Existence of tree representations in $L(V_{\lambda+1})$

Scott Cramer

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Abstract

We propagate various representations for subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$. We show that every subset of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ has a $U(j)$-representation and a j-Suslin representation. We also prove that uniform versions of these representations exist for certain subsets of $V_{\lambda+1}$. We discuss various consequences of our results, including implications for the singular cardinal hypothesis and the relationship between large cardinals and strong models of determinacy.

Representations for subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ under I_0 were first studied by H. Woodin in [8], where the notion of a $U(j)$-representation was introduced and many consequences of their existence were shown. These representations were subsequently propagated in [4] and [2], although the question of whether every subset of $V_{\lambda+1}$ had such a representation remained open. On the other hand the notion of a j-Suslin representation was introduced in [3]. We propagate these two types of representations by using aspects of both representations to propel our induction. In particular we define and propagate a representation which we call a weakly homogeneously j-Suslin representation. This representation is stronger than both $U(j)$ and j-Suslin representations and naturally combines their properties.

Our results seem very similar to the propagation of scales in $L(\mathbb{R})$, and in particular we define a representation called a j-closed game representation which seems analogous to closed game representations in the case of $L(\mathbb{R})$. Our main result is that the existence of a certain type of j-closed game representation implies the existence of both $U(j)$-representations and j-Suslin representations.

We will in general be working under the assumption that I_0 holds at λ, and we will assume familiarity with the basics of I_0 and $L(V_{\lambda+1})$. For an introduction to I_0, see [4].

1 j-Suslin representations

We first recall the definition of I_0: we say that I_0 *holds at* λ if there is a non-trivial elementary embedding

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

such that crit $(j) < \lambda$. We call such a j an I_0 *embedding*. Below we will always assume our elementary embeddings are non-trivial and that their critical points are below λ. Recall that in this context $\Theta = \Theta_\lambda$ is the sup of ordinal α such that in $L(V_{\lambda+1})$ there is a surjection of $V_{\lambda+1}$ onto α. We say that α is *good* if every element of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from elements of $V_{\lambda+1}$. Note that the good ordinals are cofinal in Θ.

We define tree representations for subsets of $V_{\lambda+1}$ which were first introduced in [3]. These tree representations seem rather similar to Suslin representations, and so their names indicate this fact.

For this section we fix $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ an elementary embedding with crit $(j) < \lambda$. For k an elementary embedding we denote by $k_{(n)}$ the nth iterate of k, and we let

$$\mathcal{F}_\kappa(k) = \{a \in L_\kappa(V_{\lambda+1}) \,|\, k(a) = a\}, \qquad \mathcal{F}^\omega_\kappa(k) = \bigcup_{n<\omega} \mathcal{F}^\omega_\kappa(k_{(n)}),$$

and let

$$E^k(\kappa) = \{k' : L_\kappa(V_{\lambda+1}) \to L_\kappa(V_{\lambda+1}) \,|\, \exists n, m(k'_{(n)} = k_{(m)})\}$$

if $k : L_\kappa(V_{\lambda+1}) \to L_\kappa(V_{\lambda+1})$ is elementary and iterable. Also for $a \in L_\kappa(V_{\lambda+1})$ let

$$E^k(\kappa, a) = \{k \in E^k(\kappa) \,|\, k(a) = a\}.$$

Note that for $\kappa < \Theta$, if $j(\kappa) = \kappa$ then $j(\mathcal{F}^\omega_\kappa(j)) = \mathcal{F}^\omega_\kappa(j)$.

Definition 1. For $\vec{\kappa} = \langle \kappa_i | \, i < \omega\rangle$ increasing and cofinal in λ, we let $\mathcal{W}^{\vec{\kappa}}$ be the set of sequences $s \in V^\omega_\lambda$ such that

1. for some $n < \omega$, $|s| = n$ and for all $i < n$, $s(i) \subseteq V_{\kappa_i}$,
2. if $i \leq m < |s|$ then $s(i) = s(m) \cap V_{\kappa_i}$.

Also let $\mathcal{W}^{\vec{\kappa}}_n = \{s \in W^{\vec{\kappa}} \,|\, |s| = n\}$. In this context if $x \in V_{\lambda+1}$, we set

$$\hat{x} = \hat{x}_{\vec{\kappa}} = \langle x \cap V_{\kappa_n} \,|\, n < \omega\rangle \in \mathcal{W}^{\vec{\kappa}},$$

where we use the first notation if the sequence $\vec{\kappa}$ is understood.

Suppose that $\kappa < \Theta$. Let $X \subseteq V_{\lambda+1}$. We say that T is a (j, κ)-*Suslin representation* for X if for some sequence $\langle \kappa_i | \, i < \omega\rangle$ increasing and cofinal in λ the following hold.

1. T is a (height ω) tree on $V_\lambda \times \mathcal{F}^\omega_\kappa(j)$ such that for all $(s, a) \in T$, $s \in \mathcal{W}^{\vec{\kappa}}_{|s|}$.
2. For all $s \in \mathcal{W}^{\vec{\kappa}}$, $T_s \in \mathcal{F}^\omega_\Theta(j)$.
3. For all $x \in V_{\lambda+1}$, $x \in X$ iff $T_{\hat{x}}$ is illfounded.

We say that X is j-Suslin if for some κ, X has a (j, κ)-Suslin representation. If T satisfies conditions 1 and 3 then we say that T is a *weak* (j, κ)-*Suslin representation for* X.

Remark 2. We note that weak j-Suslin representations are very easy to find (by considering the pointwise image of a set X in M_ω under $j_{0,\omega}$) and therefore do not seem to be of much interest. Hence, while we will see below that the existence of $U(j)$-representations immediately give weak j-Suslin representations because of the Tower Condition, this fact is not very interesting by itself, and we will have to work considerably harder to obtain j-Suslin representations from $U(j)$-representations (and vice-versa).

Similarly we say that T is a *uniform* (j,κ)*-Suslin representation* for X if the following hold.

1. T is a function on $[\lambda]^{<\omega}$ such that for all $s \in [\lambda]^\omega$, if $T(s)$ is the tree whose nth level is given by $T(s \upharpoonright n)$, then $T(s)$ is a (height ω) tree on $V_\lambda \times \mathcal{F}^\omega_\kappa(j)$.

2. For all $s \in [\lambda]^\omega$ such that s is cofinal in λ, $T(s)$ is a (j,κ)-Suslin representation for X.

We refer the reader to [3] for consequences of the existence of uniform j-Suslin representations. We will use these consequences below in Section 7.

2 Fixed point filter and $U(j)$-representations

In this section we introduce $U(j)$-representations, which were first defined by Woodin in [8]. These representations are similar to weakly homogeneously Suslin representations in the context of $\mathbb{R}$.

We first introduce some terminology. Again fix $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ an elementary embedding with $\operatorname{crit}(j) < \lambda$. We say that a sequence $\vec{a} = \langle a_n | \, n < \omega\rangle$ is *weakly fixed by* j if for all $n < \omega$, $|a_n| < \lambda$, $a_n \subseteq a_{n+1}$ and there exists an m such that $j_{(m)}(a_n) = a_n$.

For S a set of embeddings we let

$$\mathrm{Fix}(S) = \{a | \, \forall k \in S\, (k(a) = a))\}.$$

We then set $\mathcal{F}^k(\kappa, a)$ to be the filter generated by the sets $\mathrm{Fix}(S)$ where $S \in [E^k(\kappa,a)]^\lambda$. Note that these filters are λ^+-complete.

For $\vec{a} \in [L_\kappa(V_{\lambda+1})]^\omega$ weakly fixed by j, we let $\mathcal{A}(\kappa,\vec{a})$ be the set of sequences $\langle A_n | \, n < \omega\rangle$ such that for all $n < \omega$, $A_n \in \mathcal{F}^j(\kappa, a_n)$. We also set for $\vec{A} \in \mathcal{A}^j(\kappa,\vec{a})$, $T^F(\vec{A})$ to be the largest tree T such that any node s of T is such that for all large enough n, $s \in A_n$.

We now proceed to define the set of $U(j)$-measures and $U(j)$-representations.

Definition 3 (Woodin)**.** Let $U(j)$ be the set of $U \in L(V_{\lambda+1})$ such that in $L(V_{\lambda+1})$ the following hold:

1. U is a λ^+-complete ultrafilter.

2. For some $\gamma < \Theta$, $U \in L_\gamma(V_{\lambda+1})$.

3. For all sufficiently large $n < \omega$, $j_{(n)}(U) = U$ and for some $A \in U$,

$$\{a \in A |\, j_{(n)}(a) = a\} = A \cap \mathcal{F}_\Theta(j_{(n)}) \in U.$$

For each ordinal κ, let $\Theta^{L_\kappa(V_{\lambda+1})}$ denote the supremum of the ordinals α such that there is a surjection $\rho : V_{\lambda+1} \to \alpha$ such that $\{(a,b)|\, \rho(a) < \rho(b)\} \in L_\kappa(V_{\lambda+1})$.

The following lemma gives a method for generating lots of $U(j)$-measures by considering the $\mathcal{F}^j(\kappa, a)$ filter on a fine enough partition.

Lemma 4 (Woodin)**.** *Suppose $\kappa < \Theta$, $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$, $a \in L_\kappa(V_{\lambda+1})$ and that $j(\kappa, a) = (\kappa, a)$. Then there is $\delta < crit(j)$ and a partition $\{S_\alpha |\, \alpha < \delta\} \in L(V_{\lambda+1})$ of $L_\kappa(V_{\lambda+1})$ into $\mathcal{F}^j(\kappa, a)$-positive sets such that for each $\alpha < \delta$,*

$$\mathcal{F}^j(\kappa, a) \restriction S_\alpha \in U(j).$$

Proof. First, we have that since $j(\kappa) = \kappa$ that

$$j(E^j(\kappa, a)) = E^j(\kappa, a) \text{ and } j(\mathcal{F}^j(\kappa, a)) = \mathcal{F}^j(\kappa, a).$$

Now we show that there is no sequence $\langle S_\alpha |\, \alpha < \mathrm{crit}\,(j)\rangle \in L(V_{\lambda+1})$ of pairwise disjoint $\mathcal{F}^j(\kappa, a)$-positive sets. This follows since

$$\{a \in L_\kappa(V_{\lambda+1}) |\, j(a) = a\} \in \mathcal{F}^j(\kappa, a),$$

and hence if

$$j(\langle S_\alpha |\, \alpha < \mathrm{crit}\,(j)\rangle) = \langle T_\alpha |\, \alpha < j(\mathrm{crit}\,(j))\rangle\,,$$

then there exists a β such that $\beta \in T_{\mathrm{crit}\,(j)}$ and $j(\beta) = \beta$. But then by elementarity, there exists an $\alpha < \mathrm{crit}\,(j)$ such that $\beta \in S_\alpha$. But then $j(\beta) = \beta \in T_\alpha$, a contradiction.

Now, since $\mathcal{F}^j(\kappa, a)$ is λ^+-complete, there must exists a $\delta < \mathrm{crit}\,(j)$ and a partition $\{S_\alpha |\, \alpha < \delta\} \in L(V_{\lambda+1})$ of $L_\kappa(V_{\lambda+1})$ into $\mathcal{F}^j(\kappa, a)$-positive sets such that for each $\alpha < \delta$, $\mathcal{F}^j(\kappa, a) \restriction S_\alpha$ is an ultrafilter.

For $\alpha < \delta$, let U_α be the ultrafilter given by $\mathcal{F}^j(\kappa, a) \restriction S_\alpha$. We have that U_α is λ^+-complete since $\mathcal{F}^j(\kappa, a)$ is λ^+-complete. Furthermore we have that

$$B_\alpha := \{a \in S_\alpha |\, j(a) = a)\} \in U_\alpha.$$

And hence we have that $j(U_\alpha) = U_\alpha$, since for all $\beta \in B_\alpha$, $\beta \in S_\alpha \iff \beta \in j(S_\alpha)$. So we have that for all $\alpha < \delta$, $U_\alpha \in U(j)$. $\square$

Suppose that $\kappa < \Theta$ and $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$ and $\langle a_i |\, i < \omega\rangle$ is weakly fixed by j. Let $U(j, \kappa, \langle a_i |\, i < \omega\rangle)$ denote the set of $U \in U(j)$ such that there exists $n < \omega$ such that for all $k \in E^j(\kappa, \langle a_i |\, i \leq n\rangle)$,

$$\mathrm{Fix}(\{k\}) \in U.$$

We can now define $U(j)$-representations for subsets of $V_{\lambda+1}$.

Definition 5 (Woodin)**.** Suppose $\kappa < \Theta$, κ is weakly inaccessible in $L(V_{\lambda+1})$, and $\langle a_i |\, i < \omega\rangle \in (L_\kappa(V_{\lambda+1}))^\omega$ is weakly fixed by j.

Suppose that $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$. Then Z is $U(j, \kappa, \langle a_i |\, i < \omega\rangle)$*-representable* if there exists an increasing sequence $\langle \lambda_i |\, i < \omega\rangle$, cofinal in λ and a function

$$\pi : \bigcup \{V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} |\, i < \omega\} \to U(j, \kappa, \langle a_i |\, i < \omega\rangle)$$

such that the following hold:

1. For all $i < \omega$ and $(a, b, i) \in \mathrm{dom}(\pi)$ there exists $A \subseteq (L(V_{\lambda+1}))^i$ such that $A \in \pi(a, b, i)$.
2. For all $i < \omega$ and $(a, b, i) \in \mathrm{dom}(\pi)$, $\pi(a, b, i) \in U(j, \kappa, a_i)$[1].
3. For all $i < \omega$ and $(a, b, i) \in \mathrm{dom}(\pi)$, if $m < i$ then
$$(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m) \in \mathrm{dom}(\pi)$$
and $\pi(a, b, i)$ projects to $\pi(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m)$.
4. For all $x \subseteq V_\lambda$, $x \in Z$ if and only if there exists $y \subseteq V_\lambda$ such that
   (a) for all $m < \omega$, $(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) \in \mathrm{dom}(\pi)$,
   (b) the tower
   $$\langle \pi(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) |\, m < \omega\rangle$$
   is well founded.

For $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$ we say that Z is $U(j)$*-representable* if there exists $(\kappa, \langle a_i |\, i < \omega\rangle)$ such that Z is $U(j, \kappa, \langle a_i |\, i < \omega\rangle)$-representable.

One important property of $U(j)$-representations is a continuous ill-foundedness condition called the Tower Condition which Woodin [8] showed implies that they are closed under complements.

Definition 6 (Woodin)**.** Suppose $A \subseteq U(j)$, $A \in L(V_{\lambda+1})$, and $|A| \leq \lambda$. The *Tower Condition for* A is the following statement: There is a function $F : A \to L(V_{\lambda+1})$ such that the following hold:

1. For all $U \in A$, $F(U) \in U$.
2. Suppose $\langle U_i |\, i < \omega\rangle \in L(V_{\lambda+1})$ and for all $i < \omega$, there exists $Z \in U_i$ such that $Z \subseteq L(V_{\lambda+1})^i$, $U_i \in A$, and U_{i+1} projects to U_i. Then the tower $\langle U_i |\, i < \omega\rangle$ is wellfounded in $L(V_{\lambda+1})$ if and only if there exists a function $f : \omega \to L(V_{\lambda+1})$ such that for all $i < \omega$, $f \restriction i \in F(U_i)$.

[1]This condition is slightly stronger than what is found in the definition of $U(j)$-representations in [8]. This strengthening is convenient for us, and does not change the collection of $U(j)$-representable sets.

The *Tower Condition for* $U(j)$ is the statement that for all $A \subseteq U(j)$ if $A \in L(V_{\lambda+1})$ and $|A| \leq \lambda$ then the Tower Condition holds for A.

Theorem 7. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be elementary such that crit$(j) < \lambda$. Then the following hold:*

1. *Tower Condition for $U(j)$ holds in $L(V_{\lambda+1})$ (C. [4]).*

2. *The set of $U(j)$-representable sets is closed under complements (Woodin [8]).*

Remark 8. By definition of the Tower Condition, if π is a $U(j)$-representation for X and F is a tower function for $\operatorname{rng} \pi$, then we immediately obtain a weak j-Suslin representation for X from F. However, as remarked above, obtaining weak j-Suslin representations is not particularly difficult (and apparently not useful).

3 Closure of j-Suslin representations

We now define a representation which is a natural combination of a j-Suslin and a $U(j)$-representation. Not only is this a stronger representation, but its stronger properties will help below in its propagation throughout $L(V_{\lambda+1})$.

Definition 9. Let $X \subseteq V_{\lambda+1}$. We say that T is a *weakly $(\delta, \vec{a})$-homogeneously (j, κ)-Suslin representation* for X if T is a (j, κ)-Suslin representation for X, and the following hold.

1. $\vec{a} \in [L_\kappa(V_{\lambda+1})]^\omega$ is weakly fixed by j.

2. For all $x \in V_{\lambda+1}$ and $\vec{A} \in \mathcal{A}^j(\delta, \vec{a})$, if $[T_{\hat{x}}] \neq \emptyset$ then
$$[T_{\hat{x}}] \cap [T^F(\vec{A})] \neq \emptyset.$$

We say that X is *weakly homogeneously j-Suslin* if for some $\delta, \vec{a}$ and κ, X is weakly $(\delta, \vec{a})$-homogeneously (j, κ)-Suslin.

The next definition shows a natural method for obtaining a $U(j)$-representation from a weakly homogeneously j-Suslin representation.

Definition 10. First fix $\vec{\lambda}$ which is increasing cofinal in λ. Suppose that $\kappa < \kappa'$ and T is a (j, κ)-Suslin representation for X. Let $\vec{\lambda}$ be increasing cofinal in λ. A $U(j, \kappa', \vec{a})$-representation π^T is *derived from* T if π^T is maximal satisfying the following:

1. For all $(x, y, i) \in V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\}$, $T_x \in \pi^T(x, y, i)$

2. π^T is a $U(j, \kappa', \vec{a})$-representation.

We similarly say that π^T_n is a *partial* $U(j,\kappa',\vec{a})$*-representation derived from* T[2] if π^T_n has domain $\bigcup_{i<n} V_{\lambda_i+1}\times V_{\lambda_i+1}\times\{i\}$ and is maximal[3] such that for all $i<n$ and $(x,y,i)\in V_{\lambda_i+1}\times V_{\lambda_i+1}\times\{i\}$, $T_x\in\pi^T_n(x,y,i)$

Suppose that $\kappa,\kappa'<\delta_1<\delta_2$. Then we define the tree

$$S = MS^{\delta_1,\delta_2}_{\kappa',\vec{a}}(T)$$

as follows. For $x\in\mathcal{W}^{\vec{\lambda}}_n$, $\langle(\pi^T_0,f_0),\dots,(\pi^T_n,f_n)\rangle\in S_x$ if the following hold.

1. π^T_n is a partial $U(j,\kappa',\vec{a})$-representation derived from T and for $i<n$, $\pi^T_i=\pi^T_n\restriction\operatorname{dom}(\pi^T_i)$.

2. $\operatorname{dom}(f_n)=\operatorname{rng}\pi^T_n\restriction V_{\lambda_n+1}\times V_{\lambda_n+1}\times\{n\}$ and $\operatorname{rng} f\subseteq\delta_2$. Also $f\in\mathcal{F}_{\delta_2}(j)$.

3. For all $i<n$ and $x,y\in V_{\lambda_i+1}$ and $x',y'\in V_{\lambda_{i+1}+1}$ which extend x and y respectively, if σ_i is the ultrapower embedding given by computing the ultrapower in $L_{\delta_2}(V_{\lambda+1})$ of $\pi^T_n(x,y,i)$ with functions in $L_{\delta_1}(V_{\lambda+1})$, then

$$\sigma_i(f_i(\pi^T_n(x,y,i)))>f_{i+1}(\pi^T_n(x',y',i+1)).$$

Lemma 11. *Suppose that $X\subseteq V_{\lambda+1}$ is in $L(V_{\lambda+1})$. If X is weakly $(\delta,\vec{a})$-homogeneously (j,κ)-Suslin then X is $U(j,\delta,\vec{a})$-representable.*

Proof. Let T witness that X is weakly $(\delta,\vec{a})$-homogeneously (j,κ)-Suslin. Then any $U(j,\delta,\vec{a})$-representation π^T derived from T must be a $U(j)$-representation for X by definition of weakly $(\delta,\vec{a})$-homogeneously (j,κ)-Suslin. Since such π^T always exist, we have the desired result. □

The following proof is very similar to the proof that $U(j)$-representations are closed under complements from the Tower Condition (see [8], Lemma 128).

Lemma 12. *Assume j witnesses I_0 holds at λ. Suppose that $X\subseteq V_{\lambda+1}$ is a weakly $(\delta,\vec{a})$-homogeneously (j,κ)-Suslin representable in $L_\Theta(V_{\lambda+1})$ by T. Then there is a $\kappa'>\kappa$, $\delta'>\delta$, and $\vec{b}$ such that $V_{\lambda+1}\setminus X$ is a weakly $(\delta',\vec{b})$-homogeneously (j,κ')-Suslin representable in $L(V_{\lambda+1})$.*

Proof. Let $T\in L_\Theta(V_{\lambda+1})$ be a tree which witnesses that X is weakly $(\delta,\vec{a})$-homogeneously (j,κ)-Suslin. Let $\delta_1<\delta_2$ be regular such that $\kappa,\delta<\delta_1<\delta_2<\Theta$, $j(\kappa')=\kappa'$ and

$$L_{\delta_1}(V_{\lambda+1})\prec L_{\delta_2}(V_{\lambda+1})\prec L_\Theta(V_{\lambda+1})$$

(see [8], Lemma 22). Let $S=MS^{\delta_1,\delta_2}_{\delta,\vec{a}}(T)$. Let $\vec{b}$ be defined by $b_i=a_i\cup\{\delta,\delta_1\}$ for $i<\omega$. We want to see that S is a weakly $(\delta_2,\vec{b})$-homogeneously (j,δ_2)-Suslin representation for

[2]Note that π^T_n depends only on T up to its nth level. This fact is important to keep in mind below.

[3]This is the main reason we defined condition (2) for a $U(j)$-representation in the way we did. Otherwise this maximality condition would not make sense.

$V_{\lambda+1} \setminus X$. To see this, let $\vec{A} \in \mathcal{A}^j(\delta_2, \vec{b})$. We first show that if $x \in V_{\lambda+1} \setminus X$ then $S_{\hat{x}} \cap T^F(\vec{A})$ is illfounded. Assume that $x \in V_{\lambda+1} \setminus X$, which implies that $T_{\hat{x}}$ is wellfounded. This implies of course that there exists $\vec{B} \in \mathcal{A}^j(\delta, \vec{a})$ such that $T_{\hat{x}} \cap T^F(\vec{B})$ is wellfounded (in fact every $\vec{B}$ must satisfy this).

Now we have that for all $i < \omega$, since $a_i \cup \{\delta\} \subseteq b_i$, $U(j, \delta, a_i) \subseteq A_i$. Hence there is π^T, a $U(j, \delta, \vec{a})$-representations derived from T such that

$$\left\langle \pi^T \restriction \bigcup_{i<n} V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} | \, n < \omega \right\rangle \in T^F(\vec{A}).$$

Now by definition of being derived from T we have that for all $i < \omega$, and $y \in V_{\lambda_i+1}$,

$$T_{x \cap V_{\lambda_i}} \in \pi^T(x \cap V_{\lambda_i}, y, i).$$

Hence since $T_{\hat{x}}$ is wellfounded we have that for all $y \in V_{\lambda+1}$,

$$\left\langle \pi^T(x \cap V_{\lambda_i}, y \cap V_{\lambda_i}, i) | \, i < \omega \right\rangle$$

is an illfounded tower. Hence by the Tower Condition we can define functions f_i for $i < \omega$ such that $\mathrm{dom}(f_i) = V_{\lambda_i+1}$ and for all $y \in V_{\lambda+1}$ and $i < \omega$,

$$\sigma_i^y(f_i(\pi^T(x \cap V_{\lambda_i}, y \cap V_{\lambda_i}, i))) > f_{i+1}(\pi^T(x \cap V_{\lambda_{i+1}}, y \cap V_{\lambda_{i+1}}, i+1)),$$

where σ_i^y is the ultrapower of $\pi^T(x \cap V_{\lambda_i}, y \cap V_{\lambda_i}, i)$ with functions in $L_{\delta_1}(V_{\lambda+1})$, computed in $L_{\delta_2}(V_{\lambda+1})$. In addition we can find such function f_i such that $\langle f_i | \, i < \omega \rangle \in T^F(\vec{A})$. To see this, note that δ_2 is a regular cardinal in $L(V_{\lambda+1})$ and $L_{\delta_2}(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$, which implies that ordinals of the above ultrapower are below δ_2, and $\bigcap_{i<\omega} A_i \cap \delta_2$ is cofinal in δ_2. Hence

$$\left\langle (\pi_n^T, f_n) | \, n < \omega \right\rangle \in [S_x \cap T^F(\vec{A})],$$

where

$$\pi_n^T = \pi^T \restriction \bigcup_{i<n} V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\}.$$

Hence $S_x \cap T^F(\vec{A})$ is illfounded, which is what we wanted to show.

Now we show that if $S_{\hat{x}} \cap T^F(\vec{A})$ is illfounded then $x \in V_{\lambda+1} \setminus X$. Let

$$\left\langle (\pi_n^T, f_n) | \, n < \omega \right\rangle \in [S_x \cap T^F(\vec{A})].$$

Since the Tower Condition holds (see [4]), we have, as in the proof of Lemma 129 in [8] that for all $y \in V_{\lambda+1}$,

$$\left\langle \pi_n^T(x \cap V_{\lambda_n}, y \cap V_{\lambda_n}, n) | \, n < \omega \right\rangle$$

is an illfounded tower if $(x \cap V_{\lambda_n}, y \cap V_{\lambda_n}, n) \in \mathrm{dom} \pi_n^T$ for all $n < \omega$. By definition of a weakly homogeneously j-Suslin tree, for all $n < \omega$

$$T_{x \cap V_{\lambda_n}} \in \pi_n^T(x \cap V_{\lambda_n}, y \cap V_{\lambda_n}, n).$$

Hence there is $\vec{B} \in \mathcal{A}(\delta, \vec{a})$ such that $T_x \cap T^F(\vec{B})$ is wellfounded. But since T is $(\delta, \vec{a})$-homogeneous, we have that $T_{\hat{x}}$ is wellfounded as well. Hence $x \in V_{\lambda+1} \setminus X$, as we wanted. $\square$

We can also form the tree $MS^{\delta_1,\delta_2}_{\delta,\vec{a}}(T)$ in the case when T is not $(\delta,\vec{a})$-homogeneous. We will need to do this below, and so it is important to know what this tree projects to in this situation. The following lemma gives us this information, and in fact the above proof actually proves this more general fact. It says basically that this tree projects to the points where T is ill founded on a sequence of measure one sets.

Lemma 13. *Suppose that $X \subseteq V_{\lambda+1}$ is (j,κ)-Suslin representable in $L_\Theta(V_{\lambda+1})$ by T. Let $\delta_1 < \delta_2$ be regular such that $\kappa, \delta < \delta_1 < \delta_2 < \Theta$, $j(\kappa') = \kappa'$ and*

$$L_{\delta_1}(V_{\lambda+1}) \prec L_{\delta_2}(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1}).$$

Let $S = MS^{\delta_1,\delta_2}_{\delta,\vec{a}}(T)$ and set $Y = p[S]$. Then $x \in Y$ iff there is $\vec{A} \in \mathcal{A}(\delta,\vec{a})$ such that

$$[T_{\hat{x}}] \cap [T^F(\vec{A})] = \emptyset.$$

We now define another uniformity notion which allows us to preserve being j-Suslin when taking existential quantifications.

Definition 14. Let $\kappa < \Theta$. Suppose that S is a tree on $\mathcal{F}^\omega_\kappa(j)$ and that for all $a \in [S]$, $X_a \subseteq V_{\lambda+1}$ is (j,κ)-Suslin as witnessed by a tree T^a. Furthermore assume that there is a tree T on $\mathcal{F}^\omega_\kappa(j) \times \mathcal{F}^\omega_\kappa(j) \times \mathcal{F}^\omega_\kappa(j)$ such that for all $a \in [S]$, $T^a = T_a$ and for all $\vec{x} \in \mathcal{W}^{\vec{\kappa}}$,

$$\{(a,b) |\, (a,x,b) \in T\} \in \mathcal{F}^\omega_\kappa(j).$$

Then we say that $\langle X_a |\, a \in [S]\rangle$ is *sequentially (j,κ)-Suslin on S as witnessed by T.*

For the rest of this section we fix $\langle \kappa_i |\, i < \omega\rangle$ increasing and cofinal in λ.

Lemma 15. *Let $\kappa < \Theta$. Suppose that S is a tree on $\mathcal{F}^\omega_\kappa(j)$ and that for all $a \in [S]$, $X_a \subseteq V_{\lambda+1}$, and $\langle X_a |\, a \in [S]\rangle$ is sequentially (j,κ)-Suslin as witnessed by T. Then*

$$\{x \in V_{\lambda+1} |\, \exists a \in [S](x \in X_a)\}$$

is (j,κ)-Suslin. Furthermore if S is a tree on $\mathcal{F}^\omega_\kappa(j) \times \mathcal{F}^\omega_\kappa(j)$ and for all $b \in \mathcal{F}^\omega_\kappa(j)$ and $x \in \mathcal{W}^{\vec{\kappa}}$,

$$\{(a,c) |\, ((a,b),x,c) \in T\} \in \mathcal{F}^\omega_\kappa(j)$$

then

$$\left\langle \{x \in V_{\lambda+1} |\, \exists a((a,b) \in [S] \wedge x \in X_{(a,b)})\} |\, b \in \mathcal{F}^\omega_\kappa(j)^\omega \wedge \exists a((a,b) \in [S])\right\rangle,$$

is sequentially (j,κ)-Suslin by a tree T'.

Proof. Let $Y = \{x \in V_{\lambda+1} |\, \exists a \in [S](x \in X_a)\}$. To see that Y is (j,κ)-Suslin, let T witness that $\langle X_a |\, a \in [S]\rangle$ is sequentially (j,κ)-Suslin. Then we let T' be defined by $(x,a,b) \in T'$ iff $(a,x,b) \in T$. The fact that T' is a (j,κ)-Suslin representation for Y follows then by the definition of sequentially Suslin. In particular note that we have for all $\vec{x} \in \mathcal{W}^{\vec{\kappa}}$ that $T'_{\vec{x}} \in \mathcal{F}^\omega_\kappa(j)$.

The last part of the lemma is very similar. $\square$

For T' the tree as defined in the proof, we will say that T' is defined by

$$T'_b = \exists a\, T_{(a,b)}.$$

We will be somewhat sloppy with this notation below, though the meaning will always be clear.

We now define a j-closed game representation for subsets of $V_{\lambda+1}$.

Definition 16. Fix $\kappa < \Theta$ and $\langle \kappa_i | \, i < \omega \rangle$ increasing and cofinal in λ. Suppose that $X \subseteq V_{\lambda+1}$ and for each $x \in V_{\lambda+1}$, G_x is a game where I and II combine to play $\langle x^i | \, i < \omega \rangle$ such that the following hold.

1. For all $i < \omega$, $x^i = \langle x^i_n | \, n < \omega \rangle$ and for all $n < \omega$, $x^i_n \in \mathcal{F}^\omega_\kappa(j)$.

2. II must abide by certain rules, but she always has a legal move at every stage[4].

3. There is a tree T on $V_\lambda \times \mathcal{F}^\omega_\kappa(j)$ such that if $x \in V_{\lambda+1}$, then $\langle x_i | \, i < \omega \rangle$ is a winning run for I of G_x iff either II does not follow the rules or

$$\langle (x_0 \restriction i, \ldots, x_i \restriction i) | \, i < \omega \rangle \in [T_{\hat{x}_{\vec{\kappa}}}].$$

   Also for T^i, the ith level of T, we have that $T^i \in \mathcal{F}^\omega_\kappa(j)$ for all $i < \omega$.

4. $x \in X$ iff I has a quasi-winning strategy in G_x.

We say that G is a j-closed game representation for X as witnessed by T.

We introduce some terminology for j-closed game representations. We define $pp(T)$, called the *tree of partial plays*, as follows. A sequence $(x_0, \ldots, x_n) \in [pp(T)]$ iff $x_0, \ldots, x_n$ is a legal position in the game as determined by T. We let $pp_n(T)$ be the tree of partial plays of length n.

For G_x a j-closed game and γ an ordinal we let $G_x[\gamma]$ have the same rules as G_x in the sense that every position must restrict to a legal position in G_x, but with the additional requirement that II play ordinals $\gamma_0 > \gamma_1 > \cdots$ such that $\gamma > \gamma_0$.

We now proceed to define a sequence of trees based on a j-closed game representation which will eventually lead to a possible j-Suslin representation for the given set X. For κ an ordinal, let $R_\kappa(\gamma)$ be the γth ordinal $\alpha > \kappa$ such that $L_\alpha(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

Fix $X \subseteq V_{\lambda+1}$, a j-closed game representation G and a tree T which witnesses this. For a fixed γ we will define for $\bar{\gamma} \leq \gamma$, $\vec{b}$ and $\vec{c}$ both weakly fixed by j, and trees $\tilde{T}^{\bar{\gamma},\vec{b}}$, $\tilde{W}^{\bar{\gamma},\vec{b}}$, $\tilde{Z}^{\bar{\gamma},\vec{c}}$ and $\tilde{S}^{\bar{\gamma},\vec{c}}$ by induction on $\bar{\gamma}$. The idea is that illfoundedness of $\tilde{T}$ will show that I has a quasi-winning position, and it is I's turn to play, illfoundedness of $\tilde{W}$ will show that II has a quasi-winning position and it is I's turn to play, illfoundedness of $\tilde{Z}$ will show that II has a qusai-winning position and it is II's turn to play, and illfoundedness of $\tilde{S}$ will show that I has a quasi-winning position and it is II's turn to play.

[4] We only assume this for convenience below, as it makes certain trees easier to define.

1. For $u \in [pp(T)]$ such that it is I's turn to play in u, $\tilde{T}^{0,\vec{b}}_u = pp(T)_u$.

2. For $\bar{\gamma} < \gamma$ and $u \in [pp(T)]$ such that it is I's turn to play in u,
$$\tilde{W}^{\bar{\gamma},\vec{b}}_u = MS^{R_\kappa(6\cdot\bar{\gamma}+4),R_\kappa(6\cdot\bar{\gamma}+5)}_{R_\kappa(6\cdot\bar{\gamma}+3),\vec{b}}(\tilde{T}^{\bar{\gamma},\vec{b}}_u).$$

3. For $0 < \bar{\gamma} < \gamma$, $\tilde{Z}^{\bar{\gamma},\vec{c}}$ is defined by
$$\tilde{Z}^{\bar{\gamma},\vec{c}}_u = \exists x\, \exists \beta < \bar{\gamma}\, \exists \vec{b} \trianglerighteq \vec{c}\, \tilde{W}^{\beta,\vec{b}}_{u^\frown\langle x\rangle},$$
where $u \in [pp(T)]$ is such that it is II's turn to play and $u^\frown\langle x\rangle \in [pp(T)]$, and by $\vec{b} \trianglerighteq \vec{c}$ we mean that for all $n < \omega$, $b_n \supseteq c_n$, where both are weakly fixed by j. We put $\tilde{Z}^{0,\vec{c}}_u = \emptyset$.

4. For $0 < \bar{\gamma} < \gamma$ and $u \in [pp(T)]$ such that it is II's turn to play in u,
$$\tilde{S}^{\bar{\gamma},\vec{c}}_u = MS^{R_\kappa(6\cdot\bar{\gamma}+1),R_\kappa(6\cdot\bar{\gamma}+2)}_{R_\kappa(6\cdot\bar{\gamma}),\vec{c}}(\tilde{Z}^{\bar{\gamma},\vec{c}}_u).$$

5. For $0 < \bar{\gamma} < \gamma$, $\tilde{T}^{\bar{\gamma},\vec{b}}$ is defined by[5]
$$\tilde{T}^{\bar{\gamma},\vec{b}}_u = \exists x\, \exists \vec{c}\, \tilde{S}^{\bar{\gamma},\vec{c}}_{u^\frown\langle x\rangle},$$
where $u \in [pp(T)]$ is such that it is I's turn to play and $u^\frown\langle x\rangle \in [pp(T)]$.

We now introduce a game which is intimately related to the above definition. Given a j-closed game representation G for X, the corresponding measure game $G^*_x[\kappa, \bar{\gamma}, \vec{a}, \vec{A}^*]$ is the following.
$$\begin{array}{lllll} I: & x_0, \vec{A}^0, \vec{c}^0 & & x_2, \vec{A}^1, \vec{c}^1 & & \cdots \\ II: & & x_1, \bar{\gamma}_0, \vec{a}^0, \vec{B}^0 & & x_3, \bar{\gamma}_1, \vec{a}^1, \vec{B}^1 & \cdots \end{array}.$$
With the following rules.

1. The sequence $x_0, x_1, x_2, \ldots$ must abide by the rules of G_x.

2. For all $n < \omega$, $A^*_n \supseteq A^0_n \supseteq B^0_n \supseteq A^1_n \supseteq B^1_n \supseteq \cdots$.

3. $\bar{\gamma} > \bar{\gamma}_0 > \bar{\gamma}_1 > \cdots$.

4. $\vec{A}^* \in \mathcal{A}(R_\kappa(6\cdot\bar{\gamma}+3), \vec{a})$ and $\vec{A}^0 \in \mathcal{A}(R_\kappa(6\cdot\bar{\gamma}), \vec{a})$.

5. For all $i \geq 0$, $\vec{B}^i \in \mathcal{A}(R_\kappa(6\cdot\bar{\gamma}^i+3), \vec{a}^i)$ and $\vec{A}^{i+1} \in \mathcal{A}(R_\kappa(6\cdot\bar{\gamma}^i), \vec{a}^i)$.

[5] Note that in the game below I might as well play $\vec{c}$ such that $\vec{b} \trianglelefteq \vec{c}$, and that it why the definition of $\tilde{T}^{\bar{\gamma},\vec{b}}_u$ does not depend on $\vec{b}$.

6. For all $i \geq 0$,
$$x_{2i}, \vec{c}^i \in [T^F(\vec{A}^*)] \cap \bigcap_{n<i} [T^F(\vec{A}^n)] \cap \bigcap_{n<i} [T^F(\vec{B}^n)]$$
and
$$x_{2i+1}, \vec{a}^i, \langle \bar{\gamma}_i, \bar{\gamma}_i, \ldots \rangle \in [T^F(\vec{A}^*)] \cap \bigcap_{n\leq i} [T^F(\vec{A}^n)] \cap \bigcap_{n<i} [T^F(\vec{B}^n)].$$

7. For all $i < \omega$ and $n < \omega$, $\vec{c}^i \in (L_\kappa(V_{\lambda+1}))^\omega$ and $c_n^i \cup \{\kappa\} \subseteq a_n^i$.

The first one to violate the rules loses.

The main point is the following lemma which gives an equivalence between this game and the above trees. We need some terminology to state this lemma. We say that $(u^*, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*)$ is $(\bar{\gamma}, \vec{b}, \vec{c}, \vec{A}^*, x)$-compatible with u for $u \in [pp(T)]$ if the following hold.

1. u^* is a legal play of $G_x^*[\kappa, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*]$, and the length of u^* is the same as the length of u.

2. We have $x_i[u^*] = x_i[u]$ for all $i < |u|$.

3. The minimum of $\bar{\gamma}^*$ and any $\bar{\gamma}_i$ played in u^* is $\bar{\gamma}$. If the minimum is $\bar{\gamma}^*$, then $\vec{a}^* = \vec{b}$ and $\vec{B}^* = \vec{A}^*$, and otherwise if i is such that $\bar{\gamma}_i$ is minimal, then $\vec{a}^i[u^*] = \vec{b}$ and $\vec{B}^i[u^*] = \vec{A}^*$, and for $\vec{c}'$ in the last move played by I, $\vec{c}' = \vec{c}$.

This terminology basically relates plays in G_x^* to plays in G_x in the natural way.

Lemma 17. *For any $\bar{\gamma}$, $\vec{a}$ weakly fixed by j, $\vec{A}^* \in \mathcal{A}[\bar{\gamma}, \vec{a}]$, $X \subseteq V_{\lambda+1}$, G a j-closed game representation for X as witnessed by T, and $x \in V_{\lambda+1}$, we have the following:*

1. *I has a quasi-winning strategy in $G_x^*[\kappa, \bar{\gamma}, \vec{a}, \vec{A}^*]$ iff*
$$x \in p[\tilde{T}_\emptyset^{\bar{\gamma}, \vec{a}} \cap T^F(\vec{A}^*)].$$

2. *if $u \in [pp(T)]$ is such that it is I's turn to play in u, then*
$$x \in p[\tilde{T}_u^{\bar{\gamma}, \vec{a}} \cap T^F(\vec{A}^*)]$$
iff for any $(u^*, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*)$ which for some $\vec{c}$ is $(\bar{\gamma}, \vec{a}, \vec{c}, \vec{A}^*, x)$-compatible with u, u^* is a quasi-winning position for I in $G_x^*[\kappa, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*]$.

3. *if $u \in [pp(T)]$ is such that it is II's turn to play in u and $\vec{c}$ is weakly fixed by j, then*
$$x \in p[\tilde{Z}_u^{\bar{\gamma}, \vec{c}} \cap T^F(\vec{A}^*)]$$
iff for any $(u^*, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*)$, $(\bar{\gamma}, \vec{a}, \vec{c}, \vec{A}^*, x)$-compatible with u, u^* is a quasi-winning position for II in $G_x^*[\kappa, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*]$.

Proof. We prove this by induction on $\bar{\gamma}$, proving $(3)_{\bar{\gamma}}$ and then $(2)_{\bar{\gamma}}$.

The base case is that $\bar{\gamma} = 0$ and $u \in [pp(T)]$ is such that it is II's turn to play and $\vec{c}$ is weakly fixed by j. In that case II is not in a winning position for any such u^* and $\tilde{Z}_u^{0,\vec{c}} = \emptyset$.

Now assume $(2)_{\bar{\gamma}'}$ and $(3)_{\bar{\gamma}'}$ hold for all $\bar{\gamma}' < \bar{\gamma}$. We prove $(3)_{\bar{\gamma}}$. Suppose that $u \in [pp(T)]$ and that it is II's turn to play in u, $\vec{c}$ is weakly fixed by j, and $\vec{A}^* \in \mathcal{A}[\bar{\gamma}, \vec{a}]$. Suppose that

$$x \in p[\tilde{Z}_u^{\bar{\gamma},\vec{c}} \cap T^F(\vec{A}^*)]$$

and $(u^*, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*)$ are $(\bar{\gamma}, \vec{a}, \vec{c}, \vec{A}^*, x)$-compatible with u. Then by Lemmas 15 and 13 there are $u' = u^\frown \langle x' \rangle$, $\beta < \bar{\gamma}$, $\vec{b}$, and $\vec{B}'$ such that

1. $\hat{x}', \vec{b}, \langle \beta, \beta, \dots, \rangle \in [T^F(\vec{A}^*)]$,
2. $\vec{B}' \in \mathcal{A}[R_\kappa(6 \cdot \beta + 3), \vec{b})]$,
3. $u^{**} := u^{*\frown} \left\langle (x', \vec{b}, \beta, \vec{B}') \right\rangle$ is a legal move in $G_x^*[\kappa, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*]$
4. $(\tilde{T}_{u'}^{\beta,\vec{b}} \cap T^F(\vec{B}'))_{\hat{x}}$ is well-founded.

Hence by induction applying $(2)_\beta$, since $(u^{**}, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*)$ is $(\beta, \vec{b}, \vec{c}, \vec{B}')$-compatible with u', we have that u^{**} is a quasi-winning position for II in $G_x^*[\kappa, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*]$. So u^* is a quasi-winning position for II.

In the other direction, let $(u^*, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*)$ be $(\bar{\gamma}, \vec{a}, \vec{c}, \vec{A}^*, x)$-compatible with u and assume that u^* is a quasi-winning position for II in $G_x^*[\kappa, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*]$. Then by the definition of a quasi-winning position we have $\tilde{u}$ and $u' = u^\frown \langle x' \rangle$ such that

1. $\tilde{u} = u^{*\frown} \left\langle x', \vec{b}, \beta, \vec{B}' \right\rangle$ is a quasi-winning position in $G_x^*[\kappa, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*]$ for II.
2. $x', \vec{b}, \langle \beta, \beta, \dots, \rangle \in [T^F(\vec{A}^*)]$,
3. $\vec{B}' \in \mathcal{A}[R_\kappa(6 \cdot \beta + 3), \vec{b}]$,

Hence applying $(2)_\beta$ by induction, since $(\tilde{u}, \bar{\gamma}^*, \vec{a}^*, \vec{B}^*)$ is $(\beta, \vec{b}, \vec{c}, \vec{B}')$-compatible with u' we have $(\tilde{T}_{u'}^{\beta,\vec{b}} \cap \vec{B}')_{\hat{x}}$ is well-founded. But then by Lemma 13, $\tilde{W}_{u'}^{\beta,\vec{b}}$ is ill-founded. So by Lemma 15, $(\tilde{Z}_u^{\bar{\gamma},\vec{c}} \cap T^F(\vec{A}^*))_x$ is ill-founded, which is what we wanted.

Showing that $(2)_{\bar{\gamma}}$ holds is very similar, using Lemmas 15 and 13, and we leave this to the reader. □

Definition 18. We say that a j-closed game representation G_x for $X \subseteq V_{\lambda+1}$ is *normal* if there are $\bar{\gamma}$ and $\vec{a}$ such that for all $x \in V_{\lambda+1}$ and $\vec{A}^* \in \mathcal{A}(\bar{\gamma}, \vec{a})$, I has a quasi-winning strategy in G_x iff I has a quasi-winning strategy in $G_x^*[\kappa, \bar{\gamma}, \vec{a}, \vec{A}^*]$.

Theorem 19. *Suppose that $X \subseteq V_{\lambda+1}$ has a normal j-closed game representation G_x. Then X has a weakly homogeneous j-Suslin representation. In fact for $(\bar{\gamma}, \vec{a})$ witnessing that G_x is normal, we have that $X = p[\tilde{T}_\emptyset^{\bar{\gamma},\vec{a}}]$ and that $\tilde{T}_\emptyset^{\bar{\gamma},\vec{a}}$ is a weakly $(\bar{\gamma}, \vec{a})$-homogeneous $(j, \bar{\gamma})$-Suslin representation for X.*

Proof. The theorem follows immediately from the previous lemma and the definition of a normal j-closed game representation. □

4 j-Closed game representations

We now describe how to obtain a j-closed game representation for an $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$. The argument we give is very similar to the corresponding argument in [7], although our weaker version of the closed game representation allows us to prove that every subset in $L(V_{\lambda+1})$ has such a representation. We fix such an X and let α be good such that X is definable over $L_\alpha(V_{\lambda+1})$ from $a_X \in V_{\lambda+1}$ by a formula ϕ_X.

We define the game G_x^α. We want player I to describe a size λ model of

$$V = L(V_{\lambda+1}) + \phi_X[a_X, x]$$

which contains all the reals played by player II, while using ordinals less than α to prove that this model is wellfounded. In addition, I will have to prove to II that its model is actually a by winning rank games against II playing ordinals below α.

Player I describes his model in the language $\mathcal{L}_\lambda$, which has relations $\in, =$ and constant symbols $\underline{\mathrm{x}}_i$ for $i < \lambda$ and constants $\dot{a}$ for all $a \in V_\lambda$ and a constant $\dot{\lambda}$. Define $\mathcal{L}_\kappa$ to be the same as $\mathcal{L}_\lambda$ but restricting to constants $\underline{\mathrm{x}}_i$ and $\dot{a}$ for $i < \kappa$ and $a \in V_\kappa$. The constant symbols $\underline{\mathrm{x}}_i$ represent the ith element of $V_{\lambda+1}$ played x_i during a run of G_x^β. Fix maps

$$m^*, n^* : \{\theta |\, \theta \text{ is an } \mathcal{L}_\lambda\text{-formula}\} \to \lambda - 2$$

which are one to one, have disjoint ranges, $\kappa - \operatorname{rng} m^* - \operatorname{rng} n^*$ has size κ for every cardinal $\kappa \leq \lambda$, and are such that whenever $\underline{\mathrm{x}}_i$ occurs in θ, $i < \min(m(\theta), n(\theta))$. Also, for all cardinals $\kappa \leq \lambda$,

$$(m^*)^{-1}[\kappa] = (n^*)^{-1}[\kappa] = \{\theta |\, \theta \text{ is an } \mathcal{L}_\kappa\text{-formula}\}.$$

Player I's description must extend the $\mathcal{L}$-theory T with the following axioms:

(1) Extensionality

(2) $V = L(V_{\dot{\lambda}+1})$

$(3)_\phi$ $\exists v\, \phi(v) \to \exists v\, (\phi(v) \wedge \forall u \in v \neg\phi(u))$ (for $\phi \in \mathcal{L}_\lambda$).

$(4)_i$ $\underline{\mathrm{x}}_i \in V_{\dot{\lambda}+1}$ (for $i < \lambda$).

$(5)_{\phi,a_1,\ldots,a_n}$ $V_{\dot{\lambda}} \models \phi[\dot{a}_1, \ldots, \dot{a}_n]$ (where ϕ and $(a_1, \ldots, a_n) \in V_\lambda^n$ are such that $V_\lambda \models \phi[a_1, \ldots, a_n]$).

$(6)_\phi$ $\exists v\, \phi(v) \to \exists v\, \exists F\, (\phi(v) \wedge \theta_0(F, \underline{\mathrm{x}}_{m^*(\phi)}, v))$ (for $\phi \in \mathcal{L}_\lambda$).

$(7)_\phi$ $\exists v(\phi(v) \wedge v \in V_{\dot{\lambda}+1}) \to \phi(\underline{\mathrm{x}}_{n^*(\phi)})$ (for $\phi \in \mathcal{L}_\lambda$).

(8) $\phi_X[\underline{\mathrm{x}}_0, \underline{\mathrm{x}}_1]$.

$(9)_a$ $\dot{a} \in V_{\dot{\lambda}}$ (for $a \in V_\lambda$).

Here we define θ_0 as follows: for f_γ the usual uniformily definable surjective maps

$$f_\gamma : [\gamma]^{<\omega} \times V_{\lambda+1} \to L_\gamma(V_{\lambda+1})$$

we have $\phi_0(v_0, v_1, v_2)$ is a formula describing the graph of f_γ over $L_\gamma(V_{\lambda+1})$ for all γ.

First fix $\langle \kappa_i |\, i < \omega\rangle$ an increasing cofinal sequence of ordinals below λ. A run of G^α_x has the form

$$\begin{array}{cc}
I & II \\
\langle i_n, x_n, \eta_n |\, n < \kappa_0\rangle & x_{\kappa_0}, \beta^0_0 \\
\langle i_n, x_n, \eta_n |\, \kappa_0 < n < \kappa_1\rangle, s^0_0, & x_{\kappa_1}, \beta^0_1, \beta^1_0 \\
\langle i_n, x_n, \eta_n |\, \kappa_1 < n < \kappa_2\rangle, s^0_1, s^1_0 & x_{\kappa_2}, \beta^0_2, \beta^1_1, \beta^2_0 \\
\langle i_n, x_n, \eta_n |\, \kappa_2 \le n < \kappa_3\rangle, s^0_2, s^1_1, s^2_0 & x_{\kappa_3}, \beta^0_3, \beta^1_2, \beta^2_1, \beta^3_0 \\
\vdots & \vdots
\end{array}$$

where for all $k < \lambda$, $i_k \in \{0, 1\}$, $x_k \in V_{\lambda+1}$, and $\eta_k < \alpha$. If u is a position in the game, then we let

$$T^*(u) = \{\theta |\, \theta \text{ is a sentence of } \mathcal{L}_\lambda \text{ and } i_{n^*(\theta)}(u) = 0\}.$$

If p is a full run of G^β_x, then we set

$$T^*(p) = \bigcup_{n<\omega} T^*(p \restriction n).$$

For p a full run of G^α_x, p is winning for player I iff

1. $x_0 = x$, $x_1 = a_X$.
2. $T^*(p)$ is a complete, consistent extension of T such that for all $i < \lambda$ and $a \in V_\lambda$

$$\text{`}\dot{a} \in \underline{\mathrm{x}}_i\text{'} \in T^*(p) \iff a \in x_i$$

3. If ϕ and ψ are $\mathcal{L}_\lambda$-formulas of one free variable and

$$\text{`}\iota v\, \phi(v) \in \mathrm{Ord} \wedge \iota v\, \psi(v) \in \mathrm{Ord}\text{'} \in T^*(p)$$

   then $'\iota v\, \phi(v) \le \iota v\, \psi(v)' \in T^*(p)$ iff $\eta_{n^*(\phi)} \le \eta_{n^*(\psi)}$.

4. If ϕ is an $\mathcal{L}_\lambda$-formula of one free variable and

$$\text{`}\iota v\, \phi(v) \in \mathrm{Ord} \wedge \iota v\, \phi(v) < \dot{\lambda}\text{'} \in T^*(p)$$

   then $\eta_{n^*(\phi)} < \lambda$.

5. If ϕ is the $\mathcal{L}_\lambda$-formula of one free variable $\phi(v) =$ '$v = \dot{\alpha}$' for some $\alpha \leq \lambda$ then $\eta_{n^*(\phi)} = \alpha$.

6. For all n, m such that for all $m' < m$ $\beta^n_{m'} \neq 0$, $\beta^n_0 > \beta^n_1 > \cdots > \beta^n_m$ and
$$\eta_{s^n_0} > \eta_{s^n_1} > \cdots > \eta_{s^n_m}.$$
Also for all $i \leq m$, there is ϕ such that $n^*(\phi) = s^n_i$ and '$\iota v \phi(v) \in \mathrm{Ord}$' $\in T^*(p)$.

7. For all $n_1 < n_2$ and m such that for all $m' < m$
$$\beta^{n_1}_{m'} = \beta^{n_2}_{m'} \neq 0$$
then for all $m' < m$
$$s^{n_1}_{m'} = s^{n_2}_{m'}.$$

Here we use $\iota v\, \phi(v)$ to mean 'the unique v such that $\phi(v)$ holds'.

Lemma 20. *Let $X \subseteq V_{\lambda+1}$ and let α be good such that X is definable over $L_\alpha(V_{\lambda+1})$ from $a_X \in V_{\lambda+1}$ by a formula ϕ_X. Then for G^α_x as defined above, G^α_x is a j-closed game representation for X.*

Proof. We first claim that if $x \in X$ then I has a quasi-winning strategy in G^α_x. The strategy is as follows. I chooses elementary substructures
$$M_0 \prec M_1 \prec \cdots \prec L_\alpha(V_{\lambda+1})$$
as the game progresses such that the following hold:

1. $x, a_X \in M_0$.

2. For all $i < \omega$, $|M_i| = \kappa_i$ and $V_{\kappa_i} \subseteq M_i$.

3. If u is a play in the game of length $2n$ such that it is I's turn to play then I chooses M_n such that $u \in M_n$.

I then plays in the obvious way using the information given by the M_n. So for instance $\langle x_\xi | \, \xi < \kappa_i \rangle$ is an enumeration of $M_n \cap V_{\lambda+1}$, $\langle i_\gamma | \, \gamma < \kappa_0 \rangle$ codes the theory of M_n with parameters in $V_{\kappa_n} \cup V^{M_n}_{\lambda+1} \cup \{\lambda\}$, and η_γ is exactly the ordinal defined from the γth formula in M_n. And we let $s^n_m = \gamma$ where γ is such that if $n^*(\gamma) = \theta$ then the least ordinal satisfying θ is β^n_m (keeping in mind that since $u \in M_n$, $\beta^n_m \in M_n$). Clearly this quasi-strategy works for I.

Now we show that if $x \in V_{\lambda+1}$ is such that I has a quasi-winning strategy in G^α_x then $x \in X$. To see this, let p be a generic run of G^α_x. Then $T^*(p)$ is a complete, consistent extension of T. Let $\mathfrak{B} \models T^*(p)$ and set $\mathfrak{A}$ to be the substructure of $\mathfrak{B}$ with universe
$$|\mathfrak{A}| = \{b \in |\mathfrak{B}| : \exists \alpha < \lambda \, \exists \psi (\psi \text{ is a formula with no constant symbols but } \underline{x}_\alpha \text{ and } b = \iota v(\mathfrak{B} \models \psi[v])\}.$$

Then $\mathfrak{A} \prec \mathfrak{B}$ by the properties of T. Furthermore we define a function f as follows. If $b \in \mathrm{Ord}^{\mathfrak{A}}$ and $b = \iota v(\mathfrak{A} \models \psi[v])$ then let $f(b) = \eta_{n^*(\psi)}$. The map $f : \mathrm{Ord}^{\mathfrak{A}} \to \alpha$ is well-defined and order-preserving because of our requirements on I in the game. Similarly we must have $f(\dot{\lambda}^{\mathfrak{A}}) = \lambda$ by conditions of the game. Hence we can assume that $|\mathfrak{A}| = J_\gamma(V^{\mathfrak{A}}_{\dot{\lambda}+1})$ for some γ. Furthermore since f is a function into α we have that $\gamma \leq \alpha$.

Now we want to see that $V_\lambda = V^{\mathfrak{A}}_{\dot{\lambda}}$. To see this we prove by induction that for all $\kappa \leq \lambda$, $V_\kappa = V^{\mathfrak{A}}_{\dot{\kappa}}$. The base and limit cases are obvious (note that we must have $\dot{\kappa}^{\mathfrak{A}} = \kappa$ by the rules of our game). Now assume that $V_\kappa = V^{\mathfrak{A}}_{\dot{\kappa}}$ and let us show that $V_{\kappa+1} = V^{\mathfrak{A}}_{\dot{\kappa}+1}$. Obviously if $a \in V_{\kappa+1}$ then by induction $\dot{a}^{\mathfrak{A}} = a \in V^{\mathfrak{A}}_{\dot{\kappa}+1}$. Now let $b \in V^{\mathfrak{A}}_{\dot{\kappa}+1}$. We have that $b \in V^{\mathfrak{A}}_{\dot{\lambda}+1}$ and hence by the way we chose $\mathfrak{A}$, there is some $\alpha < \lambda$ such that $\underline{x}^{\mathfrak{A}}_\alpha = b$. Let $n < \omega$ be large enough so that $\kappa_n > \kappa, \alpha$. We then have that by the rules of the game that $p \upharpoonright 2n$ determines the theory of $\mathfrak{A}$ with parameters in

$$\{\underline{x}_\alpha\} \cup \{\dot{c} |\, c \in V_\kappa\}.$$

But this implies that $b \in V_{\kappa+1}$ since $\dot{c}^{\mathfrak{A}} = c$ for all $c \in V_\kappa$. So we have $V_{\kappa+1} = V^{\mathfrak{A}}_{\dot{\kappa}+1}$.

Now we have that by definition of $\mathfrak{A}$ and the rules of our game (in particular rule (7)) that

$$V^{\mathfrak{A}}_{\dot{\lambda}+1} = {V_{\lambda+1}}^{\mathfrak{A}} = \{\underline{x}^{\mathfrak{A}}_\alpha |\, \alpha < \lambda\} = \{x_\alpha |\, \alpha < \lambda\}.$$

On the other hand, since p was a generic run of G^α_x, and II can play whatever elements of $V_{\lambda+1}$ she wants to, we have that ${V_{\lambda+1}}^{\mathfrak{A}} = V_{\lambda+1}$. And hence $|\mathfrak{A}| = J_\gamma(V_{\lambda+1})$ for some $\gamma \leq \alpha$.

The final fact to see is that $\gamma \geq \alpha$ so in fact $\gamma = \alpha$. To see this, suppose that $\gamma < \alpha$. By genericity there is some n_0 such that $\beta^{n_0}_0 = \gamma$. Let $s^{n_0}_0$ be the corresponding response by I. Let $\gamma_0 = \iota v \phi(v)$ where ϕ is such that $n^*(\phi) = s^{n_0}_0$. Now by genericity there is some n_1 such that $\beta^{n_1}_0 = \gamma$ and $\beta^{n_1}_1 = \gamma_1$. By the rules of the game I's first response $s^{n_1}_0 = s^{n_0}_0$. Let $s^{n_1}_1$ be I's corresponding second response and choose γ_2 similarly. In this way choose $\gamma > \gamma_1 > \gamma_2 > \cdots > \gamma_{n_m} = 0$ and $s^{n_0}_0, s^{n_1}_1, \ldots s^{n_m}_m$. We have that the ordinal corresponding to $s^{n_m}_m$ must be below γ_{n_m} by induction. But $\gamma_{n_m} = 0$, so this is a contradiction.

Hence $\gamma = \alpha$ and $L_\alpha(V_{\lambda+1}) \models \phi_X[x, a_X]$ and hence $x \in X$ as we wanted.

Finally, finding a tree T which witnesses that G_x is a j-closed game representation and verifying the other requirements is fairly routine, and we leave this to the reader. □

5 The rank game for fixed point measures

We now recall results from [2] which we need in order to show that our j-closed game representation is in fact normal.

Definition 21. Suppose $\gamma < \Theta^{L(V_{\lambda+1})}$, $\gamma \leq \Theta^{L_\gamma(V_{\lambda+1})}$, and $\langle a_i |\, i < \omega\rangle \in (L_\gamma(V_{\lambda+1}))^\omega$ is weakly fixed by j. Then let $G(j, \gamma, \langle a_i |\, i < \omega\rangle)$ denote the following game. A typical run of the game is as follows:

$$\begin{array}{lccccc} I: & \gamma_0, \langle b^0_m |\, m < \omega\rangle & & \gamma_1, \langle b^1_m |\, m < \omega\rangle & & \cdots \\ II: & & \mathcal{E}_0 & & \mathcal{E}_1 & \cdots \end{array}$$

The rules of the game are as follows.

1. $\mathcal{E}_i \subseteq E^j(\gamma_i)$, $|\mathcal{E}_i| \leq \lambda$, and for each $k \in \mathcal{E}_i$ there exists $m < \omega$ such that $k(b^i_m) = b^i_m$.
2. $\gamma_0 = \gamma$, and for each $i < \omega$, $\gamma_{i+1} < \gamma_i$ and there exists $m < \omega$ such that for all $k \in \mathcal{E}_i$
$$k(b^i_m) = b^i_m \Rightarrow k(\gamma_{i+1}) = \gamma_{i+1}.$$
3. for all $i < \omega$, $\gamma_i \leq \Theta^{L_{\gamma_i}(V_{\lambda+1})}$,
4. $\langle b^0_m : m < \omega\rangle = \langle a_m : m < \omega\rangle$.
5. for all $m < \omega$, $b^i_m \subseteq b^i_{m+1} \subseteq \gamma_i$ and $|b^i_m| < \lambda$
6. for all $i < \omega$ and $m < \omega$ there exists $m^* < \omega$ such that for all $k \in \mathcal{E}_i$
$$k(b^i_{m^*}) = b^i_{m^*} \Rightarrow k(b^{i+1}_m) = b^{i+1}_m.$$

The first one to violate the rules loses.

Of course II always wins this game, but we are interested in the rank of this game, which we define as follows.

Definition 22. Let $G_\delta(j, \gamma, \langle a_i | \, i < \omega\rangle)$ have the same definition as $G(j, \gamma, \langle a_i | \, i < \omega\rangle)$ except that II must also play ordinals $\delta_0 > \delta_1 > \cdots$ such that $\delta_0 < \delta$. If δ is least such that II has a quasi-winning strategy in $G_\delta(j, \gamma, \langle a_i | \, i < \omega\rangle)$, then we set $\delta = \mathrm{rank}(j, \gamma, \langle a_i | \, i < \omega\rangle)$.

The following was proved in [2].

Theorem 23. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an I_0 embedding. Fix $\kappa < \Theta$ good in $L(V_{\lambda+1})$. Then there exists $\delta \geq \kappa$ and $\vec{a}$ such that rank$(j, \delta, \vec{a}) \geq \kappa$.*

By the proof of this theorem we have the following.

Theorem 24. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an I_0 embedding. Then for cofinally many good $\kappa < \Theta$ there is $\vec{a}$ such that rank$(j, \kappa, \vec{a}) = \kappa$.*

We first introduce some terminology.

Definition 25. Suppose that $\vec{a}$ and $\vec{b}$ are both weakly fixed by j. We put $\vec{a} \trianglelefteq \vec{b}$ if for all $i < \omega$, there is an $n < \omega$ such that $a_i \subseteq b_n$.

The following lemma is immediate.

Lemma 26. *Suppose that $\langle \vec{a}^\alpha | \, \alpha < \lambda\rangle$ is such that for all $\alpha < \lambda$, $\vec{a}^\alpha$ is weakly fixed by j. Then there is $\vec{b}$ such that for all $\alpha < \lambda$, $\vec{a}^\alpha \trianglelefteq \vec{b}$.*

Lemma 27. *Let $X \subseteq V_{\lambda+1}$ and let κ be good such that X is definable over $L_\kappa(V_{\lambda+1})$ from $a_X \in V_{\lambda+1}$ by a formula ϕ_X and for some $\vec{a}$, rank$(j, \kappa, \vec{a}) = \kappa$. Then for G^α_x as defined in the previous section, G^α_x is a normal j-closed game representation for X.*

Proof. Fix $\langle \kappa_i | \, i < \omega \rangle$ increasing cofinal in λ. Let κ, $\vec{a}$, a_X and ϕ_X be as in the hypothesis. First note that since there is no cofinal function $f : V_{\lambda+1} \to \Theta$ in $L(V_{\lambda+1})$, if $x \in X$ is such that II has a quasi-winning strategy in G^{κ}_x, then there is a $\gamma < \Theta$ such that II has a quasi-winning strategy in $G^{\kappa}_x[\gamma]$, the clocked game. And hence, for the same reason, for large enough $\gamma < \Theta$ we have that for any $x \in V_{\lambda+1}$, II has a quasi-winning strategy in $G^{\kappa}_x[\gamma]$ iff II has a quasi-winning strategy in G^{κ}_x. Fix such a $\gamma < \Theta$ which is good, and $\vec{a}$ such that $\gamma > \kappa$, $\mathrm{rank}(j, \gamma, \vec{a}) = \gamma$, $\kappa \in a_0$, and

$$\mathrm{rank}(j, \kappa, \langle a_i \cap \kappa | \, i < \omega \rangle = \kappa.$$

We can find such γ and $\vec{a}$ by the proof of Theorem 23. Also assume that for all $i < \omega$, $\kappa_i \in a_i$.

We claim that for any $\vec{A}^* \in \mathcal{A}^j(\gamma, \vec{a})$, $x \in X$ iff I has a quasi-winning strategy in $G^*_x[\kappa, \gamma, \vec{a}, \vec{A}^*]$.

First suppose that $x \in X$, so I has a quasi-winning strategy in $G^{\kappa}_x[\gamma]$. We describe a quasi[6]-winning strategy for I in the corresponding measure game $G^*_x[\kappa, \gamma, \vec{a}, \vec{A}^*]$. First note that since $i_n, x_n \in V_{\lambda+1}$ for all $n < \lambda$, and $s^n_m \in \lambda$ for all $n, m < \omega$, we only have to describe how to play the ordinals η_n, since every other aspect of I's strategy can remain the same. This follows since for all $i < \omega$, $\kappa_i \in a_i$.

We have I play according to the following strategy. We keep track along the way of a play u' of the game $G^{\kappa}_x[\gamma]$ which we are playing, and a modified version u of this game $G^{\kappa}_x[\gamma]$ which is the subplay of the play u^* of the game $G^*_x[\kappa, \gamma, \vec{a}, \vec{A}^*]$, which we are actually interested in playing. The basic idea is that once II makes a move, we extend u' by that move and see what I's strategy says in the $G^{\kappa}_x[\gamma]$ game. Then we translate I's move to a move extending u, and then use that move to extend u^*, along with an appropriate choice of $\vec{c}$.

We define our strategy by induction on the length of the play u^* of $G^*_x[\kappa, \gamma, \vec{a}, \vec{A}^*]$. Suppose that $i < \omega$ and u^* has length $2i$, so it is I's turn to play. Assume the following hold.

1. u' and u are plays of the game $G^{\kappa}_x[\gamma]$ of length $2i$ such that II's moves are the same in each play and are the restriction of II's moves in u^* to $G^{\kappa}_x[\gamma]$.

2. u' is a winning position for I in $G^{\kappa}_x[\gamma]$.

3. The i, x, s components of u', u, u^* are all the same.

4. There is an order-preserving bijection f_{i-1},

$$f_{i-1} : \{\eta_\alpha(u') | \, \alpha < \kappa_{i-1}\} \cup \{\kappa\} \to \{\eta_\alpha(u) | \, \alpha < \kappa_{i-1}\} \cup \{\kappa\}$$

   such that for all $\beta \in \mathrm{dom} f_{i-1}$,

$$\mathrm{rank}(j, f_{i-1}(\beta), \left\langle c^{i-1}_n \cap f_{i-1}(\beta) | \, n < \omega \right\rangle) \geq \beta$$

[6]Below we will just say ‘strategy’, though we mean ‘quasi-strategy’.

and $f_{i-1}(\beta) \in \bigcup_n c_n^{i-1}$. Here we take $\vec{c}^{-1} = \vec{a}$. Also assume that for all $\beta \in \mathrm{dom} f_{i-1}$, $f_{i-1}(\beta)$ is the least $\beta^* \in \bigcup_n B_n^{i-2}$ such that for some $\vec{d} \in T^F(\vec{B}^{i-2})$,

$$\mathrm{rank}(j, \beta^*, \vec{d}) \geq \beta.$$

Here $\vec{B}^{-2} = \vec{B}^{-1} = \vec{A}^*$.

Now, since u' is winning for I, let $\bar{u}'$ extend u' by some winning move by I. We now define $\bar{u}$ from $\bar{u}'$ by making I's move the same except for the η component. We define the η component by first defining f_i. For each $\beta \in \{\eta_\alpha(\bar{u}') | \, \alpha < \kappa_i\} \cup \{\kappa\}$ we set $f_i(\beta)$ to be the least $\beta^* \in \bigcup_n B_n^{i-1}$ such that for some $\vec{d} \in T^F(\vec{B}^{i-1})$,

$$\mathrm{rank}(j, \beta^*, \vec{d}) \geq \beta.$$

Note that f_i is indeed an order-preserving bijection onto its range which extends f_{i-1}. To see this note that $\bigcup_n B_n^{i-2} \supseteq \bigcup_n B_n^{i-1}$. And hence for any $\beta \in \mathrm{dom}(f_i) \cap \mathrm{dom}(f_{i-1})$, $f_i(\beta) \geq f_{i-1}(\beta)$. On the other hand we have by induction that

$$\mathrm{rank}(j, f_{i-1}(\beta), \langle c_n^{i-1} \cap f(\beta) | \, n < \omega \rangle) \geq \beta$$

and

$$\langle c_n^{i-1} \cap f(\beta) \cup \{f(\beta)\} | \, n < \omega \rangle \in T^F(\vec{B}^{i-1}).$$

Hence this witnesses that $f_i(\beta) \leq f_{i-1}(\beta)$ and so they are equal. We have that f_i is order-preserving since for $\beta_1 < \beta_2$, the least $\beta_1^* \in \bigcup_n B_n^{i-1}$ such that for some $\vec{d} \in T^F(\vec{B}^{i-1})$, $\mathrm{rank}(j, \beta_1^*, \vec{d}) \geq \beta_1$ must be below the least $\beta_2^* \in \bigcup_n B_n^{i-1}$ such that for some $\vec{d'} \in T^F(\vec{B}^{i-1})$, $\mathrm{rank}(j, \beta_2^*, \vec{d'}) \geq \beta_2$ by definition of the rank function.

Now to define the η component of $\bar{u}$, for $\alpha < \kappa_i$, set

$$\eta_\alpha(\bar{u}) = f_i(\eta_\alpha(\bar{u}')).$$

We can now define $\bar{u}^*$ extending u^*. We define the $G_x^\kappa[\gamma]$ component of $\bar{u}^*$ to be the same as $\bar{u}$. Set

$$\vec{A}^i[\bar{u}^*] = \langle B_n^{i-1} \cap L_{R_\kappa(6 \cdot \bar{\gamma}^{i-1})}(V_{\lambda+1}) | \, n < \omega \rangle$$

(where $\vec{B}^{-1} = \vec{A}^*$ and $\bar{\gamma}^{-1} = \bar{\gamma}$). Finally let $\vec{c}^i[\bar{u}^*] \in T^F(\vec{B}^{i-1})$ be such that for all $\beta \in \mathrm{dom} f_i$, $f_i(\beta) \in \bigcup_n c_n^i[\bar{u}^*]$ and

$$\mathrm{rank}(j, f_i(\beta), \langle c_n^i[\bar{u}^*] \cap f_i(\beta) | \, n < \omega \rangle) \geq \beta.$$

By Lemma 26 and the definition of f_i, we can indeed find such a choice of $\vec{c}^i[\bar{u}^*]$.

Now we can extend $\bar{u}^*$ by any legal play by II, correspondingly extending $\bar{u}$ and $\bar{u}'$, and clearly our induction hypothesis is satisfied at $i+1$. Hence we have specified a quasi-winning strategy for I in $G_x^*[\kappa, \gamma, \vec{a}, \vec{A}^*]$.

Now suppose that I has a quasi-winning strategy in $G_x^*[\kappa, \gamma, \vec{a}, \vec{A}^*]$. We will show that I has a quasi-winnning strategy in $G_x^\kappa[\gamma]$. We do this by taking the plays by II and playing them (appropriately modified) against the strategy of I in $G^*_x[\kappa,\gamma,\vec{a},\vec{A}^*]$ which we call σ^*. Then our play in $G^\kappa_x[\gamma]$ is just the restriction of the winning play in $G^*_x[\kappa,\gamma,\vec{a},\vec{A}^*]$. The proof is very similar to the above proof of the converse.

We define our strategy by induction on i such that the length of the play u of $G^\kappa_x[\gamma]$ has length $2i$. We assume by induction that the following hold.

1. u^* is a play of $G^*_x[\kappa,\gamma,\vec{a},\vec{A}^*]$ according to σ^* of length $2i$ such that its restriction to $G^\kappa_x[\gamma]$ is exactly u in terms of I's moves and in terms of the x component of II's moves.

2. There are order-preserving bijections g_{i-1} and h_{i-1},

$$g_{i-1}:\{\beta^n_m[u]|\, n+m<i\}\cup\{\kappa\}\to\{\beta^n_m[u^*]|\, n+m<i\}\cup\{\kappa\}$$

and

$$h_{i-1}:\{\bar{\gamma}_n[u]|\, n<i\}\cup\{\gamma\}\to\{\bar{\gamma}_n[u^*]|\, n+m<i\}\cup\{\gamma\}$$

such that for all $\beta\in\mathrm{dom}g_{i-1}$,

$$\mathrm{rank}(j,g_{i-1}(\beta),\langle a^{i-1}_n\cap g_{i-1}(\beta)|\, n<\omega\rangle)\geq\beta$$

and $g_{i-1}(\beta)\in\bigcup_n a^{i-1}_n$. Also for all $\beta\in\mathrm{dom}h_{i-1}$,

$$\mathrm{rank}(j,h_{i-1}(\beta),\langle a^{i-1}_n\cap h_{i-1}(\beta)|\, n<\omega\rangle)\geq\beta$$

and $h_{i-1}(\beta)\in\bigcup_n a^{i-1}_n$. Here we take $\vec{a}^{-1}=\vec{a}$. Also assume that for all $\beta\in\mathrm{dom}g_{i-1}$, $g_{i-1}(\beta)$ is the least $\beta^*\in\bigcup_n A^{i-1}_n$ such that for some $\vec{d}\in T^F(\vec{A}^{i-1})$,

$$\mathrm{rank}(j,\beta^*,\vec{d})\geq\beta.$$

Here $\vec{A}^{-1}=\vec{A}^*$. We make the similar assumption about h_{i-1}.

Now, we let $\bar{u}^*$ be an extension of u^* with a play by I according to σ^*. We extend u to $\bar{u}$ by the corresponding response by I. Then given any extension of $\bar{u}$ to u by some play by II we extend (exactly as we did above) the functions g_{i-1} and h_{i-1} to g_i and h_i to define $\bar{\gamma}[v^*]$ and the β component of v^*, where v^* is the extension of $\bar{u}^*$ by a move by II. And we similarly define $\vec{B}^{i+1}[v^*]$ and $\vec{a}^{i+1}[v^*]$ as we did before.

In this way we have a quasi-winning strategy in $G^\kappa_x[\gamma]$, which we wanted. $\square$

Now as an immediate consequence of Theorem 19 and Lemmas 11, 20 and 27 we have our first main theorem.

Theorem 28. *Assume that $j:L(V_{\lambda+1})\to L(V_{\lambda+1})$ is an I_0 embedding. Then every subset $X\subseteq V_{\lambda+1}$ such that $X\in L(V_{\lambda+1})$ satisfies the following in $L(V_{\lambda+1})$.*

1. *X is $U(j)$-representable.*

2. *X is j-Suslin.*

3. *X is weakly homogeneously j-Suslin.*

6 Uniform j-Suslin representations

We will now be working to show when uniform j-Suslin representations exist. Luckily it turns out that all of our arguments from the previous sections uniformly depended on the sequence $\vec{\lambda}$ which we chose to be increasing and cofinal in λ. The only other non-uniform aspect of the above proofs was the parameter a_X which we needed to define X. Hence it seems that if we can eliminate this parameter we can obtain uniform versions of the above representations for X. This is exactly what we will see below.

Since the arguments are exactly as before, we simply give the uniform definitions and leave the modifications of the above arguments to the reader.

Suppose that $\kappa, \kappa' < \delta_1 < \delta_2$, $\vec{a}$ is weakly fixed by j, and T is a uniform (j, κ)-Suslin representation for X. Then we define the tree

$$S = MS^{\delta_1,\delta_2}_{\kappa',\vec{a}}(T)$$

by

$$S(s) = MS^{\delta_1,\delta_2}_{\kappa',\vec{a}}(T(s))$$

when $s \in [\lambda]^\omega$ is cofinal. Note that if $s, s' \in [\lambda]^\omega$ and n are such that $s \upharpoonright n = s' \upharpoonright n$ then the trees $MS^{\delta_1,\delta_2}_{\kappa',\vec{a}}(T(s))$ and $MS^{\delta_1,\delta_2}_{\kappa',\vec{a}}(T(s'))$ are the same below their nth level. Hence this definition makes sense.

The following lemma is then immediate.

Lemma 29. *Suppose that $X \subseteq V_{\lambda+1}$ is uniformly weakly $(\delta, \vec{a})$-homogeneously (j, κ)-Suslin representable in $L_\Theta(V_{\lambda+1})$ by T and $j(T) = T$. Then there is a $\kappa' > \kappa$, $\delta' > \delta$, and $\vec{b}$ such that $V_{\lambda+1} \backslash X$ is uniformly weakly $(\delta', \vec{b})$-homogeneously (j, κ')-Suslin representable in $L(V_{\lambda+1})$ by S and $j(S) = S$.*

We can generalize the definition of sequentially j-Suslin as follows.

Definition 30. Let $\kappa < \Theta$. Suppose that S is a tree on $\mathcal{F}^\omega_\kappa(j)$ and that for all $a \in [S]$, $X_a \subseteq V_{\lambda+1}$ is uniformly (j, κ)-Suslin as witnessed by a tree T^a. Furthermore assume that there is a tree T on $\mathcal{F}^\omega_\kappa(j) \times \mathcal{F}^\omega_\kappa(j)$ such that for all $a \in [S]$, $T^a = T_a$, and for all $\vec{\kappa} \in [\lambda]^\omega$ and $\vec{x} \in \mathcal{W}^{\vec{\kappa}}$,

$$\{(\vec{\kappa} \upharpoonright |\vec{x}|, a, b) |\, (a, \langle \vec{\kappa} \upharpoonright |\vec{x}|, (\vec{x}, b)\rangle \in T\} \in \mathcal{F}^\omega_\kappa(j).$$

Then we say that $\langle X_a |\, a \in [S]\rangle$ is *sequentially uniformly (j, κ)-Suslin as witnessed by T*.

The corresponding definition for the j-closed game representation is the following.

Definition 31. Fix $\kappa < \Theta$. Suppose that $X \subseteq V_{\lambda+1}$ and for each $\vec{\kappa}$ increasing cofinal in λ, $G^{\vec{\kappa}}$ is a (j, κ)-closed game representation for X as witnessed by $T^{\vec{\kappa}}$. Suppose T is such that for all $s \in [\lambda]^{<\omega}$ and $b \in \mathcal{W}^s$, $T(s)_b = T^{\vec{\kappa}}_b$ for any $\vec{\kappa}$ such that $\vec{\kappa} \upharpoonright |s| = s$. Then we say that G is a uniform (j, κ)-closed game representation as witnessed by T.

We say that a uniform (j, κ)-closed game representation G is *normal* if there exists $\bar{\gamma}$ and $\vec{a}$ such that for all $\vec{\kappa} \in [\lambda]^\omega$, $G^{\vec{\kappa}}$ is normal, and this is witnessed by $(\bar{\gamma}, \vec{a})$.

We then define $\tilde{T}, \tilde{W}, \tilde{Z}$, and $\tilde{S}$ as before, and exactly as before we obtain the following.

Theorem 32. *Suppose that $X \subseteq V_{\lambda+1}$ has a normal uniform j-closed game representation G. Then X has a uniform weakly homogeneously j-Suslin representation. In fact for $(\bar{\gamma}, \vec{a})$ witnessing that G is normal, we have $X = p[\tilde{T}(s)^{\bar{\gamma},\vec{a}}_{\emptyset}]$ for any $s \in [\lambda]^{\omega}$ cofinal, and that $\tilde{T}^{\bar{\gamma},\vec{a}}_{\emptyset}$ is a uniform weakly $(\bar{\gamma}, \vec{a})$-homogeneously $(j, \bar{\gamma})$-Suslin representation for X.*

Now we have the key fact of when normal uniform j-closed game representations exist.

Lemma 33. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an I_0 embedding. If $X \subseteq V_{\lambda+1}$ is such that for some κ good, X is definable over $L_{\kappa}(V_{\lambda+1})$ from parameters in $V_{\lambda} \cup \kappa$ then there is a uniform normal j-closed game representation for X.*

So as before we have the following theorem.

Theorem 34. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be an I_0 embedding. If $X \subseteq V_{\lambda+1}$ is such that for some κ good, X is definable over $L_{\kappa}(V_{\lambda+1})$ from parameters in $V_{\lambda} \cup \kappa$ then X is uniformly weakly homogeneously j-Suslin representable in $L(V_{\lambda+1})$.*

7 Applications

We now come to consequences of our results above. First of all by results in [3] we have the following theorems.

Theorem 35. *Suppose that I_0 holds at λ and $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then for $\vec{\kappa}$ the critical sequence of j, if $\alpha < \Theta$ is good then for some $\bar{\alpha} < \lambda$ there is an elementary embedding*

$$L_{\bar{\alpha}}(M_{\omega}[\vec{\kappa}] \cap V_{\lambda+1}) \to L_{\alpha}(V_{\lambda+1}).$$

Theorem 36. *Suppose that I_0 holds at λ,*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

is elementary. Suppose $g \in V$ is $\mathbb{P}$-generic over M_{ω} where $\mathbb{P} \in M_{\omega}$. Also assume that $cof(\lambda)^{M_{\omega}[g]} = \omega$. Then if $\alpha < \Theta$ is good, for some $\bar{\alpha} < \lambda$ there is an elementary embedding

$$L_{\bar{\alpha}}(M_{\omega}[g] \cap V_{\lambda+1}) \to L_{\alpha}(V_{\lambda+1}).$$

We can also generalize the above results to the context of $I_0^{\#}$. This generalization is fairly standard and can be seen for instance in the proof of Inverse Limit Reflection at I_0 in [4]. We only state the results here and leave the proof to the reader, which involves a straightforward generalization of the above results as well as the results of [2].

Theorem 37. *Assume that $j : L(V^{\#}_{\lambda+1}, V_{\lambda+1}) \to L(V^{\#}_{\lambda+1}, V_{\lambda+1})$ be an $I_0^{\#}$ embedding. Then every subset $X \subseteq V_{\lambda+1}$ such that $X \in L(V^{\#}_{\lambda+1}, V_{\lambda+1})$ satisfies the following in $L(V^{\#}_{\lambda+1}, V_{\lambda+1})$.*

1. *X is $U(j)$-representable.*

2. *X is j-Suslin.*

3. *X is weakly homogeneously j-Suslin.*

Theorem 38. *Let $j : L(V^{\#}_{\lambda+1}, V_{\lambda+1}) \to L(V^{\#}_{\lambda+1}, V_{\lambda+1})$ is an $I^{\#}_0$ embedding. If $X \subseteq V_{\lambda+1}$ is such that for some κ good, X is definable over $L_\kappa(V^{\#}_{\lambda+1}, V_{\lambda+1})$ from parameters in $V_\lambda \cup \kappa$ then X is uniformly weakly homogeneously j-Suslin representable in $L(V^{\#}_{\lambda+1}, V_{\lambda+1})$.*

Hence as above we obtain the following.

Theorem 39. *Suppose that $j : L(V^{\#}_{\lambda+1}, V_{\lambda+1}) \to L(V^{\#}_{\lambda+1}, V_{\lambda+1})$ is elementary, and $g \in V$ is $\mathbb{P}$-generic over M_ω, the ωth iterate of j, where $\mathbb{P} \in M_\omega$. Also assume that cof$(\lambda)^{M_\omega[g]} = \omega$. Then for some $\bar{\alpha} < \lambda$ there is an elementary embedding*

$$L_{\bar{\alpha}}((M_\omega[g] \cap V_{\lambda+1})^{\#}, M_\omega[g] \cap V_{\lambda+1}) \to L_\alpha(V^{\#}_{\lambda+1}, V_{\lambda+1}).$$

As a corollary to this theorem and results independently shown by Dimonte-Friedman [5] and Woodin (unpublished), we have the following result.

Theorem 40. *Assume that there is an elementary embedding $j : L_\omega(V^{\#}_{\lambda+1}) \to L_\omega(V^{\#}_{\lambda+1})$. Then it is consistent that I_0 holds at λ and the singular cardinal hypothesis fails at λ.*

In addition, by a theorem of Shi-Woodin[6] the above results give a proof (alternative to that found in [1]) of the perfect set property for subsets of $V_{\lambda+1}$.

The final consequence we mention is given by results in [8]. Let Γ^∞ be the set of universally Baire sets of reals. The determinacy axiom LSA states that the largest Suslin cardinal exist and is a Θ_α, that is a member of the Solovay sequence. By Corollary 168 of [8] we have the following theorem.

Theorem 41. *Suppose that λ is a limit of supercompact cardinals and there is a proper class of Woodin cardinals. Suppose that I_0 holds at λ. Let $G \subseteq$ Coll(ω, λ) be V-generic. Then for*

$$\Gamma^\infty_G = (\Gamma^\infty)^{V[G]} \cap L(V_{\lambda+1})[G]$$

we have that $L(\Gamma^\infty_G)$ satisfies LSA.

Although G. Sargsyan recently showed that $Con(LSA)$ follows from a Woodin limit of Woodins, the above theorem gives an alternative proof of $Con(LSA)$ from large cardinals. We also obtain the following theorem, which shows a strong relationship between $L(V_{\lambda+1})$ and models of determinacy after collapsing λ to ω.

Theorem 42. *Suppose that λ is a limit of supercompact cardinals and there is a proper class of Woodin cardinals. Suppose that I_0 holds at λ. Let $G \subseteq$ Coll(ω, λ) be V-generic. Then for*

$$\Gamma^\infty_G = (\Gamma^\infty)^{V[G]} \cap L(V_{\lambda+1})[G]$$

we have that

$$\Theta^{L(V_{\lambda+1})} = \Theta^{L(\Gamma^\infty_G)}.$$

Proof. To see that
$$\Theta^{L(V_{\lambda+1})} \leq \Theta^{L(\Gamma^\infty_G)}$$
let $\alpha < \Theta$ and let $X \subseteq V_{\lambda+1}$ code a prewellordering of $V_{\lambda+1}$ of ordertype at least α. We have that X is $U(j)$-representable in $L(V_{\lambda+1})$, which implies that in $L(V_{\lambda+1})[G]$, X can be coded as a subset $Y \subseteq \mathbb{R}^{L(V_{\lambda+1})[G]}$ such that Y is weakly homogeneously Suslin. Hence since there is a proper class of Woodin cardinals, Y is universally Baire. And hence $\alpha < \Theta^{L(\Gamma^\infty_G)}$.

To see that
$$\Theta^{L(V_{\lambda+1})} \geq \Theta^{L(\Gamma^\infty_G)}$$
we show that in $L(V_{\lambda+1})[G]$ there is no surjection $f : V_{\lambda+1} \to \Theta^{L(V_{\lambda+1})}$. Suppose this is not the case, and let $\tau \in L(V_{\lambda+1})$ be a term for f. Then we have that $g : V_{\lambda+1} \times \mathrm{Coll}(\omega, \lambda) \to \Theta^{L(V_{\lambda+1})}$ defined by $g(x, p) = \alpha$ iff $p \Vdash \tau(x) = \alpha$ is clearly a surjection onto $\Theta^{L(V_{\lambda+1})}$. But $g \in L(V_{\lambda+1})$, which is a contradiction. Hence the theorem follows. $\square$

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