{\lambda}$.

Let $T \subseteq V_{\bar{\lambda}}$ be a tree defined as follows. For $i < \omega$ let

$$T_i = \{B \in V_{\kappa_i+1} : |\{A \in V_{\bar{\lambda}+1} | A \in \bar{X}, B = A \cap V_{\kappa_i}\}| > \bar{\lambda}\}$$

and

$$T = \{(A_{i_0}, \ldots, A_{i_n}) | \forall m \leq n (A_{i_m} \in T_{i_m} \text{ and } \forall s < m (A_{i_s} = A_{i_m} \cap V_{\kappa_{i_s}}))\}.$$

Let I be the set

$$I = \{\vec{s} \in [\lambda]^{<\omega} | \forall i < \text{len}(\vec{s})(s_i < \lambda_i)\}.$$

Now let

$$F : \{(\vec{A}, s) | \exists n, \vec{i}(\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T, s \in I, |s| = n)\} \to E$$

have the following properties:

1. For all $\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T$, $s \in I$, $|s| = n - 1$, if
$$F(\vec{A}, s^\frown \langle \xi \rangle) = (K, \vec{k}) \text{ and } F((\vec{A}, s^\frown \langle \beta \rangle)) = (K', \vec{k}')$$
for $\xi < \beta < \lambda_n$, then $K(A_{i_n}) \neq K'(A_{i_n})$.

2. For all $\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T$, $s \in I$, $|s| = n$, and $m < n$ if
$$F(\vec{A}, s) = (K, \vec{k}) \text{ and } F(\vec{A} \restriction m + 1, s \restriction m) = (K', \vec{k}'),$$
then $K(A_{i_m}) = K'(A_{i_m})$.

3. For all $\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T$, $s \in I$, $|s| = n - 1$ then $F(\vec{A}, s^\frown \langle \alpha \rangle)$ and $F(\vec{A} \restriction n, s)$ agree up to n.

Also assume that F is maximal with these properties, in the sense that F cannot be extended to some F' also satisfying these properties.

Let Z be the set of $A \in \bar{X}$ such that there exists a sequence $\langle i_n | \, n < \omega \rangle$ such that for $A_{i_n} = A \cap V_{\kappa_{i_n}}$, for all $n < \omega$, and $s \in I$, if $|s| = n$ then $((A_{i_0}, \ldots, A_{i_n}), s) \in \mathrm{dom}(F)$. We claim that $|\bar{X} \setminus Z| \leq \bar{\lambda}$. To see this, suppose that $A \in \bar{X} \setminus Z$. Then there exists $\vec{A}$ and s such that $F(\vec{A}, s) = (K, \vec{k})$, and
$$|\{K'(A)| \, (K', \vec{k}') \in E \text{ agrees with } K \text{ up to } |s| + 1 \ \wedge K'(\vec{A}) = K(\vec{A})\}| < \lambda.$$
Furthermore, by the proof of Lemma 5.5, for all t with $|t| = |s|$, we have that $(\vec{A}, t)$ has this property as well. But by Lemma 5.4 for every K, there are $\leq \bar{\lambda}$ many such A with this property. Hence $|\bar{X} \setminus Z| \leq \bar{\lambda}$.

So finally, let $A \in Z$, and let $\langle i_n | n < \omega \rangle$ be such that for all $n < \omega$, $A_{i_n} = A \cap V_{\kappa_{i_n}}$, and for $s \in I$, if $|s| = n$ then $((A_{i_0}, \ldots, A_{i_n}), s) \in \mathrm{dom}(F)$. Set
$$K^{s,n} = F((A_{i_0}, \ldots, A_{i_n}), s).$$
Also for $x \in \lambda^\omega$, let K^x be the common part of $\left\langle K^{x \restriction n, n} | \, n < \omega \right\rangle$, and set
$$P = \{K^x(A) | \, x \in \lambda^\omega, \forall i < \omega (x_i < \lambda_i)\}.$$
Clearly P is a perfect subset of X by definition of E and the fact that $A \in Z \subseteq \bar{X}$. Furthermore by definition of F we have $|P| > \lambda$. Note that for any $s \in I$, if we set
$$P^s = \{K^x(A) | \, x \in \lambda^\omega, \forall i < \omega (x_i < \lambda_i), \forall i < |s| (s_i = x_i)\}$$
then P^s is a perfect subset of P, $|P^s| > \lambda$ and
$$P^s = P \cap O_{(A_{i_n}, \kappa_{i_n})}$$
where $n = |s|$. And hence we have the final part of the conclusion. $\square$

6 The Tower Condition

We introduce the notion of a $U(j)$-representation, which Woodin introduced as an analogue of being weakly homogeneous Suslin in the context of $L(V_{\lambda+1})$. Woodin [10] showed that if there exists an elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}),$$

then every subset of $V_{\lambda+1}$ in $L_\lambda(V_{\lambda+1})$ is $U(j)$-representable. We extend this result by proving the Tower Condition, which, by a theorem of Woodin, shows that every subset of $V_{\lambda+1}$ in $L_{\lambda^+}(V_{\lambda+1})$ has a $U(j)$-representation (in fact it shows more, see Theorem 6.11).

For the rest of this section we fix $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ elementary. We will use the notation $j_{(i)}$ to denote the i-th iterate of j to distinguish it from our inverse limit notation.

Definition 6.1 (Woodin). *Let $U(j)$ be the set of $U \in L(V_{\lambda+1})$ such that in $L(V_{\lambda+1})$ the following hold:*

1. *U is a λ^+-complete ultrafilter.*
2. *For some $\gamma < \Theta$, $U \subseteq P(L_\gamma(V_{\lambda+1}))$.*
3. *For some $A \in U$ and all sufficiently large $n < \omega$,*

$$j_{(n)}(U) = U \qquad \text{and} \qquad \{a \in A |\, j_{(n)}(a) = a\} \in U.$$

For each ordinal κ, let $\Theta^{L_\kappa(V_{\lambda+1})}$ denote the supremum of the ordinals α such that there is a surjection $\rho : V_{\lambda+1} \to \alpha$ such that

$$\{(a, b) |\, \rho(a) < \rho(b)\} \in L_\kappa(V_{\lambda+1}).$$

Suppose that $\kappa < \Theta$ and $\kappa \le \Theta^{L_\kappa(V_{\lambda+1})}$. Then $\mathcal{E}(j, \kappa)$ is the set of all elementary embeddings $k : L_\kappa(V_{\lambda+1}) \to L_\kappa(V_{\lambda+1})$ such that there exists $n, m < \omega$ such that $k_{(n)} = j_{(m)} \restriction L_\kappa(V_{\lambda+1})$.

Suppose that $\kappa < \Theta$ and $\kappa \le \Theta^{L_\kappa(V_{\lambda+1})}$. Suppose that $\langle a_i |\, i < \omega\rangle$ is a sequence of elements of $L_\kappa(V_{\lambda+1})$ such that for all $i < \omega$, there exists an $n < \omega$ such that $j_{(n)}(a_i) = a_i$. Let $U(j, \kappa, \langle a_i |\, i < \omega\rangle)$ denote the set of $U \in U(j)$ such that there exists $n < \omega$ such that for all $k \in \mathcal{E}(j, \kappa)$, if $k(a_i) = a_i$ for all $i \le n$, then

$$\{a \in L_\kappa(V_{\lambda+1}) |\, k(a) = a\} \in U.$$

Remark 6.2. If $U \in U(j) \cap L_\kappa(V_{\lambda+1})$ where κ is good, $k \in \mathcal{E}(j, \kappa+1)$ and $k(U) = U$, then we have that for some $A \in U$, $\{a \in A | \, k(a) = a)\} \in U$. To see this, note that for any n, $0 < n < \omega$ such that there is $A \in U$ such that $\{a \in A | \, k_{(n)}(a) = a\} \in U$, there is such an $A \in \operatorname{rng} k$. And hence, pulling back by k we have that for such an A,

$$\{a \in k^{-1}(A) | \, k_{(n-1)}(a) = a\} \in U.$$

So by induction we have that there is an $A^k \in U$ such that $\{a \in A^k | \, k(a) = a\} \in U$. Hence, while it does not appear as though every $U \in U(j)$ appears in some $U(j, \kappa, \langle a_i | \, i < \omega \rangle$, this is not far from the truth in the above sense.

Also note that if $n < \omega$ and $A \in U$ are such that $\{a \in A | \, j_{(n)}(a) = a\} \in U$, then for all $B \in U$, $\{a \in B | \, j_{(n)}(a) = a\} \in U$. This follows by simply noting

$$B \cap \{a \in A | \, j_{(n)}(a) = a\} \subseteq \{a \in B | \, j_{(n)}(a) = a\}.$$

Definition 6.3 (Woodin)**.** *Suppose $\kappa < \Theta$ is weakly inaccessible in $L(V_{\lambda+1})$, and $\langle a_i | \, i < \omega \rangle$ is an ω-sequence of elements of $L_\kappa(V_{\lambda+1})$ such that for all $i < \omega$ there is an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.*

Suppose that

$$Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}.$$

Then Z is $U(j, \kappa, \langle a_i | \, i < \omega \rangle)$-representable *if there exists an increasing sequence $\langle \lambda_i | \, i < \omega \rangle$, cofinal in λ and a function*

$$\pi : \bigcup \{V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} | \, i < \omega\} \to U(j, \kappa, \langle a_i | \, i < \omega \rangle)$$

such that the following hold:

1. *For all $i < \omega$ and $(a, b, i) \in dom(\pi)$ there exists $A \subseteq (L(V_{\lambda+1}))^i$ such that $A \in \pi(a, b, i)$.*
2. *For all $i < \omega$ and $(a, b, i) \in dom(\pi)$, if $m < i$ then*

   $$(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m) \in dom(\pi)$$

   and $\pi(a, b, i)$ projects to $\pi(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m)$.
3. *For all $x \subseteq V_\lambda$, $x \in Z$ if and only if there exists $y \subseteq V_\lambda$ such that*

   (a) for all $m < \omega$, $(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) \in dom(\pi)$,

(b) the tower

$$\langle \pi(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) |\, m < \omega\rangle$$

is wellfounded.

For $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$ we say that Z is $U(j)$-representable *if there exists $(\kappa, \langle a_i |\, i < \omega\rangle)$ such that Z is $U(j, \kappa, \langle a_i |\, i < \omega\rangle)$-representable.*

$U(j)$-representations are important for a number of reasons. One example of their importance is the following theorem of Woodin which gives a version of generic absoluteness using a uniform version of $U(j)$-representations.

Theorem 6.4 (Woodin)**.** *Suppose that $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is a proper elementary embedding (see [10]). Let M_ω be the ω-th iterate of $L(V_{\lambda+1})$ by j, and let $j_{0,\omega} : L(V_{\lambda+1}) \to M_\omega$. Suppose that $g \in V$, g is M_ω-generic for a partial order $\mathbb{P} \in j_{0,\omega}(V_\lambda)$ and that cof$(\lambda) = \omega$ in $M_\omega[g]$. Then for all $\alpha < \lambda$ there exists an elementary embedding*

$$\pi : L_\alpha(M_\omega[g] \cap V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$$

such that $\pi \restriction \lambda$ is the identity.

Definition 6.5 (Woodin)**.** *Suppose $A \subseteq U(j)$, $A \in L(V_{\lambda+1})$, and $|A| \leq \lambda$. The* Tower Condition for *A is the following statement: There is a function $F : A \to L(V_{\lambda+1})$ such that the following hold:*

1. *For all $U \in A$, $F(U) \in U$.*

2. *Suppose $\langle U_i |\, i < \omega\rangle \in L(V_{\lambda+1})$ and for all $i < \omega$, there exists $Z \in U_i$ such that $Z \subseteq L(V_{\lambda+1})^i$, $U_i \in A$, and U_{i+1} projects to U_i. Then the tower $\langle U_i |\, i < \omega\rangle$ is wellfounded in $L(V_{\lambda+1})$ if and only if there exists a function $f : \omega \to L(V_{\lambda+1})$ such that for all $i < \omega$, $f \restriction i \in F(U_i)$.*

The Tower Condition for *$U(j)$ is the statement that for all $A \subseteq U(j)$ if $A \in L(V_{\lambda+1})$ and $|A| \leq \lambda$ then the Tower Condition holds for A.*

For the proof of the Tower Condition we do not actually use inverse limit reflection. Instead, we use the structure of the inverse limits together with their 'naive extensions' above λ. Because of this difference we define for $\alpha < \Theta$,

$$\mathcal{E}^e_\alpha = \{(J, \vec{j}) |\, (J, \vec{j} \restriction V_{\lambda+1}) \in \mathcal{E}, \forall i (j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

Suppose that $(J, \vec{j}) \in \mathcal{E}^e_\alpha$. Then we say that $a \in L_\alpha(V_{\lambda+1})$ is in the *extended range of* J if for all $i < \omega$, $a \in \operatorname{rng}(j_0 \circ \cdots \circ j_i)$. We set $J^{\text{ext}}(b) = a$ if for some $n < \omega$, for all $i \geq n$,

$$(j_0 \circ \cdots \circ j_i)^{-1}(a) = b.$$

Again, we omit the sequence of embeddings from our notation.

To state the next lemma recall the useful notation

$$j_m^{(m-1)} = (j_0 \circ j_1 \circ \cdots \circ j_{m-1})(j_m).$$

Lemma 6.6. *Suppose α is good and $(J, \langle j_i \rangle) \in \mathcal{E}^e_\alpha$ is an inverse limit such that for all i, $j_i(j_i) = j \restriction L_\alpha(V_{\lambda+1})$. Let $U \in L_\alpha(V_{\lambda+1})$ be in the extended range of J and such that for some i, $j_{(i)}(U) = U$. Let*

$$j_0(U^0) = U, j_1(U^1) = U^0, \ldots.$$

Then there exists an n such that for all $m \geq n$, $U^n = U^m$. Furthermore, for this n we have that for all $m \geq n$, $j_m^{(m-1)}(U) = U$.

Proof. Note that $j_{(n)}$ denotes the nth iterate of j, and j_n denotes the nth element of the inverse limit sequence. Let m be such that $j_{(m-1)}(U) = U$. We prove by induction that for $n \geq m$ we have $j_n(U^n) = U^n$. First suppose that $m = 1$. Then $j(U) = U$. We have that

$$j(U) = U \Rightarrow j_0(j_0)(U) = U \Rightarrow j_0(U^0) = U^0.$$

The first implication follows since j_0 is a square root of j, and the second implication follows by pulling back the equality by j_0. And hence $U^0 = U$. The fact that $j_n(U^n) = U^n$ follows by induction.

Now suppose that $m > 1$. Assume by induction that we have proved the result for all $m' < m$. Then we have for $n = m - 1$

$$j_{(n)}(U) = U \Rightarrow (j_0(j_0))_{(n)}(U) = U \Rightarrow (j_0)_{(n)}(U^0) = U^0 \Rightarrow j_{(n-1)}(U^0) = U^0.$$

The first implication follows since j_0 is a square root of j, the second follows by pulling back by j_0, and the third follows again by the fact j_0 is a square root of j. And then using the induction hypothesis on U^0 and $\langle j_i | \, i \geq 1 \rangle$ we have the first result.

To see the second result, note that $U^{m-1} = j_m(U^m) = U^m$, and hence

$$\begin{aligned} & j_m(U^m) = U^m \Rightarrow j_m(U^{m-1}) = U^{m-1} \Rightarrow \\ & \quad j_m^{(m-1)}((j_0 \circ \cdots \circ j_{m-1})(U^{m-1})) = (j_0 \circ \cdots \circ j_{m-1})(U^{m-1}) \Rightarrow \\ & \quad j_m^{(m-1)}(U) = U, \end{aligned}$$

for any $m \geq n$, for n satisfying the first part of the conclusion (where $U^{-1} = U$). Here the first implication follows since $U^{m-1} = U^m$, the second follows by applying $j_0 \circ \cdots \circ j_{m-1}$, and the final follows by definition of U^{m-1}. Hence we have the desired result. □

Lemma 6.7. *Suppose that $A \in L_\Theta(V_{\lambda+1})$, $|A| \leq \lambda$ and for all $a \in A$, there exists an i such that $j_{(i)}(a) = a$. Then there exists a sequence $\langle B_i | \, i < \omega\rangle$ and $(K, \langle k_i | \, i < \omega\rangle) \in \mathcal{E}_\eta^e$ for some $\eta < \Theta$ good such that,*

1. *for all $i < \omega$, $|B_i| < \lambda$.*
2. *for all $i < \omega$, $B_i = (k_0 \circ \cdots \circ k_{i-1})(B_0)$,*
3. *$A \subseteq \lim_{i \to \omega} B_i := \{a | \, \exists n \, \forall i \geq n (a \in B_i)\}$,*
4. *for all $i < \omega$, $k_i(k_i) = j \restriction L_\alpha(V_{\lambda+1})$,*
5. *for all $a \in \lim_{i \to \omega} B_i$, for all large enough $i < \omega$ we have $k_i^{(i-1)}(a) = a$,*
6. *for all $a \in \lim_{i \to \omega} B_i$, for all large enough $i < \omega$ we have*

$$a \in rng\,(K_i^{(i-1)})^{ext}.$$

Proof. Let $C = \langle U_\alpha | \, \alpha < \lambda\rangle$ be an enumeration of A, and let $\eta < \Theta$ be good and large enough so that $C, A \in L_\eta(V_{\lambda+1})$. Let $(K, \langle k_i | \, i < \omega\rangle) \in \mathcal{E}_\eta^e$ be such that for all $i < \omega$, $k_i(k_i) = j \restriction L_\alpha(V_{\lambda+1})$,

$$k_0(C_0) = C, \; k_0(A_0) = A$$

and for $i > 0$,

$$k_i(C_i) = C_{i-1}, \; k_i(A_i) = A_{i-1}.$$

Let $\bar{\lambda} = \bar{\lambda}_K$. Set

$$B_0 = \lim_{i \to \omega} C_i \restriction \bar{\lambda}.$$

Let $B_i = (k_0 \circ \cdots \circ k_{i-1})(B_0)$.

We want to show that for $\alpha < \mathrm{crit}\,(k_0)$, that $U_\alpha \in \lim_{i\to\omega} B_i$. But this follows by Lemma 6.6. To see this, by induction define U^i_α for $i < \omega$ as follows: $k_0(U^0_\alpha) = U_\alpha$ and for $i \geq 0$, $k_{i+1}(U^{i+1}_\alpha) = U^i_\alpha$. Then by the lemma we have that for some n, $U^i_\alpha = U^n_\alpha$ for all $i \geq n$. Hence $U^n_\alpha \in B_0$. We want that for all $i \geq n$, $U_\alpha \in B_{i+1}$. But this follows since

$$U^n_\alpha = U^i_\alpha \in B_0 \Rightarrow (k_0 \circ \cdots \circ k_i)(U^i_\alpha) = U_\alpha \in B_{i+1}.$$

Similarly, we have that for $\alpha < (k_0 \circ \cdots \circ k_{i-1})(\mathrm{crit}\,k_i)$, $U_\alpha \in \lim_{i\to\omega} B_i$. For ease of notation we prove this for $i = 1$. The proof for $i > 1$ is very similar. So we want for $\alpha < k_0(\mathrm{crit}\,k_1)$, that $U_\alpha \in \lim_{i\to\omega} B_i$. To see this, by induction define U^i_α for $i < \omega$ as follows: $k^{(0)}_1(U^1_\alpha) = U_\alpha$ and for $i \geq 1$, $k^{(0)}_{i+1}(U^{i+1}_\alpha) = U^i_\alpha$. Then by Lemma 6.6 we have that for some n, $U^i_\alpha = U^n_\alpha$ for all $i \geq n$. We want to see that $U^n_\alpha \in B_1$. We have

$$B_1 = k_0(\lim_{i\to\omega} C_i \upharpoonright \bar{\lambda}) = \lim_{i\to\omega} k_0(C_i) \upharpoonright k_0(\bar{\lambda}),$$

and furthermore

$$k_0(k_1 \circ \cdots \circ k_i)(k_0(C_i)) = C.$$

Hence using that $k_0(\mathrm{crit}\,k_1) = \mathrm{crit}\,(k_0(k_1 \circ \cdots \circ k_i))$ and $\alpha < k_0(\mathrm{crit}\,k_1)$ we have that $U^n_\alpha \in B_1$. We show that for all $i \geq n$, $U_\alpha \in B_{i+1}$. But this follows since

$$U^n_\alpha = U^i_\alpha \in B_1 \Rightarrow k_0(k_1 \circ \cdots \circ k_i)(U^i_\alpha) = U_\alpha \in B_{i+1}.$$

Note that

$$B_i = k_0(k_1 \circ \cdots \circ k_{i-1})(k_0(B_0)) = k_0(k_1 \circ \cdots \circ k_{i-1})(B_1).$$

But

$$\sup_{i<\omega}(k_0 \circ \cdots \circ k_{i-1})(\mathrm{crit}\,k_i) = \lambda.$$

So $A \subseteq \lim_i B_i$.

Note that we have for all $U \in \lim_{i\to\omega} B_i$, for all large enough i we have $j^{(i-1)}_i(U) = U$ and therefore $U \in \mathrm{rng}\,(K^{(i-1)}_i)^{\mathrm{ext}}$, using the proof of Lemma 6.6 together with above argument. □

Theorem 6.8. *Suppose $A \subseteq U(j)$, $A \in L(V_{\lambda+1})$, $|A| \leq \lambda$. Then the tower condition for A holds.*

Proof. Let $A \subset U(j)$, $|A| = \lambda$, and $A \in L(V_{\lambda+1})$. By Lemma 6.7 there are $\eta < \Theta$, $(J, \langle j_i | \, i < \omega \rangle) \in \mathcal{E}^e_{\eta+2}$ and $\langle B_i | \, i < \omega \rangle$ such that the following hold

1. η is good,

$$L_\eta(V_{\lambda+1}) \prec_{\Sigma_1} L_\Theta(V_{\lambda+1}) \text{ and } A \subseteq L_\eta(V_{\lambda+1}).$$

2. $A \subseteq \lim_{i \to \omega} B_i$,

3. for all $i < \omega$, $|B_i| < \lambda$.

4. for all $i < \omega$, $j_i(j_i) = j \upharpoonright L_\eta(V_{\lambda+1})$,

5. for all $i < \omega$, $B_i = (j_0 \circ \cdots \circ j_{i-1})(B_0)$.

6. for all $U \in \lim_{i \to \omega} B_i$, for all large enough $i < \omega$ we have

$$j_i^{(i-1)}(U) = U,$$

7. for all $U \in \lim_{i \to \omega} B_i$, for all large enough $i < \omega$ we have

$$U \in \operatorname{rng}(J_i^{(i-1)})^{\text{ext}}.$$

Claim 6.9. *There is a tower function $F_0 \in L_\eta(V_{\lambda+1})$ for B_0.*

Proof. We have that $|B_0| < \lambda$ and λ is a strong limit. Hence $|B_0^\omega| < \lambda$. So by λ-DC in $L(V_{\lambda+1})$, we can choose a function g such that for tower of measures $\langle U_n | \, n < \omega \rangle$ where $U_n \in B_0$ for all $n < \omega$, if $\langle U_n | \, n < \omega \rangle$ is an ill founded tower then this is witnessed by $\langle g(\langle U_n | \, n < \omega \rangle, i) | \, i < \omega \rangle$. So for all $i < \omega$, $g(\langle U_n | \, n < \omega \rangle, i) \in U_i$ and there is no f such that for all $i < \omega$,

$$f \upharpoonright i \in g(\langle U_n | \, n < \omega \rangle, i).$$

Let F_0 be defined by

$$F_0(U) = \bigcap \{ g(\vec{U}, i) | \, \vec{U} \in B_0^\omega, g(\vec{U}, i) \text{ is defined, and } U_i = U \}$$

where i is such that U concentrates on i-sequences (if there is such a $\vec{U}$; otherwise let $F_0(U)$ be any element of U). Since for all $U \in B_0$, U is λ^+-complete, $F_0(U) \in U$, and clearly F_0 is a tower function for B_0. □

Since $|B_0| < \lambda$, λ-DC holds in $L(V_{\lambda+1})$, and each measure in A is λ^+-complete, there is a tower function $F_0 \in L_\eta(V_{\lambda+1})$ for B_0. Define for $i > 0$,

$$(j_0 \circ \cdots \circ j_{i-1})(F_0) = F_i.$$

Let $B := \lim_{i\to\omega} B_i$, and for $U \in B$ define

$$F(U) = \bigcap\{F_i(U) \cap \{a \in L(V_{\lambda+1}) |\, j_i^{(i-1)}(a) = a\}| \\ i < \omega, \forall n \geq i(U \in B_n, j_n^{(n-1)}(U) = (U))\}.$$

Note that $F(U) \in U$ by Remark 6.2 and the conditions on B_i.

We want to show that F is a tower function for $B := \lim_{i\to\omega} B_i$. To see this suppose $\langle U_i |\, i < \omega\rangle$ is an illfounded tower with $U_i \in B$ for all $i < \omega$, and $f \in L_\eta(V_{\lambda+1})$ is such that

$$\forall i(f \restriction i \in F(U_i)).$$

Let $\langle \alpha_i |\, i < \omega\rangle \in L_\eta(V_{\lambda+1})$ be such that

$$j_{U_i,U_{i+1}}(\alpha_i) > \alpha_{i+1}.$$

For $i < \omega$, let m_i be least such that $U_i \in B_n$ for all $n \geq m_i$.

One key point is that for all $i < \omega$, there is an $n < \omega$ such that for all $m \geq n$, $f \restriction i \in \operatorname{rng}(J_m^{(m-1)})^{\text{ext}}$. This follows since for n large enough, if $m \geq n$, then $j_m^{(m-1)}(f \restriction i) = f \restriction i$, by our definition of F. And hence

$$(J_n^{(n-1)})^{\text{ext}}(f \restriction i) = f \restriction i.$$

Now let $(K, \langle k_i | i < \omega\rangle) \in \mathcal{E}^e_{\eta+1}$ be a 0-close limit root of J such that the following hold:

1. For all i, $(k_0 \circ \cdots \circ k_i)(F_0) = F_i$ and $(k_0 \circ \cdots \circ k_i)(B_0) = B_i$.
2. For all $i < \omega$, $\alpha_i, f(i) \in \operatorname{rng}(K^{\text{ext}})$. Let α_i^n and $f^n(i)$ be such that

   $$k_0(\alpha_i^0) = \alpha_i,\ k_1(\alpha_i^1) = \alpha_i^0,\ k_2(\alpha_i^2) = \alpha_i^1, \ldots$$

   and

   $$k_0(f^0(i)) = f(i),\ k_1(f^1(i)) = f^0(i),\ k_2(f^2(i)) = f^1(i), \ldots$$

3. For all $i < n$, $k_i(j_n \restriction L_\eta(V_{\lambda+1})) = j_i(j_n \restriction L_\eta(V_{\lambda+1}))$.

4. For all n, let i_n be least such that $U_n \in \operatorname{rng}((j_0 \circ \cdots j_{i_n-1})(J^{\text{ext}}_{i_n}))$. Let $U_{n,i}$ be defined as follows:

$$(j_0 \circ \cdots j_{i_n-1})(j_{i_n})(U_{n,0}) = U_n,\ (j_0 \circ \cdots j_{i_n-1})(j_{i_n+1})(U_{n,1}) = U_{n,0}, \ldots,$$
$$(j_0 \circ \cdots j_{i_n-1})(j_{i_n+i+1})(U_{n,i+1}) = U_{n,i}, \ldots$$

Then for $i, n < \omega$ and $m < i_n$, there are $U^m_{n,i}$ such that

$$k_0(U^0_{n,i}) = U_{n,i},\ k_1(U^1_{n,i}) = U^0_{n,i}, \ldots, k_{i_n-1}(U^{i_n-1}_{n,i}) = U^{i_n-2}_{n,i}.$$

Furthermore, for $i \geq i_n$

$$k_i(U^{i_n-1}_{n,i-i_n}) = j_i(U^{i_n-1}_{n,i-i_n}).$$

It is easy to find such a $(K, \vec{k})$ using the proof of Lemma 2.9.

We have for all i that

$$\alpha_i \geq \alpha^0_i \geq \alpha^1_i \geq \cdots .$$

Let α^ω_i be the stable value. For $n < \omega$, by Lemma 6.6 $\langle U_{n,i} | \, i < \omega \rangle$ and $\langle f^i \restriction n | \, i < \omega \rangle$ must stabilize for some i (here we use what we noted above: that $f \restriction n$ is in the extended range of J). Let $U_{n,\omega}$ and f^ω be the stable values, defining $U^m_{n,\omega}$ in the obvious way as above. Note that we have for all $n, i < \omega$,

$$\begin{aligned}
&(k_0 \circ \cdots \circ k_{i_n-1})(k_{i_n+i})(U_{n,i}) \\
&\qquad = (k_0 \circ \cdots \circ k_{i_n-1})(k_{i_n+i})((k_0 \circ \cdots \circ k_{i_n-1})(U^{i_n-1}_{n,i})) \\
&\qquad = (k_0 \circ \cdots \circ k_{i_n-1})(k_{i_n+i}(U^{i_n-1}_{n,i})) \\
&\qquad = (k_0 \circ \cdots \circ k_{i_n-1})(j_{i_n+i}(U^{i_n-1}_{n,i})) \\
&\qquad = (k_0 \circ \cdots \circ k_{i_n-1})(j_{i_n+i})((k_0 \circ \cdots \circ k_{i_n-1})(U^{i_n-1}_{n,i})) \\
&\qquad = (j_0 \circ \cdots \circ j_{i_n-1})(j_{i_n+i})(U_{n,i}) = U_{n,i-1}
\end{aligned}$$

where $U_{n,-1} = U_n$.

We want to show that $\left\langle U^{i_n-1}_{n,\omega} | \, n < \omega \right\rangle = \langle U'_n | \, n < \omega \rangle$ is an illfounded tower as witnessed by $\langle \alpha^\omega_i \rangle$, and for all $n < \omega$, $U'_n \in B_0$. But also that for all n, $f^\omega \restriction n \in F_0(U'_n)$, contradicting the fact that F_0 is a tower function for B_0.

The fact that for all n, $U'_n \in B_0$ follows since for all large enough i, $U_n \in B_i$ and $(k_0 \circ \cdots \circ k_{i-1})(U'_n) = U_n$.

To see that $\langle U'_n | \, n < \omega \rangle$ is illfounded, fix n and let n_0 be such that

$$f \restriction n \in F_{n_0}(U_n)$$

and for all $i \geq n_0$,

$$(k_0 \circ \cdots \circ k_i)(\alpha^\omega_n, \alpha^\omega_{n+1}, U'_n, U'_{n+1}, f^\omega \restriction n) = (\alpha_n, \alpha_{n+1}, U_n, U_{n+1}, f \restriction n).$$

Then we have that

$$j_{U_n, U_{n+1}}(\alpha_n) > \alpha_{n+1} \Rightarrow j_{U'_n, U'_{n+1}}(\alpha^\omega_n) > \alpha^\omega_{n+1}.$$

And since $f \restriction n \in F_{n_0}(U_n)$ we have that $f^\omega \restriction n \in F_0(U'_n)$.

Hence we have a contradiction to the fact that F_0 is a tower function for B_0. So F is a tower function for B, and the theorem follows as $A \subseteq B$. $\square$

In fact, since we did not actually use inverse limit reflection, exactly the same proof gives the tower condition for $L(X, V_{\lambda+1})$. In this situation we start with assuming an elementary embedding $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$, and we make the same definition for a $U(j)$-representation and the Tower Condition, replacing each $L(V_{\lambda+1})$ with $L(X, V_{\lambda+1})$. We then have the following:

Theorem 6.10. *Suppose there exists an elementary embedding*

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1}).$$

Then the Tower Condition for $U(j)$ holds in $L(X, V_{\lambda+1})$.

Finally by a Theorem of Woodin (see [10] Corollary 149) we have the following:

Corollary 6.11. *Suppose there exists an elementary embedding*

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1}).$$

Let Y be $U(j)$-representable in $L(X, V_{\lambda+1})$. Let $\kappa = \lambda^+$ and set

$$\eta = \sup\{(\kappa^+)^{L[A]} | \, A \subseteq \lambda\}.$$

Then every set

$$Z \in L_\eta(Y, V_{\lambda+1}) \cap V_{\lambda+2}$$

is $U(j)$-representable in $L(X, V_{\lambda+1})$.

We end by noting that the proof of Theorem 6.8 can be altered so that it appears as more of a generalization of the proof of Lemma 123 of [10]. In particular we consider the direct limit of the system for a fixed η,

$$L_\eta(V_{\lambda+1}) \xrightarrow{j_0} L_\eta(V_{\lambda+1}) \xrightarrow{j_0(j_1)} L_\eta(V_{\lambda+1}) \xrightarrow{j_0(j_1(j_2))} \cdots$$

which we denote M^J_ω. We let $J^\omega = \cdots \circ j_0(j_1) \circ j_0$ be the corresponding map from $L_\eta(V_{\lambda+1}) \to M^J_\omega$. Similarly define

$$J^\omega_n = \cdots \circ j_0(j_1(\cdots j_{n-1}(j_n(j_{n+1}))\cdots)) \circ j_0(j_1(\cdots j_{n-1}(j_n)\cdots))$$

which maps $L_\eta(V_{\lambda+1}) \to M^J_\omega$. The proof then proceeds similarly as above, but we 'push up' our contradiction to the tower condition to M^J_ω as in [10], rather than 'pulling it down.' We leave the details to the reader.

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