Strong Inverse Limit Reflection

Scott Cramer

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Abstract

We show that the axiom Strong Inverse Limit Reflection holds in $L(V_{\lambda+1})$ assuming the large cardinal axiom I_0. This reflection theorem both extends results of [4], [5], and [3], and has structural implications for $L(V_{\lambda+1})$, as described in [3]. Furthermore, these results together highlight an analogy between Strong Inverse Limit Reflection and the Axiom of Determinacy insofar as both act as fundamental regularity properties.

The study of $L(V_{\lambda+1})$ was initiated by H. Woodin in order to prove properties of $L(\mathbb{R})$ under large cardinal assumptions. In particular he showed that $L(\mathbb{R})$ satisfies the Axiom of Determinacy (AD) if there exists a non-trivial elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\operatorname{crit}(j) < \lambda$ (an axiom called I_0). We investigate an axiom called Strong Inverse Limit Reflection for $L(V_{\lambda+1})$ which is in some sense analogous to AD for $L(\mathbb{R})$. Our main result is to show that if I_0 holds at λ then Strong Inverse Limit Reflection holds in $L(V_{\lambda+1})$.

Strong Inverse Limit Reflection is a strong form of a reflection property for inverse limits. Axioms of this form generally assert the existence of a collection of embeddings reflecting a certain amount of $L(V_{\lambda+1})$, together with a largeness assumption on the collection. There are potentially many different types of axioms of this form which could be considered, but we concentrate on a particular form which, by results in [3], has certain structural consequences for $L(V_{\lambda+1})$, such as a version of the perfect set property. Woodin [6] introduced a structure called a $U(j)$-representation which has similar structural implications for subsets of $V_{\lambda+1}$, and recently the author showed that I_0 implies that every subset of $V_{\lambda+1}$ has a $U(j)$-representation (see [1]). Strong inverse limit reflection and $U(j)$-representations are therefore two alternative methods for obtaining structural results for $L(V_{\lambda+1})$.

We highlight two applications of our results. The first is the following theorem which has a reduced large cardinal assumption from the corresponding theorem in [3].

Theorem 1. *Assume I_0 holds at λ. Let $S_\omega = \{\alpha < \lambda^+ \mid \mathit{cof}(\alpha) = \omega\}$. Then there are no disjoint stationary sets $S_1, S_2 \subseteq S_\omega$ such that $S_1, S_2 \in L(V_{\lambda+1})$.*

The second application is more conceptual, but perhaps more important. The following theorem is proved in [3].

Theorem 2. *Let $X \subseteq V_{\lambda+1}$. Suppose that strong inverse limit X-reflection holds at 1. Then X has the λ-splitting perfect set property.*

This theorem could be viewed as strong inverse limit X-reflection acting as a kind of fundamental regularity property for subsets of $V_{\lambda+1}$, similar to the role of AD in the context of $L(\mathbb{R})$. The following theorem, which we will prove, further strengthens this analogy.

Theorem 3. *Suppose I_0 holds at λ. Then for all $X \subseteq V_{\lambda+1}$ such that $X \in L(V_{\lambda+1})$, strong inverse limit X-reflection holds at* 1.

This theorem further strengthens the argument that strong inverse limit X-reflection appropriately generalizes some of the aspects of AD in the $L(V_{\lambda+1})$ context, as it is a property held by all subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$. In fact we will show that a more local version of Theorem 3 holds giving us a very detailed view of the propagation of strong inverse limit X-reflection in $L(V_{\lambda+1})$. For a more detailed discussion of AD-like axioms in $L(V_{\lambda+1})$ see [2].

The outline of the article is as follows. In section 1 we review and strengthen some facts on inverse limits and define the notion of (strong) inverse limit reflection. In section 2 we consider extensions of inverse limits, improving on results of [3]. In section 3 we prove our main result on strong inverse limit reflection. The main technical result needed is Lemma 40 which extends the pointwise monotonicity of square roots of elementary embeddings to inverse limits. Finally in section 4 we clear up some minor details about the definability of saturated sets witnessing strong inverse limit reflection.

1 Inverse Limits

In this section we give a brief outline of the theory of inverse limits as developed in [3] proving some useful additional properties as well. These structures were originally used for reflecting large cardinal hypotheses of the form: there exists an elementary embedding $L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$. The use of inverse limits in reflecting such large cardinals is originally due to Laver [4]. For an introduction to the theory of inverse limits see [4], [5], [3], or [2].

Suppose that $\langle j_i |\, i < \omega\rangle$ is a sequence of elementary embeddings such that the following hold:

1. For all $i < \omega$, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is an elementary embedding with $\operatorname{crit}(j_i) < \lambda$ [1].
2. There exists $\bar{\lambda} < \lambda$ such that $\operatorname{crit} j_0 < \operatorname{crit} j_1 < \cdots < \bar{\lambda}$ and $\lim_{i<\omega} \operatorname{crit} j_i = \bar{\lambda} =: \bar{\lambda}_J$.

Then we can form the inverse limit

$$J = j_0 \circ j_1 \circ \cdots : V_{\bar{\lambda}_J} \to V_\lambda$$

by setting

$$J(a) = \lim_{i\to\omega} (j_0 \circ \cdots \circ j_i)(a)$$

for any $a \in V_{\bar{\lambda}_J}$. Note that this limit makes sense since a is fixed by all but finitely many of the j_i. Now $J : V_{\bar{\lambda}_J} \to V_\lambda$ is elementary, and can be extended to a Σ_0-embedding

[1] We will always be assuming that the critical points of our elementary embeddings are below λ, even if we do not explicitly say so.

$J^* : V_{\bar\lambda+1} \to V_{\lambda+1}$ by $J(A) = \bigcup_i J(A \cap V_{\bar\lambda_i})$ for $\left\langle \bar\lambda_i | \, i < \omega \right\rangle$ any cofinal sequence in $\bar\lambda$. We refer to the pair $(J, \langle j_i | \, i < \omega \rangle)$ as an *inverse limit*, and we will write simply

$$J = j_0 \circ j_1 \circ \cdots$$

to mean that $(J, \langle j_i | \, i < \omega \rangle)$ is an inverse limit. Note that it is important that we keep track of the sequence $\langle j_i | \, i < \omega \rangle$ since it is not unique[2] for a given J, although we sometimes suppress this in our notation; we will many times be sloppy and refer to an inverse limit as 'J' or '$(J, \vec{j})$' instead of '$(J, \langle j_i | \, i < \omega \rangle)$', especially in our notation.

Suppose $J = j_0 \circ j_1 \circ \cdots$ is an inverse limit. Then for $i < \omega$ we write $J_i := j_i \circ j_{i+1} \circ \cdots$, the inverse limit obtained by 'chopping off' the first i embeddings. For $i < \omega$ we write

$$J^{(i)} := (j_0 \circ \cdots \circ j_i)(J),$$

that is the inverse limit $(J^{(i)}, \left\langle j_n^{(i)} | \, n < \omega \right\rangle)$ where $j_n^{(i)} \restriction V_\lambda = (j_0 \circ \cdots \circ j_i)(j_n \restriction V_\lambda)$ for $n < \omega$. Similarly for $n < \omega$,

$$J_n^{(i)} := (j_0 \circ \cdots \circ j_i)(J_n), \; j_n^{(i)} := (j_0 \circ \cdots \circ j_i)(j_n).$$

Then we can rewrite J in the following useful ways[3]:

$$\begin{aligned} J = j_0 \circ j_1 \circ \cdots &= \cdots (j_0 \circ j_1)(j_2) \circ j_0(j_1) \circ j_0 \\ &= \cdots j_2^{(1)} \circ j_1^{(0)} \circ j_0 \end{aligned}$$

and

$$\begin{aligned} J &= j_0 \circ J_1 = j_0(J_1) \circ j_0 = J_1^{(0)} \circ j_0 \\ &= (j_0 \circ \cdots \circ j_{i-1})(J_i) \circ j_0 \circ \cdots \circ j_{i-1} = J_i^{(i-1)} \circ j_0 \circ \cdots \circ j_{i-1} \end{aligned}$$

for any $i > 0$. Hence we can view an inverse limit J as a direct limit.

We let $\mathcal{E}$ be the set of inverse limits. So

$$\mathcal{E} = \{(J, \langle j_i | \, i < \omega \rangle) | \, J = j_0 \circ j_1 \circ \cdots : V_{\bar\lambda_J+1} \to V_{\lambda+1}\}.$$

Various subcollections of $\mathcal{E}$ can be defined as follows for α an ordinal:

$$\mathcal{E}_\alpha = \{(J, \vec{j}) \in \mathcal{E} | \, \forall i < \omega \, (j_i \text{ extends to an elementary embedding } L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

We say that α is *good* if every element of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from elements of $V_{\lambda+1}$. Note that the good ordinals are cofinal in Θ. These ordinals are of particular interest, as we have the following local existence fact.

[2]Consider grouping the embeddings as $(j_0 \circ j_1) \circ j_2 \circ \cdots$ instead of $j_0 \circ j_1 \circ j_2 \circ \cdots$ for instance.

[3]Here and below we write $j_0(j_1)$ for the unique extension of $j_0(j_1 \restriction V_\lambda)$ to an elementary embedding $V_{\lambda+1} \to V_{\lambda+1}$. See [4] for a more detailed discussion of this phenomenon.

Lemma 4 (Laver [5])**.** *Suppose there exists an elementary embedding*

$$j : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

where α is good. Then $\mathcal{E}_\alpha \neq \emptyset$.

The following shows a weak result about how the inverse limits $J_n^{(n-1)}$ fall into the sets $\mathcal{E}_\alpha$.

Lemma 5. *Suppose that α is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$. Then for all $n < \omega$,*

$$(J_n^{(n-1)}, \left\langle j_i^{(n-1)} | \, i \geq n \right\rangle) \in \mathcal{E}_\alpha.$$

Proof. This follows immediately from the fact that for any $j, k : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$,

$$j(k \restriction L_\alpha(V_{\lambda+1})) : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$$

is an elementary embedding. Note that the fact that α is good allows us to conclude that $k \restriction L_\alpha(V_{\lambda+1}) \in L_{\alpha+1}(V_{\lambda+1})$. □

A key question regarding inverse limits a they relate to reflection is to what extent J can be extended beyond $V_{\bar{\lambda}+1}$. The point is that such extensions allow the transfer of properties of $L(V_{\lambda+1})$ to $L(V_{\bar{\lambda}+1})$ for $\bar{\lambda} < \lambda$. The following summarizes some results in this direction.

Theorem 6. *Let $(J, \langle j_i | \, i < \omega \rangle) \in \mathcal{E}$ be an inverse limit.*

1. *(Laver [4]) Suppose for all $i < \omega$, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary. Then*

   $$J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

   is elementary.

2. *(Laver [5]) Suppose for all $i < \omega$, j_i extends to an elementary embedding*

   $$L_{\lambda^+ + \omega}(V_{\lambda+1}) \to L_{\lambda^+ + \omega}(V_{\lambda+1}).$$

   Then J extends to an elementary embedding

   $$L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1}) \to L_{\lambda^+}(V_{\lambda+1}).$$

3. *(C. [3]) Let α be below the first Σ_1-gap of $L(V_{\lambda+1})$ and suppose there exists an elementary embedding $L_{\alpha+\omega+1}(V_{\lambda+1}) \to L_{\alpha+\omega+1}(V_{\lambda+1})$. Then there exists an inverse limit $(K, \langle k_i | \, i < \omega \rangle)$ such that for some $\bar{\alpha} < \lambda$, K extends to an elementary embedding*

   $$L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1}).$$

Notice that in this theorem the third result is not quite as strong as the first two, in the sense that it does not give directly that every inverse limit whose embeddings are sufficiently strong is able to reflect to α. This distinction is at the heart of the theorems in the present paper, as the strongest forms of reflection become more difficult to achieve for α large below Θ. We define some notation in order to give a stronger form of the third result which more closely mirrors the first two.

First for elementary embeddings $j, k : V_{\lambda+1} \to V_{\lambda+1}$ we say that k *is a square root of* j if $k(k \restriction V_\lambda) = j \restriction V_\lambda$. We also write $k(k) = j$, slightly abusing notation as we did above.

There is a corresponding notion for inverse limits. Suppose

$$(J, \langle j_i |\, i < \omega\rangle), (K, \langle k_i |\, i < \omega\rangle) \in \mathcal{E}.$$

Then we say that $(K, \langle k_i |\, i < \omega\rangle$ *is a limit root of* $(J, \langle j_i |\, i < \omega\rangle)$ if there is $n < \omega$ such that $\bar{\lambda}_J = \bar{\lambda}_K$ and

$$\forall i < n\, (k_i = j_i) \text{ and } \forall i \geq n\, (k_i(k_i) = j_i).$$

We say $(K, \vec{k})$ *is an* n*-close limit root of* $(J, \vec{j})$ if n witnesses that $(K, \vec{k})$ is a limit root of $(J, \vec{j})$. We also say that $(K, \vec{k})$ and $(J, \vec{j})$ *agree up to* n if for all $i < n$, $j_i = k_i$.

The next lemma is a basic fact about the existence of inverse limits, which is proved using the following lemma about the existence of elementary embeddings.

Lemma 7 (Laver [5][4])**.** *Suppose $\alpha < \beta$ are good. If $(J, \vec{j}) \in \mathcal{E}_\beta$ then for all $\bar{A} \in V_{\bar{\lambda}_J+1}$ and $B \in V_{\lambda+1}$ there exists a $(K, \vec{k}) \in \mathcal{E}_\alpha$ such that K is a* 0*-close limit root of J, $K(\bar{A}) = J(\bar{A})$ and $B \in rng\, K$.*

This lemma follows from the following existence lemma for square roots.

Lemma 8 (Martin, see [5])**.** *Suppose $\alpha < \beta$ are good and $j : L_\beta(V_{\lambda+1}) \to L_\beta(V_{\lambda+1})$ is an elementary embedding. Then for any $a, b \in V_{\lambda+1}$ there is $k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that $a \in rng\, k$, $j(b) = k(b)$ and $k(k \restriction V_\lambda) = j \restriction V_\lambda$.*

The inverse limits $J_n^{(n-1)}$ which we defined above behave nicely with respect to limit roots.

Lemma 9. *Suppose that $(J, \vec{j}), (K, \vec{k}) \in \mathcal{E}$ and K is a* 0*-close limit root of J. Then for all $n < \omega$, $(K_n^{(n-1)}, \left\langle k_i^{(n-1)} |\, i \geq n\right\rangle)$ is a* 0*-close limit root of $(J_n^{(n-1)}, \left\langle j_i^{(n-1)} |\, i \geq n\right\rangle)$.*

Proof. It is enough to show the following: if k_1 and k_2 are square roots of j_1 and j_2 respectively and $j_2 \in \operatorname{rng} k_1$, then $k_1(k_2)$ is a square root of $j_1(j_2)$. But by Σ_0-elementarity, we have that $k_1(k_2)$ is a square root of $k_1(j_2)$. But $k_1(j_2) = j_1(j_2)$ since $j_2 \in \operatorname{rng} k_1$. Hence we have the fact we wanted.

To see that the lemma follows, we see for instance that $(K_1^{(0)}, \left\langle k_i^{(0)} |\, i \geq 0\right\rangle)$ is a 0-close limit root of $(J_1^{(0)}, \left\langle j_i^{(0)} |\, i \geq 0\right\rangle)$ by applying the above fact to k_0, k_n, j_0, j_n for all $n \geq 1$. $\square$

[4]The lemma as stated appears in [3], although the proof is the same as the corresponding lemma in [5], which says nothing about limit roots.

By a very similar proof we have the following fact, which shows how arbitrary limit roots behave when passing to inverse limits $J_n^{(n-1)}$ and $K_n^{(n-1)}$.

Lemma 10. *Suppose that $(J,\vec{j}),(K,\vec{k}) \in \mathcal{E}$ and K is an i-close limit root of J. Then for all $n<\omega$, $(K_n^{(n-1)}, \left\langle k_i^{(n-1)} | \, i \geq n \right\rangle)$ is an $f(i,n)$-close limit root of $(J_n^{(n-1)}, \left\langle j_i^{(n-1)} | \, i \geq n \right\rangle)$, where f is defined as follows.*

$$f(i,n) = \begin{cases} i-n & \text{if } n \leq i \\ 0 & \text{otherwise.} \end{cases}$$

One theme in the study of inverse limits is that the behavior of an inverse limit mimics that of its constituent embeddings. One example of this phenomenon is given by the next two lemmas. We will see this theme in the next section as well, and it is at the heart of our proof of strong inverse limit reflection (see Lemmas 38 and 40 for instance).

Lemma 11. *Suppose that $j,k : V_{\lambda+1} \to V_{\lambda+1}$ are elementary and k is a square root of j. I if $A \in rng\, k$, then $k(A) = j(A)$. Also, if $C \in V_{\lambda+1}$ and $j(C) \in rng\, k$ then $k(C) = j(C)$.*

Proof. To see the first part suppose $k(B) = A$, and notice (taking liberties with our notation)

$$k(A) = k(k(B)) = k(k)(k(B)) = j(k(B)) = j(A).$$

For the second part, since $j \in \operatorname{rng} k$, we have that $C \in \operatorname{rng} k$, and hence the result follows from the first part of the lemma. □

We can show a very similar property for inverse limits.

Lemma 12. *Suppose that $(K,\vec{k}),(J,\vec{j}) \in \mathcal{E}$ and $(K,\vec{k})$ is a limit root of $(J,\vec{j})$. Let $\bar{\lambda} = \bar{\lambda}_J$. Suppose $\bar{A} \in V_{\bar{\lambda}}$ and $A = J(\bar{A})$. Then if $A \in rng\, K$, we have*

$$K(\bar{A}) = A = J(\bar{A}).$$

Proof. Let A_n for $n<\omega$ be defined by induction as

$$A_0 = (j_0)^{-1}(A) \text{ and for } n \geq 0,\ A_{n+1} = (j_{n+1})^{-1}(A_n).$$

Then we have (case 1)

$$\begin{aligned} &k_0 \text{ is a squareroot of } j_0 \text{ and } A \in \operatorname{rng} k_0 \cap \operatorname{rng} j_0 \\ &\qquad \Rightarrow A_0 = j_0^{-1}(A) \in \operatorname{rng} k_0 \Rightarrow k_0(A_0) = j_0(A_0) \Rightarrow A_0 \in \operatorname{rng} K_1 \end{aligned}$$

and (case 2)

$$k_0 = j_0 \Rightarrow k_0(A_0) = j_0(A_0) \Rightarrow A_0 \in \operatorname{rng} K_1.$$

Similarly, for $n \geq 0$, (case 1)

$$\begin{aligned} &k_{n+1} \text{ is a squareroot of } j_{n+1} \text{ and } A_n \in \operatorname{rng} k_{n+1} \cap \operatorname{rng} j_{n+1} \\ &\qquad \Rightarrow A_{n+1} = j_{n+1}^{-1}(A_n) \in \operatorname{rng} k_{n+1} \Rightarrow k_{n+1}(A_{n+1}) = j_{n+1}(A_{n+1}) \Rightarrow A_{n+1} \in \operatorname{rng} K_{n+2} \end{aligned}$$

and (case 2)

$$k_{n+1} = j_{n+1} \Rightarrow k_{n+1}(A_{n+1}) = j_{n+1}(A_{n+1}) \Rightarrow A_{n+1} \in \operatorname{rng} K_{n+2}.$$

Hence we have that

$$K(\bar{A}) = A = J(\bar{A})$$

as in the proof of Lemma 7 (see Lemma 2.9 of [3]). □

A very similar proof shows the following lemma, whose proof we leave to the reader.

Lemma 13. *Suppose that $(K, \vec{k}), (J, \vec{j}) \in \mathcal{E}$ and $(K, \vec{k})$ is a limit root of $(J, \vec{j})$. Let $\bar{\lambda} = \bar{\lambda}_J$. Suppose $a \in V_{\lambda+1}$ and $a \in rng\,(k_0 \circ k_1 \circ \cdots \circ k_n)$ for all $n < \omega$. Then we have for any $n < \omega$,*

$$(k_0 \circ k_1 \circ \cdots \circ k_n)(a) = (j_0 \circ j_1 \circ \cdots \circ j_n)(a).$$

The next lemma gives a more complete picture of the phenomenon in the previous two lemmas on the agreement between an inverse limit and a limit root. In particular it is informative to realize that, using the notation as in the statement below, there is an $i < \omega$ such that $\operatorname{crit} K_i^{(i-1)} > \lambda_0 + 1$, and hence $\bar{A} \in \operatorname{rng} K_i^{(i-1)}$ for any $\bar{A} \in V_{\lambda_0+1}$. The conclusion of this lemma must therefore be limited in its scope, since otherwise it would demand too much agreement between J and K.

Lemma 14. *Suppose that $(K, \vec{k}), (J, \vec{j}) \in \mathcal{E}$, $(K, \vec{k})$ is a limit root of $(J, \vec{j})$ and for all i,*

$$k_0 \upharpoonright V_\lambda, \ldots, k_i \upharpoonright V_\lambda \in rng\, k_{i+1}.$$

Let $\bar{\lambda} = \bar{\lambda}_0 = \bar{\lambda}_J$ and

$$\lambda_i = (j_0 \circ \cdots \circ j_{i-1})(\bar{\lambda}).$$

Suppose $\bar{A} \in V_{\lambda_0+1}$ and $A = J(\bar{A})$. Then if i is such that $\bar{A} \in rng\, K_i^{(i-1)}$, then

$$K_i^{(i-1)}((j_0 \circ \cdots j_{i-1})(\bar{A})) = A = J(\bar{A}).$$

Proof. Without loss of generality we assume $i = 1$. Then we have that $\bar{A} \in \operatorname{rng} k_1^{(0)}$. But since $k_0 \upharpoonright V_\lambda \in \operatorname{rng} k_1$, we have $j_0 \upharpoonright V_\lambda \in \operatorname{rng} k_1^{(0)}$. Hence $j_0(\bar{A}) \in \operatorname{rng} k_1^{(0)}$. And so since $k_1^{(0)}$ is a square root of $j_1^{(0)}$, we have that

$$k_1^{(0)}(j_0(\bar{A})) = j_1^{(0)}(j_0(\bar{A})) = (j_0 \circ j_1)(\bar{A}).$$

And since

$$(k_1^{(0)})^{-1}(\bar{A}), k_1^{(0)} \in \operatorname{rng} k_2^{(0)}$$

we have $\bar{A} \in \operatorname{rng} k_2^{(0)}$. Furthermore $k_0 \upharpoonright V_\lambda \in \operatorname{rng} k_2$ implies that $j_0 \upharpoonright V_\lambda \in \operatorname{rng} k_2^{(0)}$, so we have that $j_0(\bar{A}) \in \operatorname{rng} k_2^{(0)}$. And hence that

$$k_1^{(0)}(j_0(\bar{A})) \in \operatorname{rng} k_1^{(0)}(k_2^{(0)}) = k_2^{(1)}.$$

But this shows that

$$k_2^{(1)}(k_1^{(0)}(j_0(\bar{A}))) = j_2^{(1)}(k_1^{(0)}(j_0(\bar{A}))) = j_2^{(1)}(j_1^{(0)}(j_0(\bar{A}))) = (j_0 \circ j_1 \circ j_2)(\bar{A})$$

since $k_2^{(1)}$ is a square root of $j_2^{(1)}$.

Continuing this way we have that

$$(j_0 \circ \cdots \circ j_{i-1})(\bar{A}) = (k_1^{(0)} \circ \cdots \circ k_{i-1}^{(0)})(j_0(\bar{A}))$$

for all $i > 0$, which proves the lemma. $\square$

The final variation on this theme, which combines the proofs of Lemma 7 and the previous few lemmas is the following, whose proof we leave to the reader.

Lemma 15. *Suppose α is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$. Then for any $a \in V_{\lambda+1}$, there is a $(K, \vec{k}) \in \mathcal{E}_\alpha$ satisfying the following:*

1. *For any $s < n \leq m$, $(k_n^{(s)} \circ k_{n+1}^{(s)} \circ \cdots \circ k_m^{(s)})(a) = (j_n^{(s)} \circ j_{n+1}^{(s)} \circ \cdots \circ j_m^{(s)})(a)$.*
2. *For any $s < n \leq m$, $a \in rng(k_n^{(s)} \circ k_{n+1}^{(s)} \circ \cdots \circ k_m^{(s)})$.*

We now come to an important definition for inverse limits, which highlights a useful type of collection of inverse limits. This definition arises out of the useful difference between square roots and limit roots: that being n-close for larger and larger n allows the existence of a sequence $\left\langle (J^i, \vec{j}^i) | \, i < \omega \right\rangle$ of inverse limits with $(J^{i+1}, \vec{j}^{i+1})$ a limit root of $(J^i, \vec{j}^i)$ for all $i < \omega$.

Definition 16. Suppose $E \subseteq \mathcal{E}$. Then we say that E is *saturated* if for all $(J, \vec{j}) \in E$ there exists an $i < \omega$ such that for all $A \in V_{\bar{\lambda}_J+1}$, and $B \in V_{\lambda+1}$, there exists $(K, \vec{k}) \in E$ such that $(K, \vec{k})$ is an i-close limit root of $(J, \vec{j})$, $K_i(A) = J_i(A)$ and $B \in \operatorname{rng} K_i$. We set $i(E, (J, \vec{j})) =$ the least such i.

Note that if $(K, \vec{k})$ is an i-close limit root of $(J, \vec{j})$ and $K_i(A) = J_i(A)$ then $K(A) = J(A)$. However, we cannot conclude that $B \in \operatorname{rng} K$ if $B \in \operatorname{rng} K_i$. For instance if $i = 1$ then we always have that $\operatorname{crit}(J) = \operatorname{crit}(K) \notin \operatorname{rng} K$, while of course $\operatorname{crit} K \in \operatorname{rng} K_1$.

We now define a natural closure operation on sets of inverse limits.

Definition 17. Suppose $E \subseteq \mathcal{E}$ is a set of inverse limits. We say that $(K, \vec{k})$ is the *common part of* $\left\langle (K^n, \vec{k}^n) | \, n < \omega \right\rangle$ if for all $i < \omega$ there is an $n < \omega$ such that for all $n' \geq n$, $k_i = k_i^{n'}$. We define $CL(E)$ to be the set

$$CL(E) = \{(K, \vec{k}) \in \mathcal{E} | \, \exists \left\langle (K^i, \vec{k}^i) | \, i < \omega \right\rangle ((K, \vec{k}) \text{ is the common part of } \left\langle (K^i, \vec{k}^i) | \, i < \omega \right\rangle, \\ \forall i < \omega (\bar{\lambda}_K = \bar{\lambda}_{K^i} \text{ and } ((K^i, \vec{k}^i) \in E)))\}.$$

The set $CL(E)$ arises naturally in the study of inverse limits as a direct result of considering sequences of inverse limits which are limit roots of one another. That is, for a sequence $\left\langle (K^i, \vec{k}^i) | \, i < \omega \right\rangle$ of inverse limits such that $(K^{i+1}, \vec{k}^{i+1})$ is a limit root of $(K^i, \vec{k}^i)$ for all $i < \omega$, it must be the case that this sequence has a common part inverse limit $(K, \vec{k})$. This follows from the fact that there cannot be an infinite sequence $j_0, j_1, \ldots$ where j_{i+1} is a squareroot of j_i for all $i < \omega$, since their critical points must be decreasing. Considering such sequences and their common part are important, for instance, in the proof of Theorem 25 which appears in [3]. There it is important that certain properties of the inverse limits along the sequence are maintained in the common part $(K, \vec{k})$. This is in a sense what the property strong inverse limit reflection will say below.

Lemma 18 ([3])**.** *Suppose that α is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Then there exists a saturated set $E \subseteq \mathcal{E}_\alpha$ such that $(J, \vec{j}) \in E$.*

See Lemma 37 below for the proof of an even stronger result. Saturated sets are in a sense large, so the following theorem extends part 3 of Theorem 6 and is along the lines of parts 1 and 2.

Theorem 19 (C. [3])**.** *Let α be such that $L_\alpha(V_{\lambda+1}) \not\prec_1^{V_{\lambda+1} \cup \{V_{\lambda+1}\}} L_{\alpha+1}(V_{\lambda+1})$ and suppose there exists an elementary embedding $L_{\alpha+\omega}(V_{\lambda+1}) \to L_{\alpha+\omega}(V_{\lambda+1})$. Then there exists a saturated set E of inverse limits, a $\bar{\lambda}$ and an $\bar{\alpha}$ such that for all $(K, \langle k_i | \, i < \omega \rangle) \in CL(E)$, K extends to an elementary embedding*

$$L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1}).$$

We make the following definitions which capture the conclusions of many of the reflection theorem above.

Definition 20. We define *inverse limit reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in E$, J extends to

$$\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$$

which is elementary.

We define *strong inverse limit reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to

$$\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$$

which is elementary.

We will also need the notion of inverse limit X-reflection where $X \subseteq V_{\lambda+1}$. Similarly as before we let

$$\begin{aligned}\mathcal{E}(X) = \{(J, \langle j_i | \, i < \omega \rangle) | \, \forall i \, (j_i : (V_{\lambda+1}, X) \to (V_{\lambda+1}, X)) \text{ and} \\ J = j_0 \circ j_1 \circ \cdots : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X) \text{ is } \Sigma_0\}.\end{aligned}$$

Here we let $\bar{X} = J^{-1}[X]$. We modify the definition of saturated to X-saturated, requiring in addition that $J^{-1}[X] = K^{-1}[X]$.

Definition 21. Suppose $X \subseteq V_{\lambda+1}$. We define *inverse limit X-reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X-saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in E$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \to L_{\alpha}(X, V_{\lambda+1})$ which is elementary.

We define *strong inverse limit X-reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X-saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \to L_{\alpha}(X, V_{\lambda+1})$ which is elementary.

The following result (whose proof we will use below) was shown in [3].

Theorem 22. *Suppose that there exists an elementary embedding*

$$j : L_{\Theta}(V_{\lambda+1}) \to L_{\Theta}(V_{\lambda+1}).$$

Then inverse limit reflection holds at α for all $\alpha < \Theta$.

We can also rephrase Theorem 19 as follows.

Theorem 23. *For any α such that $L_{\alpha}(V_{\lambda+1}) \not\prec_1^{V_{\lambda+1} \cup \{V_{\lambda+1}\}} L_{\alpha+1}(V_{\lambda+1})$, if there exists an elementary embedding*

$$L_{\alpha+\omega}(V_{\lambda+1}) \to L_{\alpha+\omega}(V_{\lambda+1})$$

then strong inverse limit reflection holds at α.

Our main result will be to generalize this result to any $\alpha < \Theta$. Hence we will obtain the following.

Theorem 24. *For any good $\alpha < \Theta^{L(V_{\lambda+1})}$, if there exists an elementary embedding*

$$L_{\alpha+\omega}(V_{\lambda+1}) \to L_{\alpha+\omega}(V_{\lambda+1})$$

then strong inverse limit reflection holds at α.

Strong inverse limit reflection has stronger consequences that inverse limit reflection. For instance the following is proved in [3].

Theorem 25. *Let $\alpha < \Theta$ be good and suppose that strong inverse limit reflection holds at α. Then there are no disjoint sets $S_1, S_2 \in L_{\alpha}(V_{\lambda+1})$ such that $S_1, S_2 \subseteq \lambda^+$ and both S_1 and S_2 are stationary (in V).*

Hence these two theorems immediately give Theorem 1 above.

2 Coherent extension of inverse limits

As our goal is to obtain strong inverse limit reflection, we are very much interested in the extending the embedding J for an inverse limit $(J, \vec{j})$. We first want to show that (for strong enough $(J, \vec{j})$) if J extends to α for α good, then the extension is in a sense unique. We then show that if a limit root of J extends, so does J. We will ultimately exploit this for long sequences of limit roots. First we prove the following lemma.

Lemma 26. *Suppose that α is good, $(K, \vec{k}) \in \mathcal{E}_\alpha$ and K extends to an elementary embedding*

$$\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1}).$$

Define $M_K^\alpha \subseteq J_\alpha(V_{\lambda+1})$ as

$$M_K^\alpha = \{a \in J_\alpha(V_{\lambda+1}) |\, \exists A \in V_{\lambda+1}(a \textit{ is definable from } A \textit{ over } J_\alpha(V_{\lambda+1}) \textit{ and } A \in \mathit{rng}\, K)\}.$$

Then

$$M_K^\alpha = \hat{K}[J_{\bar{\alpha}}(V_{\bar{\lambda}+1})]$$

and $\hat{K}$ is given by the inverse of the transitive collapse of M_K^α.

Proof. To see that $M_K^\alpha = \hat{K}[J_{\bar{\alpha}}(V_{\bar{\lambda}+1})]$, suppose that $a \in \hat{K}[J_{\bar{\alpha}}(V_{\bar{\lambda}+1})]$. Then since α is good, there exists an $A \in V_{\lambda+1}$ and a formula ϕ such that a is definable from A and ϕ over $J_\alpha(V_{\lambda+1})$. So $J_\alpha(V_{\lambda+1})$ satisfies that there exists an element $B \in V_{\lambda+1}$ such that a is definable by ϕ from B. Hence for $\bar{a} = \hat{K}^{-1}(a)$, we have that $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ satisfies that there is $\bar{B} \in V_{\bar{\lambda}+1}$ such that $\bar{a}$ is definable by ϕ from $\bar{B}$. Let $\bar{B}'$ witness this statement. Then by applying $\hat{K}$ we have that a is definable from $\hat{K}(\bar{B}')$ by ϕ over $J_\alpha(V_{\lambda+1})$. On the other hand, if a is definable from some $A \in \operatorname{rng} K \cap V_{\lambda+1}$ over $J_\alpha(V_{\lambda+1})$ then clearly $a \in \operatorname{rng} \hat{K}$ by elementarity.

The fact that $\hat{K}$ is the inverse of the transitive collapse of M_K^α follows immediately from the fact that $M_K^\alpha = \hat{K}[J_{\bar{\alpha}}(V_{\bar{\lambda}+1})]$. □

Lemma 27. *Suppose $\alpha < \Theta$ is good, that $(J, \vec{j}) \in \mathcal{E}$ and J extends to an elementary embedding*

$$\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}_J+1}) \to J_\alpha(V_{\lambda+1})$$

for some $\bar{\alpha}$. Then for all $\beta \geq \alpha$, if J extends to an elementary embedding

$$\hat{J}^* : J_{\bar{\beta}}(V_{\bar{\lambda}_J+1}) \to J_\beta(V_{\lambda+1})$$

with $\alpha \in \mathit{rng}\, \hat{J}^*$, then $(\hat{J}^*)^{-1}(\alpha) = \bar{\alpha}$ and

$$\hat{J}^* \restriction J_{\bar{\alpha}}(V_{\bar{\lambda}_J+1}) = \hat{J}.$$

Proof. The main point is that by the previous lemma

$$\operatorname{rng} \hat{J} = M_J^\alpha = \operatorname{rng}(\hat{J}^*) \cap J_\alpha(V_{\lambda+1}),$$

and hence both $\hat{J}$ and $(\hat{J}^*) \cap J_\alpha(V_{\lambda+1})$ are given by the inverse of the transitive collapse of M_J^α. So they must be the same. □

We now want to show that if a limit root of J extends to some α, so does J. We need a slightly stronger notion of extension which we now define.

Definition 28. Suppose α is good. We say that $(J,\vec{j}) \in \mathcal{E}_\alpha$ *extends coherently to* α if for all $n < \omega$ there are $\bar{\alpha}_n$ and $\bar{\lambda}_n$ such that $J_n^{(n-1)}$ extends to an elementary embedding

$$\hat{J}_n^{(n-1)} : L_{\bar{\alpha}_n}(V_{\bar{\lambda}_n+1}) \to L_\alpha(V_{\lambda+1})$$

and such that for all $n < m$,

$$\hat{J}_n^{(n-1)} = \hat{J}_m^{(m-1)} \circ j_n^{(n-1)} \circ j_{n+1}^{(n-1)} \circ \cdots \circ j_{m-1}^{(n-1)}.$$

Lemma 29. *Suppose that α is good and $(J,\vec{j}) \in \mathcal{E}_{\alpha+1}$. Then if J extends to an elementary embedding $\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$ for some $\bar{\alpha}$, then $(J,\vec{j})$ extends coherently to α.*

Proof. First we have that for all $n < \omega$, J_n extends to an elementary embedding

$$\hat{J}_n = (j_0 \circ \cdots \circ j_{n-1})^{-1} \circ \hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

This follows since α is good and

$$J_n = (j_0 \circ \cdots \circ j_{n-1})^{-1} \circ J.$$

Now by elementarity, since $(J,\vec{j}) \in \mathcal{E}_{\alpha+1}$, we can apply $j_0 \circ \cdots \circ j_{n-1}$ to the above statement to see that $J_n^{(n-1)}$ extends to an elementary embedding

$$\hat{J}_n^{(n-1)} : J_{\bar{\alpha}_n}(V_{\bar{\lambda}_n+1}) \to J_\alpha(V_{\lambda+1})$$

where $\bar{\alpha}_n = (j_0 \circ \cdots \circ j_{n-1})(\bar{\alpha})$ and $\bar{\lambda}_n = (j_0 \circ \cdots \circ j_{n-1})(\bar{\lambda})$ and

$$\hat{J}_n^{(n-1)} = (j_0 \circ \cdots \circ j_{n-1})(\hat{J}_n).$$

To see the coherency condition, we compute for any $\bar{a} \in V_{\bar{\lambda}+1}$, if for instance $\hat{J}_1(\bar{a}) = a$, then applying j_0 to this statement we have that

$$\hat{J} = (j_0 \circ \hat{J}_1)(\bar{a}) = \hat{J}_1^{(0)}(j_0(\bar{a})).$$

Similarly we have that

$$\hat{J} = (j_0 \circ j_1 \circ \cdots j_{n-1} \circ \hat{J}_n)(\bar{a}) = \hat{J}_n^{(n-1)}((j_0 \circ \cdots \circ j_{n-1})(\bar{a})).$$

Now for $n < m < \omega$, we can apply these facts to J_n and m to see that

$$\hat{J}_n = (j_n \circ \cdots \circ j_{m-1})(\hat{J}_m) \circ j_n \circ \cdots \circ j_{m-1}.$$

But then applying $(j_0 \circ \cdots j_{n-1})$ to this statement we have that

$$\hat{J}_n^{(n-1)} = \hat{J}_m^{(m-1)} \circ j_n^{(n-1)} \circ \cdots \circ j_{m-1}^{(n-1)},$$

which is what we wanted. ∎

We introduce the notion of extending coherently, because our arguments below will involve passing to the extensions of $(J^{(n-1)}, \left\langle j_i^{(n-1)} | \, i \geq n \right\rangle)$. Clearly, if $(J, \vec{j}) \in \mathcal{E}_\alpha$ extends coherently to α, then for all $n < \omega$, $(J_n^{(n-1)}, \left\langle j_i^{(n-1)} | \, i \geq n \right\rangle) \in \mathcal{E}_\alpha$ extends coherently to α, which helps in making such arguments. On the other hand this does not seem quite true for instance for the assumption $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$.

We now show that if an inverse limit has a limit root which extends to an elementary embedding, then it extends as well and in fact factors through its limit root, in some sense.

Lemma 30. *Suppose $\alpha < \Theta$ is good and that $(K, \vec{k}), (J, \vec{j}) \in \mathcal{E}_\alpha$ and $(K, \vec{k})$ is a limit root of $(J, \vec{j})$. Suppose that $(K, \vec{k})$ extends coherently to α to an elementary embedding*

$$\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$$

and that for $\langle \bar{\alpha}_n | \, n < \omega \rangle$ defined by $\bar{\alpha}_n = (k_0 \circ \cdots \circ k_{n-1})(\bar{\alpha})$ we have for some $n < \omega$ that $\mathrm{crit}(K_n^{(n-1)}) > \bar{\lambda}$ and $\bar{\alpha}_n \in \mathrm{rng}(j_0 \circ \cdots \circ j_{n-1})$. Then for some $\bar{\beta} \geq \bar{\alpha}$, J extends to an elementary embedding

$$\hat{J} : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

Proof. Since K extends coherently to α, for all $n < \omega$, $K_n^{(n-1)}$ extends to an elementary embedding

$$\hat{K}_n^{(n-1)} : J_{\bar{\alpha}_n}(V_{\bar{\lambda}_n+1}) \to J_\alpha(V_{\lambda+1})$$

where $\bar{\lambda}_n = (k_0 \circ \cdots \circ k_{n-1})(\bar{\lambda})$.

Now let $n < \omega$ be such that $\mathrm{crit}\,(K_n^{(n-1)}) > \bar{\lambda}$ and $\bar{\alpha}_n \in \mathrm{rng}\,(j_0 \circ \cdots \circ j_{n-1})$. Set

$$\bar{\beta} = (j_0 \circ \cdots \circ j_{n-1})^{-1}(\bar{\alpha}_n).$$

Then by Lemma 14 we have that for all $\bar{A} \in V_{\bar{\lambda}+1}$,

$$K_n^{(n-1)}((j_0 \circ \cdots \circ j_{n-1})(\bar{A})) = J_n^{(n-1)}((j_0 \circ \cdots \circ j_{n-1})(\bar{A})) = J(\bar{A}).$$

Hence we have that

$$K_n^{(n-1)} \circ j_0 \circ j_1 \circ \cdots \circ j_{n-1} \restriction V_{\bar{\lambda}+1} = J : V_{\bar{\lambda}+1} \to V_{\lambda+1}.$$

And so we have that

$$\hat{K}_n^{(n-1)} \circ j_0 \circ j_1 \circ \cdots \circ j_{n-1} \restriction J_{\bar{\beta}}(V_{\bar{\lambda}+1}) : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$$

is the desired extension of J. $\square$

The previous lemma required that some $\bar{\alpha}_n$ be in the range of the fragments of J. The next lemma shows that we can always find such a K where this occurs if the embeddings constituting J are elementary enough.

Lemma 31. *Suppose $\alpha < \Theta$ is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$. Suppose further that there is a $(K', \vec{k}') \in \mathcal{E}_\alpha$, a limit root of $(J, \vec{j})$ such that for some $\bar{\alpha}$, K' extends coherently to an elementary embedding*

$$\hat{K}' : J_{\bar{\alpha}}(V_{\bar{\lambda}_{K'}+1}) \to J_\alpha(V_{\lambda+1}).$$

Then there is a $(K, \vec{k}) \in \mathcal{E}_\alpha$ a limit root of $(J, \vec{j})$ such that for some $n < \omega$ we have crit$(K_n^{(n-1)}) > \bar{\lambda}_J$, $K_n^{(n-1)}$ extends to an elementary embedding

$$\hat{K}_n^{(n-1)} : J_{\bar{\alpha}}(V_{\bar{\lambda}_n+1}) \to J_\alpha(V_{\lambda+1})$$

for some $\bar{\alpha}$ and $\bar{\lambda}_n$ with $\bar{\alpha} \in$ rng$(j_0 \circ \cdots \circ j_{n-1})$. Furthermore $(K, \vec{k})$ extends coherently to α to an elementary embedding.

Proof. Let $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$ be as in the hypothesis. Fix $\delta < \lambda$ such that $\delta > \bar{\lambda}_J$ and $\delta \in \operatorname{rng} J$ and fix n such that $\operatorname{crit}(J_n^{(n-1)}) > \delta$.

We claim that that for some $\bar{\alpha}$ and $n < \omega$ there is $(K^n, \vec{k}^n) \in \mathcal{E}_\alpha$ a 0-close limit root of $J_n^{(n-1)}$ such that $\operatorname{crit}(K^n) > \delta$ and $(K^n, \vec{k}^n)$ extends coherently to α to an elementary embedding

$$\hat{K}^n : J_{\bar{\alpha}}(V_{\bar{\lambda}_n+1}) \to J_\alpha(V_{\lambda+1}).$$

This follows since, taking $(K, \vec{k})$ witnessing our assumption on $(J, \vec{j})$, we can find n large enough so that $(K^n, \vec{k}^n) = (K_n^{(n-1)}, \left\langle k_i^{(n-1)} | \, i \le n \right\rangle)$ witnesses this claim. To see this, first of all we have that since K maps $\bar{\lambda}_K$ cofinally in λ, for $\bar{\delta} < \bar{\lambda}_K$ such that $K(\bar{\delta}) \ge \delta$ we can take n such that $\operatorname{crit}(k_n) > \bar{\delta}$. So in that case

$$\operatorname{crit}(K_n^{(n-1)}) = \operatorname{crit}(k_n^{(n-1)}) = (k_0 \circ \cdots \circ k_{n-1})(\operatorname{crit}(k_n)) \ge K(\bar{\delta}) \ge \delta.$$

On the other hand, Lemma 10 since $(K, \vec{k})$ is a limit root of $(J, \vec{j})$, for large enough n, $(K_n^{(n-1)}, \left\langle k_i^{(n-1)} | \, i \ge n \right\rangle)$ is a 0-close limit root of $(J_n^{(n-1)}, \left\langle j_i^{(n-1)} | \, i \ge n \right\rangle)$. So the claim follows.

Let $j_i^* : J_{\alpha+1}(V_{\lambda+1}) \to J_{\alpha+1}$ be the elementary extension of j_i for $i < n$. Then

$$J_n^{(n-1)}, \delta, \bar{\lambda}_n \in \operatorname{rng}(j_0^* \circ \cdots \circ j_{n-1}^*)$$

we have that $\bar{\alpha} \in \operatorname{rng}(j_0 \circ \cdots \circ j_{n-1})$, and in fact for some $(K^n, \vec{k}^n)$ satisfying the above claim, $(K^n, \vec{k}^n) \in \operatorname{rng}(j_0^* \circ \cdots \circ j_{n-1}^*)$. Now let $\langle \bar{k}_i^n | \, i < \omega \rangle$ be such that

$$(j_0^* \circ \cdots \circ j_{n-1}^*)(\langle \bar{k}_i^n | \, i < \omega \rangle) = \langle k_i^n | \, i < \omega \rangle .$$

Let $(K^*, \vec{k}^*)$ be the inverse limit

$$K^* = j_0 \circ \cdots \circ j_{n-1} \circ \bar{k}_0^n \circ \bar{k}_1^n \circ \cdots .$$

We have (by the elementarity of $j_0 \circ \cdots \circ j_{n-1}$) that $(K^*, \vec{k}^*) \in \mathcal{E}_\alpha$, and that it is a limit root of $(J, \vec{j})$. Hence since we have $(K_n^{*,(n-1)}, \left\langle k_i^{*,(n-1)} | \, i \ge n \right\rangle) = (K^n, \vec{k}^n)$ and hence $(K^*, \vec{k}^*)$ witnesses that the lemma holds. $\square$

Putting the previous two lemmas together, we obtain that a very large collection of inverse limits extend to elementary embeddings.

Lemma 32. *Suppose $\alpha < \Theta$ is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$. Also assume that there is a saturated set $E \subseteq \mathcal{E}_{\alpha+1}$ such that $(J, \vec{j}) \in E$. Then for some $\bar{\alpha}$, J extends to an elementary embedding*

$$\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}_J+1}) \to J_{\alpha}(V_{\lambda+1}).$$

Proof. Let J be as in the hypothesis. Then by the proof of Theorem 22 (see [3]) there is a sequence $\left\langle (K^n, \vec{k}^n) | \, n < \omega \right\rangle$ such that the following hold:

1. $K^0 = J$ and for all $n < \omega$, $(K^n, \vec{k}^n) \in \mathcal{E}_{\alpha+1}$,
2. for all $n < \omega$, $(K^{n+1}, \vec{k}^{n+1})$ is a limit root of $(K^n, \vec{k}^n)$,
3. there is a $\bar{\beta}$ and an n_0 such that for all $n \geq n_0$ K^n extends to an elementary embedding
$$\hat{K}^n : J_{\bar{\beta}}(V_{\bar{\lambda}_J+1}) \to J_{\alpha}(V_{\lambda+1}).$$

By applying the previous three lemmas we have that there must be some $\bar{\alpha}_{n_0-1}$ such that K^{n_0-1} extends to an elementary embedding

$$\hat{K}^{n_0-1} : J_{\bar{\alpha}_{n_0-1}}(V_{\bar{\lambda}_J+1}) \to J_{\alpha}(V_{\lambda+1}).$$

And similarly by induction we have that there are $\bar{\alpha}_{n_0-1}, \ldots, \bar{\alpha}_0$ such that for all $i < n_0$, K^i extends to an elementary embedding

$$\hat{K}^i : J_{\bar{\alpha}_i}(V_{\bar{\lambda}_J+1}) \to J_{\alpha}(V_{\lambda+1}).$$

So considering $i = 0$ the lemma follows. $\square$

Theorem 33. *Suppose $\alpha < \Theta$ is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Then for some $\bar{\alpha}$, J extends to an elementary embedding*

$$\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}_J+1}) \to J_{\alpha}(V_{\lambda+1}).$$

Proof. It is proved in [3] that for all $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$, there is a saturated set $E \subseteq \mathcal{E}_{\alpha+1}$ such that $(J, \vec{j}) \in E$. Hence by Lemma 32, J must extend to an elementary embedding

$$\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}_J+1}) \to J_{\alpha}(V_{\lambda+1})$$

for some $\bar{\alpha}$. $\square$

3 Limit root extensions

Our goal in this section is to get a more detailed picture of the relationship of the extension of J and the extensions of its limit roots. The key fact will be Lemma 40 which basically says that the extensions of sequences of limit roots are pointwise non-increasing. From this fact we will quickly obtain strong inverse limit reflection.

We first define an ordering on certain equivalence classes of elements of V_λ. This is a natural ordering generated by an inverse limit $(J, \vec{j})$, and it turns out to be a well-ordering if $(J, \vec{j}) \in \mathcal{E}_1$. It is really just the equivalence classes given by the direct limit system, when viewing J as a direct rather than inverse limit. Hence, the wellfoundedness of this ordering is simply the wellfoundedness of the corresponding linear iteration given by J.

Definition 34. Let $(J, \vec{j}) \in \mathcal{E}$, and define the ordering $\leq_J$ on tuples (α, n) for $\alpha < \lambda$ and $n < \omega$ as follows:

1. $(\alpha, n) \leq_J (\beta, n)$ if $\alpha \leq \beta$.
2. $(\alpha, n) \leq_J (\beta, m)$ if $n \leq m$ and $(j_n^{(n-1)} \circ \cdots \circ j_{m-1}^{(n-1)})(\alpha) \leq \beta$.
3. $(\alpha, n) \leq_J (\beta, m)$ if $m \leq n$ and $\alpha \leq (j_m^{(m-1)} \circ \cdots \circ j_{n-1}^{(m-1)})(\beta)$.

We put $(\alpha, n) \sim_J (\beta, m)$ if $(\beta, m) \leq_J (\alpha, n)$ and $(\alpha, n) \leq_J (\beta, n)$. Let $[\alpha, n]_J$ be the equivalence class of (α, n) under the equivalence relation $\sim_J$. Let $\mathcal{I}^J$ be the set of equivalence classes $[\alpha, n]_J$ for $\alpha < \lambda$. Let $\mathcal{I}^J_{\leq(\gamma,m)}$ be the set of equivalence classes $[\alpha, n]_J$ such that $(\alpha, n) \leq_J (\gamma, m)$.

As mentioned above, the equivalence classes $[\alpha, n]_J$ can be thought of equivalently as elements of the direct limit of the iteration $\cdots \circ j_2^{(1)} \circ j_1^{(0)} \circ j_0$. To see this, note that for any $n \leq m$ we have by a simple induction that

$$j_n^{(n-1)} \circ \cdots \circ j_{m-1}^{(n-1)} = j_{m-1}^{(m-2)} \circ j_{m-2}^{(m-3)} \circ \cdots \circ j_{n+1}^{(n)} \circ j_n^{(n-1)}.$$

To see this, consider the following computation

$$\begin{aligned}
j_0 \circ j_1 \circ \cdots \circ j_{n-1} &= j_0(j_1) \circ j_0(j_2) \circ \cdots \circ j_0(j_{n-1}) \circ j_0 \\
&= j_0(j_1)(j_0(j_2)) \circ j_0(j_1)(j_0(j_3)) \circ \cdots \circ j_0(j_1)(j_0(j_{n-1})) \circ j_0(j_1) \circ j_0 \\
&= j_0(j_1(j_2)) \circ j_0(j_1(j_3)) \circ \cdots \circ j_0(j_1(j_{n-1})) \circ j_0(j_1) \circ j_0 \\
&= j_2^{(1)} \circ j_3^{(1)} \circ \cdots \circ j_{n-1}^{(1)} \circ j_1^{(0)} \circ j_0 \\
&= \cdots \\
&= j_{n-1}^{(n-2)} \circ j_{n-2}^{(n-3)} \circ \cdots \circ j_1^{(0)} \circ j_0
\end{aligned}$$

Hence $\mathcal{I}^J$ is clearly linearly ordered by $\leq_J$ since these are equivalently the ordinals of the direct limit.

Lemma 35. *Suppose $(J, \vec{j}) \in \mathcal{E}_1$. Then $(\mathcal{I}^J, \leq_J)$ is a well-ordering.*

Proof. Suppose that $\langle [\alpha_i, n_i]_J | \, i < \omega \rangle$ are such that $(\alpha_i, n_i) >_J (\alpha_{i+1}, n_{i+1})$ for all $i < \omega$. By Lemma 15 we can find $(K, \vec{k}) \in \mathcal{E}_0$ with the following properties:

1. For all[5] $i, n, m < \omega$, and $(\alpha, s) \in [\alpha_i, n_i]_J$,
$$(j_n^{(n-1)} \circ \cdots j_{m-1}^{(n-1)})(\alpha) = (k_n^{(n-1)} \circ \cdots k_{m-1}^{(n-1)})(\alpha).$$
2. For all $i < \omega$ and $(\alpha, n) \in [\alpha_i, n_i]_J$ we have that
$$\alpha \in \operatorname{rng}(k_0 \circ \cdots \circ k_{n-1}).$$

It is then easy to see that (1) implies that for all $i < \omega$,

$$[\alpha_i, n_i]_J \subseteq [\alpha_i, n_i]_K,$$

and (2) implies that for all $i < \omega$, there exists an α_i' such that

$$(\alpha_i', 0) \in [\alpha_i, n_i]_K.$$

But then, since we can view the equivalence classes $[\alpha_i, n_i]_K$ as elements of the corresponding direct limit system, we have that for all $i < \omega$, $(\alpha_i, n_i) >_K (\alpha_{i+1}, n_{i+1})$, and hence $\alpha_i' > \alpha_{i+1}'$, a contradiction. $\square$

We now define an iterated version of being a limit root for inverse limits.

Definition 36. For $\alpha < \omega_1$ we define an *α-limit root sequence* $\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \alpha \right\rangle$ by induction on α as follows. A 1-limit root sequence is just $\left\langle (K^0, \vec{k}^0) \right\rangle$ such that $(K^0, \vec{k}^0) \in \mathcal{E}$. For $\alpha = \beta + 1$ a successor, $\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \alpha \right\rangle$ is an α-limit root sequence if $\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \beta \right\rangle$ is a β-limit root sequence and the following hold:

1. If β is a limit, then $(K^\beta, \vec{k}^\beta)$ is the common part of the sequence $\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \beta \right\rangle$.
2. If β is a successor, then $(K^\beta, \vec{k}^\beta)$ is a limit root of $(K^{\beta-1}, \vec{k}^{\beta-1})$.

If α is a limit, then
$$\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \alpha \right\rangle$$
is an α-limit root sequence if for all $\beta < \alpha$,
$$\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \beta \right\rangle$$

[5] This condition might be somewhat confusing because it is really overkill. The point is that when passing from $(J, \vec{j})$ to $(K, \vec{k})$, everything we care about is preserved. It is easier to state this condition as is than to write down the particular relationship between i, n, m, and s which we need.

is a β-limit root sequence.

We say that $(K, \vec{k})$ is an *α-limit root* of $(J, \vec{j}^\eta)$ if there is an $\alpha + 1$-limit root sequence $\left\langle (K^\eta, \vec{k}^\eta) | \, \eta \leq \alpha \right\rangle$ such that $(K^0, \vec{k}^0) = (J, \vec{j})$ and $(K^\alpha, \vec{k}^\alpha) = (K, \vec{k})$. So $(K, \vec{k})$ is a limit root of $(J, \vec{j})$ iff $(K, \vec{k})$ is a 1-limit root of $(J, \vec{j})$.

Suppose $\gamma < \Theta$ is good and suppose that $c : \alpha \to \omega$ is a function. Then we say that $\left\langle (K^\eta, \vec{k}^\eta) | \, n < \alpha \right\rangle$ is an *α-limit root sequence following c at γ* if the following hold:

1. For all $\eta < \alpha$, $(K^\eta, \vec{k}^\eta) \in \mathcal{E}_\gamma$.

2. Suppose that $\alpha = \beta + 1$ is a successor. Then $(K^\alpha, \vec{k}^\alpha)$ is a $c(\alpha)$-close limit root of $(K^\beta, \vec{k}^\beta)$.

We also say for embeddings $j, k : V_{\lambda+1} \to V_{\lambda+1}$ that k is an *n-square root of j* if for some sequence $j_0, j_1, \ldots, j_n$ of embeddings $V_{\lambda+1} \to V_{\lambda+1}$ we have that $j_0 = j$, $j_n = k$ and for all $i < n$, j_{i+1} is a square root of j_i.

The next lemma shows the existence of long limit root sequences. The proof in fact also gives Lemma 18 above.

Lemma 37. *Suppose that $\gamma < \Theta$ is good, $\alpha < \omega_1$, and $c : \alpha \to \omega$ is an injection. Suppose that $(K^0, \vec{k}^0) \in \mathcal{E}_{\gamma+\omega}$. Then there is $\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \alpha \right\rangle$ an α-limit root sequence following c at γ.*

Proof. First, using Lemma 7, let $(K^1, \vec{k}^1)$ be a 0-close limit root of $(K^0, \vec{k}^0)$ such that for all $i < \omega$, k^1_i extends to an embedding

$$J_{\gamma+i+1}(V_{\lambda+1}) \to J_{\gamma+i+1}(V_{\lambda+1}).$$

For $\alpha' < \alpha$ such that $\alpha' = \beta + 1$, having defined the sequence below α', we choose, as in the proof of Lemma 7, $(K^{\alpha'}, \vec{k}^{\alpha'})$ to be a $c(\alpha')$-close limit root of $(K^\beta, \vec{k}^\beta)$ such that for all $i \in [c(\alpha'), \omega)$, if k^β_i extends to an embedding

$$J_{\gamma+s_i+1}(V_{\lambda+1}) \to J_{\gamma+s_i+1}(V_{\lambda+1})$$

then $k^{\alpha'}_i$ extends to an embedding

$$J_{\gamma+s_i}(V_{\lambda+1}) \to J_{\gamma+s_i}(V_{\lambda+1}).$$

That is we apply Lemma 8 by induction to each element of the sequence $\vec{k}^\beta$. For $\alpha' < \alpha$ a limit, we simply take $(K^{\alpha'}, \vec{k}^{\alpha'})$ to be the common part of $\left\langle (K^\eta, \vec{k}^\eta) | \, \eta < \alpha' \right\rangle$.

Clearly this construction succeeds, as for all $i < \omega$, the set $\{\alpha' < \alpha | \, c(\alpha') \leq i\}$ has cardinality less than or equal to i, as c is injective. $\square$

We note the following fact about square roots of elementary embeddings, which we will extend to inverse limits.

Lemma 38. *Let α be good. Suppose that $j, k : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$ and k is a square root of j. Then for all $\beta < \alpha + 1$, we have that $k(\beta) \geq j(\beta)$.*

Proof. Fix α good and j and k as in the hypothesis. We prove this by induction on β. If β is a successor or a continuity point of j then there is nothing to prove. So assume that β is a discontinuity point of j. Let $\gamma = \sup j"\beta$. We have by induction that $\gamma \leq \sup k"\beta$. Suppose for a contradiction that $k(\beta) < j(\beta)$. Then $j(\beta)$ is definable in $L_{\alpha+1}(V_{\lambda+1})$ from $j \upharpoonright V_\lambda$ and $k(\beta)$, as the least image point above $k(\beta)$ of the unique extension of $j \upharpoonright V_\lambda$ to α. But then $j(\beta) \in \operatorname{rng} k$, and since $j \in \operatorname{rng} k$, we have that $\beta \in \operatorname{rng} k$. But then $k(\beta) = j(\beta)$, since k is a square root of j, a contradiction. $\square$

This lemma can of course be extended by induction to j and k such that for some sequence $j_0, j_1, \dots, j_n$ we have that $j_0 = j$, $j_n = k$ and for all $i < n$, j_{i+1} is a square root of j_i. In the next lemma we will use this slight extension of Lemma 38 which holds for the individual embeddings making up $(J, \vec{j})$ and $(K, \vec{k})$ such that $(K, \vec{k})$ is an α-limit root of $(J, \vec{j})$, for instance.

In order to show that inverse limits have the property in Lemma 38 as well, we first show the following technical result.

Lemma 39. *Suppose $\alpha < \Theta$ is good, $\delta + 1 < \omega_1$, and $\left\langle (J^\gamma, \vec{j}^\gamma) | \, \gamma < \delta + 1 \right\rangle$ is a $\delta + 1$-limit root sequence from $\mathcal{E}_1$. Then for all $[\beta, n]_{J^0}$, there is an $n_0 < \omega$ such that for all $\gamma < \delta + 1$ and $m, m' \geq n_0$, if $(\beta_1, m), (\beta_2, m') \in [\beta, n]_{J^0}$ then*

$$(\beta_1, m) \sim_{J^\gamma} (\beta_2, m')$$

and hence if J^γ extends coherently to $\alpha + 1$ to an embedding

$$\hat{J}^\gamma : J_{\bar{\alpha}_\gamma+1}(V_{\bar{\lambda}_{J^\gamma}+1}) \to J_{\alpha+1}(V_{\lambda+1})$$

then we have

$$\hat{J}_m^{\gamma,(m-1)}(\beta_1) = \hat{J}_{m'}^{\gamma,(m'-1)}(\beta_2).$$

Proof. Fixing $[\beta, n]_{J^0}$, we prove first that for each $\gamma < \delta$ there is such an n_0, the least which we call n_γ. The full lemma follows by noticing that for γ a limit, $n_\gamma \geq \sup_{\gamma' < \gamma} n_{\gamma'}$. The proofs of these two facts are basically the same.

Let $j_i = j_i^0$ and $k_i = j_i^\gamma$ for $i < \omega$. So $K = J^\gamma$. Then by Lemma 38 and the definition of a limit root sequence we have for all $\xi < \lambda$ and $i < \omega$ that $k_i(\xi) \geq j_i(\xi)$. Hence for all $n < \omega$, if $(\alpha_n, n), (\alpha_{n+1}, n+1) \in [\beta, n]_J$ and $(\alpha_n, n) \nsim_K (\alpha_{n+1}, n+1)$ then

$$k_n^{(n-1)}(\alpha_n) > \alpha_{n+1} = j_n^{(n-1)}(\alpha_n).$$

Hence, if there are infinitely many $n < \omega$ such that

$$(\alpha_n, n) \nsim_K (\alpha_{n+1}, n+1),$$

then $\langle [\alpha_n, n]_K \,|\, n < \omega\rangle$ contains an infinite decreasing subsequence in the $\leq_K$ ordering, which is a contradiction to the well-foundedness of $\leq_K$.

For the limit step, basically the same proof works, since if γ is such that

$$j_n^{\gamma,(n-1)}(\alpha_n) > j_n^{0,(n-1)}(\alpha_n),$$

then for all $\gamma' \in [\gamma, \delta]$, by Lemma 38

$$j_n^{\gamma',(n-1)}(\alpha_n) > j_n^{0,(n-1)}(\alpha_n).$$

And hence the lemma follows. $\square$

We need the following notation. Let $\sim$ be the equivalence relation defined as follows. Suppose $(K, \vec{k}), (K', \vec{k}') \in \mathcal{E}$ are such that for some n and m,

$$(k_0 \circ \cdots \circ k_{n-1})(\langle k_n, k_{n+1}, \ldots\rangle) = (k'_0 \circ \cdots \circ k'_{m-1})(\langle k'_m, k'_{m+1}, \ldots\rangle).$$

Then we put $\vec{k} \sim \vec{k}'$. To see that $\sim$ is transitive, first note that if $\vec{k} \sim \vec{k}'$ as witnessed by n and m, then for any $s < \omega$, $n+s$ and $m+s$ also witness this. To see this consider the computation:

$$\begin{aligned}
&(k_0 \circ \cdots \circ k_{n-1} \circ k_n)(\langle k_{n+1}, k_{n+2}, \ldots\rangle) \\
&\qquad = (k_0 \circ \cdots \circ k_{n-1})(k_n)((k_0 \circ \cdots \circ k_{n-1})(\langle k_{n+1}, k_{n+2}, \ldots\rangle)) \\
&\qquad = (k'_0 \circ \cdots \circ k'_{m-1})(k'_m)((k'_0 \circ \cdots \circ k'_{m-1})(\langle k_{m+1}, k_{m+2}, \ldots\rangle)) \\
&\qquad = (k'_0 \circ \cdots \circ k'_{m-1} \circ k'_m)(\langle k'_{m+1}, k_{m+2}, \ldots\rangle).
\end{aligned}$$

The general fact then follows by induction. Transitivity of $\sim$ follows immediately. We let $[\vec{k}]_\sim$ denote the equivalence class which $\vec{k}$ belongs to.

Lemma 40. *Suppose $\alpha < \Theta$ is good, $\delta < \omega_1$, $(J, \vec{j}) \in \mathcal{E}_{\alpha+3}$, and $(J, \vec{j})$ extends coherently to $\alpha + 1$ to an elementary embedding*

$$\hat{J} : J_{\bar{\alpha}+3}(V_{\bar{\lambda}_J+1}) \to J_{\alpha+3}(V_{\lambda+1})$$

for some $\bar{\alpha}$. Then if $(K, \vec{k}) \in \mathcal{E}_{\alpha+2}$ is a δ-limit root of $(J, \vec{j})$, and $(K, \vec{k})$ extends coherently to $\alpha + 1$ to an elementary embedding

$$\hat{K} : J_{\bar{\beta}+1}(V_{\bar{\lambda}_J+1}) \to J_{\alpha+1}(V_{\lambda+1})$$

for some $\bar{\beta}$ then $\bar{\beta} \leq \bar{\alpha}$ and for all $\bar{\gamma} \leq \bar{\beta}$, $\hat{K}(\bar{\gamma}) \geq \hat{J}(\bar{\gamma})$.

Proof. First we prove the lemma for $\delta = 1$. This will, in essence, prove the lemma for all δ successor (assuming the limit case is true as well).

Suppose the lemma fails, and let $[\bar{\alpha}, n]_J$ be $\leq_J$-least such that there exists $(K, \vec{k}) \in \mathcal{E}_{\alpha+2}$ a limit root of $(J, \vec{j})$ with $\hat{K}_n^{(n-1)}(\bar{\alpha}) < \hat{J}_n^{(n-1)}(\bar{\alpha})$. Assume for ease of notation that $n = 0$

and that $(K, \vec{k}) \in \mathcal{E}_{\alpha+2}$ is a 0-close limit root of $(J, \vec{j})$. Then we have that $\bar{\alpha}$ is definable over $J_{\alpha+2}(V_{\lambda+1})$ from $(J, \vec{j})$ and $\hat{K}(\bar{\alpha})$ as the least ordinal sent by $\hat{J}$ above $\hat{K}(\bar{\alpha})$. Hence by Lemma 13 for all n we have

$$(k_0 \circ \cdots \circ k_n)(\bar{\alpha}) = (j_0 \circ \cdots \circ j_n)(\bar{\alpha}).$$

So for all n we have

$$\hat{K}_n^{(n-1)}(\bar{\alpha}_n) < \hat{J}_n^{(n-1)}(\bar{\alpha}_n)$$

where $\bar{\alpha}_n = (j_0 \circ \cdots \circ j_{n-1})(\bar{\alpha})$.

Let β be least such that for some $n < \omega$, $(K^n, \vec{k}^n) \in \mathcal{E}_{\alpha+2}$ a limit root of

$$(J_n^{(n-1)}, \left\langle j_i^{(n-1)} | \, i \geq n \right\rangle)$$

which extends coherently to $\alpha + 1$,

$$\hat{K}^n((\hat{J}_n^{(n-1)})^{-1}(\hat{J}(\bar{\alpha}))) = \beta.$$

Then we have that $\beta < \hat{J}(\bar{\alpha})$ and $\beta \geq \sup_{\bar{\beta} < \bar{\alpha}} \hat{J}(\bar{\beta})$.

We claim that $\beta \in \operatorname{rng} \hat{J}$, which is a contradiction. To see this, we claim that β is definable from $[J]_\sim$ and $\hat{J}(\bar{\alpha})$ over $J_{\alpha+3}(V_{\lambda+1})$. And this follows since for any $S \in [J]_\sim \cap \mathcal{E}_{\alpha+2}$ which extends coherently to $\alpha+1$, for all large enough n, if

$$\bar{\alpha}' = (\hat{S}_n^{(n-1)})^{-1}(\hat{J}(\bar{\alpha}))$$

then β is the least β' such that for some $(K, \vec{k}) \in \mathcal{E}_{\alpha+2}$ a limit root of $S_n^{(n-1)}$ which extends coherently to $\alpha+1$, $\hat{K}(\bar{\alpha}') = \beta'$. This follows since for all large enough n, $S_n^{(n-1)} = J_m^{(m-1)}$ for some m. Hence since $[J]_\sim \in \operatorname{rng} \hat{J}$, we have $\beta \in \operatorname{rng} \hat{J}$.

Note that we have actually also shown that $\bar{\beta} \leq \bar{\alpha}$ in the statement of the lemma, since $\bar{\alpha} \leq \bar{\beta}$ implies that $\hat{K}(\bar{\beta}) = \alpha = \hat{J}(\bar{\alpha}) \leq \hat{K}(\bar{\alpha})$. So $\hat{K}(\bar{\alpha}) = \alpha$ since it cannot be any larger. Hence $\bar{\alpha} = \bar{\beta}$.

Now we prove the lemma for δ a limit, assuming the lemma is true for all $\delta' < \delta$.

Suppose the lemma fails for δ and $\left\langle K^{\delta'} | \, \delta' < \delta \right\rangle$ is a limit root sequence with K the common part witnessing this failure. Let $\delta' < \delta$ be least such that for some $(\bar{\beta}, m)$,

$$(\hat{K}^{\delta'})_m^{(m-1)}(\bar{\beta}) > \hat{K}_m^{(m-1)}(\bar{\beta}).$$

Without loss of generality, by renaming, we can assume that $\delta' = 0$. Let $[\bar{\beta}, m]_{K^0}$ be $\leq_{K^0}$-least such that for some $(\bar{\beta}_0, m_0) \in [\bar{\beta}, m]_{K^0}$,

$$(\hat{K}^0)_{m_0}^{(m_0-1)}(\bar{\beta}_0) > \hat{K}_{m_0}^{(m_0-1)}(\bar{\beta}_0).$$

Then in fact by the previous lemma, there is an $n_0 \geq m_0$ such that for all $m' \geq n_0$ and

$$(\bar{\beta}', m'), (\bar{\beta}_{n_0}, n_0) \in [\bar{\beta}, m]_{K^0},$$

we have

$$(\hat{K}^0)^{(m'-1)}_{m'}(\bar{\beta}') = (\hat{K}^0)^{(m_0-1)}_{m_0}(\bar{\beta}_0) > \hat{K}^{(m_0-1)}_{m_0}(\bar{\beta}_0) \geq \hat{K}^{(m'-1)}_{m'}(\bar{\beta}') = \hat{K}^{(n_0-1)}_{n_0}(\bar{\beta}_{n_0}).$$

The last inequality follows from the fact that $(K, \vec{k})$ is a δ-limit root of $(K^0, \vec{k}^0)$, which implies that the embeddings making up K are iterated square roots of the embeddings making up K^0. Hence using the fact that

$$(\bar{\beta}_0, m_0), (\bar{\beta}', m') \in [\bar{\beta}, m]_{K^0}$$

by Lemma 38 the inequality follows.

Again by renaming, we can assume without loss of generality that $n_0 = 0$. Let $(\bar{\beta}_0, 0) \in [\bar{\beta}, m]_{K^0}$. Then for all n we have

$$(k_0 \circ \cdots \circ k_n)(\bar{\beta}_0) = (k^0_0 \circ \cdots \circ k^0_n)(\bar{\beta}_0).$$

So for all n we have

$$\hat{K}^{(n-1)}_n(\bar{\beta}_n) < \hat{K}^{0,(n-1)}_n(\bar{\beta}_n)$$

where $\bar{\beta}_n = (k^0_0 \circ \cdots \circ k^0_{n-1})(\bar{\beta}_0)$.

Let β be least such that for some $n < \omega$, $(K^n, \vec{k}^n) \in \mathcal{E}_{\alpha+2}$ is a δ-limit root of

$$(J^{(n-1)}_n, \left\langle j^{(n-1)}_i \middle|\, i \geq n \right\rangle)$$

which extends coherently to $\alpha + 1$, $\hat{K}^n(\bar{\alpha}) = \beta$. Then we have that $\beta < \hat{J}(\bar{\alpha})$ and

$$\beta \geq \sup_{\bar{\beta} < \bar{\alpha}} \hat{J}(\bar{\beta}).$$

We claim that $\beta \in \operatorname{rng} \hat{J}$, which is a contradiction. To see this, we claim that β is definable from $[J]_\sim$ and $\hat{J}(\bar{\alpha})$ over $J_{\alpha+3}(V_{\lambda+1})$. And this follows since for any $S \in [J]_\sim$, for all large enough n, if

$$\bar{\alpha}' = (\hat{S}^{(n-1)}_n)^{-1}(\hat{J}(\bar{\alpha}))$$

then β is least such that for some $(K, \vec{k}) \in \mathcal{E}_{\alpha+2}$ a δ-limit root of $S^{(n-1)}_n$, $\hat{K}(\bar{\alpha}') = \beta$. Hence since $[J]_\sim \in \operatorname{rng} \hat{J}$, we have $\beta \in \operatorname{rng} \hat{J}$.

The fact that $\bar{\beta} \leq \bar{\alpha}$ follows as before. $\square$

We now can achieve our main result on saturated sets.

Theorem 41. *Let $\alpha < \Theta$ be good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Then for some $\gamma < \omega \cdot \omega$ there is $(K, \vec{k}) \in \mathcal{E}_\alpha$ which is a γ-limit root of $(J, \vec{j})$ such that there is a saturated set E and $\bar{\alpha}$ such that $(K, \vec{k}) \in E$ and for all $(K', \vec{k}') \in CL(E)$, K' extends to an elementary embedding*

$$\hat{K}' : J_{\bar{\alpha}}(V_{\bar{\lambda}_K+1}) \to J_\alpha(V_{\lambda+1}).$$

Proof. Let $c : \omega\cdot\omega \to \omega$ be an injection. We attempt to construct an $\omega\cdot\omega$-limit root sequence $\left\langle (K^n, \vec{k}^n) \middle| \, n < \omega\cdot\omega \right\rangle$ following c at $\alpha + 2$ such that for all $n < \omega\cdot\omega$ we have K^n extends to

$$\hat{K}^n : J_{\bar{\alpha}_n}(V_{\bar{\lambda}_K+1}) \to J_\alpha(V_{\lambda+1}),$$

and for all $i < \omega$, $\bar{\alpha}_{\omega\cdot i} > \bar{\alpha}_{\omega\cdot(i+1)}$. Although the proof of Lemma 37 allows us to extend the sequence following c, we cannot construct such a sequence satisfying this inequality. Hence our attempt must fail at some point, at which point we have $\bar{\alpha}_{\omega\cdot i} = \bar{\alpha}_{\omega\cdot(i+1)}$ by Lemma 40. This fact allows us to define a saturated set as desired.

We construct the sequence as follows by induction for $i < \omega$. Let $(K^0, \vec{k}^0) = (J, \vec{j})$. Having constructed $\left\langle (K^n, \vec{k}^n) \middle| \, n \leq \omega\cdot i \right\rangle$, if there exists an extension $\left\langle (K^n, \vec{k}^n) \middle| \, n \leq \omega\cdot (i+1) \right\rangle$ a limit root sequence following $c \restriction \omega\cdot(i+1)$ at $\alpha+2$ such that

$$\bar{\alpha}_{\omega\cdot(i+1)} < \lim_{n\to\omega\cdot(i+1)} \bar{\alpha}_n,$$

then choose any such extension. Otherwise by the proof of Lemma 37 and using Lemma 40 there is some $m < \omega$ and an extension

$$\left\langle (K^n, \vec{k}^n) \middle| \, n \leq \omega\cdot i + m \right\rangle$$

such that for all further extensions following c at $\alpha+2$, $\left\langle (K^n, \vec{k}^n) \middle| \, n \leq \omega\cdot (i+1) \right\rangle$ we have

$$\bar{\alpha}_{\omega\cdot(i+1)} = \bar{\alpha}_{\omega\cdot i+m}.$$

In this case, set E to be the set of $(K', \vec{k}') \in \mathcal{E}_\alpha$ such that for some limit root sequence $\left\langle (K^n, \vec{k}^n) \middle| \, n < \omega\cdot (i+1) \right\rangle$ following c at $\alpha+2$, $(K', \vec{k}') = (K^n, \vec{k}^n)$ for some $n > \omega\cdot i + m$. By the proof of Lemma 37, E is nonempty, and by the proof of 18, E is saturated. The main point is that for any $(K', \vec{k}') \in CL(E)$, either $(K', \vec{k}') \in E$, or $(K', \vec{k}')$ is the common part of a sequence $\left\langle (J^s, \vec{j}^s) \middle| \, s < \omega \right\rangle$ such that

$$\left\langle (K^n, \vec{k}^n) \middle| \, n \leq \omega\cdot i + m \right\rangle^\frown \left\langle (J^s, \vec{j}^s) \middle| \, s < \omega \right\rangle$$

is a limit root sequence following c at $\alpha+2$. This follows since $c : \omega\cdot\omega \to \omega$ is an injection, so if we let t be such that k'_n is a $t(n)$-square root of $k_n^{i\cdot\omega+m}$ we must have $t(n) \to \omega$ as $n \to \omega$ since $(K', \vec{k}') \notin E$. Hence in either case, by the property of i we are assuming, we have

$$\bar{\alpha}_{\omega\cdot i+m} = \bar{\alpha}_{K'}$$

where $\bar{\alpha}_{K'}$ is such that K' extends to

$$\hat{K}' : J_{\bar{\alpha}_{K'}}(V_{\bar{\lambda}_K+1}) \to J_\alpha(V_{\lambda+1}).$$

Hence setting $\bar{\alpha} = \bar{\alpha}_{\omega\cdot i+m}$ witnesses the theorem for E. ☐

Theorem 42. *Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then for all $\alpha < \Theta$ good, strong inverse limit reflection at α holds.*

Proof. The theorem follows immediately from Theorem 41 and the definition of strong inverse limit reflection. ☐

4 Definable saturated sets

One unfortunate aspect of the above results is that the saturated sets we defined were not quite simply definable in way that certain saturated sets are definable for α being rather small. For instance, when reflecting at ω or λ^+, it is very easy to define saturated sets of inverse limits which reflect these ordinals in the same way (namely because these ordinals are simply definable). In addition, we have not quite replicated the more complicated structures beyond saturated sets for arbitrary good α that exist at smaller ordinals. For instance the structure of inverse limits extending to some α such that $\lambda^+ + \omega \leq \alpha < \lambda^+ \cdot 2$ is a more complicated structure of inverse limits, all reflecting λ^+ in the same way, but we do not quite capture this structure at arbitrary good α with the above results.

In this section we try to replicate these structures at arbitrary good α by using the fact that the existence of the structures themselves are simply definable. In this way, we see for inverse limits, working at arbitrary good α is basically the same as working at small α.

Definition 43. Let $\kappa < \Theta$ be good and let $\bar{\lambda}, \bar{\kappa} < \lambda$. For $\beta < \kappa$ we define by induction a set $\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$ of inverse limits as follows.

$$\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(0) = \{(J, \vec{j}) \in \mathcal{E}_{\kappa} |\, J \text{ extends to } \hat{J} : L_{\bar{\kappa}}(V_{\bar{\lambda}+1}) \to L_{\kappa}(V_{\lambda+1}) \text{ which is elementary}\}.$$

Then for any β such that $0 < \beta < \kappa$ we set

$$\begin{aligned}\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta) = \{(J, \vec{j}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(0) \cap \mathcal{E}_{\kappa+\beta} |\, &\forall \gamma < \beta \,(\text{if } (J, \vec{j}) \in \mathcal{E}_{\kappa+\gamma} \text{ then}\\ &\forall a \in V_{\bar{\lambda}+1}\, \forall b \in V_{\lambda+1}\, \exists (K, \vec{k}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\gamma)\\ &(K(a) = J(a) \wedge b \in \operatorname{rng} K \wedge K \text{ is a 0-close limit root of } J))\}.\end{aligned}$$

Definition 44. Let $\kappa < \Theta$ be good and let $\bar{\lambda}, \bar{\kappa} < \lambda$. We define $\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$ for $\beta < \Theta$ by induction as follows.

$$\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(0) = \{(J, \vec{j}) \in \mathcal{E}_{\kappa} |\, J \text{ extends to } \hat{J} : L_{\bar{\kappa}}(V_{\bar{\lambda}+1}) \to L_{\kappa}(V_{\lambda+1}) \text{ which is elementary}\}.$$

Then for $\beta > 0$ and $\beta < \Theta$ we define

$$\begin{aligned}\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta) = \{(J, \vec{j}) \in \bigcap_{\gamma<\beta} \tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\gamma) |\, &\forall \gamma < \beta\, \exists n < \omega\, \forall a \in V_{\bar{\lambda}+1}\, \forall b \in V_{\lambda+1}\, \exists (K, \vec{k}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\gamma)\\ &(K(a) = J(a) \wedge b \in \operatorname{rng} K_n \wedge K \text{ is an } n\text{-close limit root of } J)\}.\end{aligned}$$

Let

$$\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}} = \bigcap_{\beta<\Theta} \tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta).$$

Theorem 45. *Suppose κ is good and there exists an elementary embedding*

$$j : L_{\kappa+\omega}(V_{\lambda+1}) \to L_{\kappa+\omega}(V_{\lambda+1}).$$

Then there exists $\bar{\kappa}, \bar{\lambda} < \lambda$ such that $\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}} \neq \emptyset$ is saturated and for all $(J, \vec{j}) \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}$, J extends to $\hat{J} : L_{\bar{\kappa}}(V_{\bar{\lambda}+1}) \to L_{\kappa}(V_{\lambda+1})$ which is elementary. Furthermore $\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}$ is definable over $L(V_{\lambda+1})$ from $\kappa, \bar{\lambda}$ and $\bar{\kappa}$.

Proof. By Theorem 41 there is a $\bar{\lambda}$, $\bar{\kappa}$ and a saturated set E such that for all $K \in CL(E)$, K extends to an elementary embedding

$$\hat{K} : J_{\bar{\kappa}}(V_{\bar{\lambda}+1}) \to J_{\kappa}(V_{\lambda+1}).$$

But then clearly for all β, $E \subseteq \tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$. Hence $\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}} \neq \emptyset$. The rest of the properties of $\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}} \neq \emptyset$ follow easily from the definition and the fact that there is no cofinal function $V_{\lambda+1} \to \Theta$ in $L(V_{\lambda+1})$[6], which implies that for some $\gamma < \Theta$, $\tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}} = \tilde{\mathcal{E}}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\gamma)$. □

Theorem 46. *Suppose that there exists an elementary embedding*

$$j : L_{\Theta}(V_{\lambda+1}) \to L_{\Theta}(V_{\lambda+1}).$$

Let κ be good. Then there exists $\bar{\kappa}, \bar{\lambda} < \lambda$ such that for all $\beta < \kappa$, $\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta) \neq \emptyset$. Furthermore for all $\beta < \kappa$, $\mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$ is definable over $L_{\kappa+\beta+1}(V_{\lambda+1})$ from $\bar{\lambda}, \bar{\kappa}$ and κ.

Proof. Let δ_0 be the least stable of $L(V_{\lambda+1})$, that is the least δ such that

$$L_{\delta}(V_{\lambda+1}) \prec_1^{V_{\lambda+1} \cup \{V_{\lambda+1}\}} L_{\Theta}(V_{\lambda+1}).$$

We show that the theorem holds for all good $\kappa < \delta$, which implies that the theorem holds for all good $\kappa < \Theta$, since if there were a contradiction there would be one below δ.

Now let $\kappa < \delta_0$. There is some κ' with $\kappa < \kappa' < \delta_0$, and

$$L_{\kappa'}(V_{\lambda+1}) \not\prec_1^{V_{\lambda+1} \cup \{V_{\lambda+1}\}} L_{\kappa'+1}(V_{\lambda+1}).$$

Let $a \in V_{\lambda+1}$ and ϕ be a Σ_1 formula such that $\kappa' + 1$ is the first place where ϕ has a witness with parameter a. Also suppose that κ is definable over $L_{\kappa'+1}(V_{\lambda+1})$ from the parameter $b \in V_{\lambda+1}$. Then as in the proof of Theorem 3.8 of [3], if $J \in \mathcal{E}_{\kappa'+\kappa+\omega}$ and $a, b \in \operatorname{rng} J$, then for $\bar{\lambda} = \bar{\lambda}_J$, J extends to an elementary embedding

$$\hat{J} : L_{\bar{\kappa}'+1}(V_{\bar{\lambda}+1}) \to L_{\kappa'+1}(V_{\lambda+1}),$$

with $\kappa \in \operatorname{rng} \hat{J}$. And furthermore, if $\bar{a}, \bar{b} \in V_{\bar{\lambda}+1}$ is such that $J(\bar{a}, \bar{b}) = (a, b)$ and $\bar{\kappa} = (\hat{J})^{-1}(\kappa)$, then for any $K \in \mathcal{E}_{\kappa'+\omega}$, such that $K(\bar{a}, \bar{b}) = (a, b)$, K extends to an elementary embedding

$$\hat{K} : L_{\bar{\kappa}'+1}(V_{\bar{\lambda}+1}) \to L_{\kappa'+1}(V_{\lambda+1})$$

[6] This fact follows as in the case of $L(\mathbb{R})$. The argument involves constructing from such a cofinal function τ a surjection of $V_{\lambda+1} \times V_{\lambda+1} \times V_{\lambda+1} \to \Theta$ by considering the least surjection $\sigma_b : V_{\lambda+1} \to \tau(a)$ definable from b, where $a, b \in V_{\lambda+1}$.

and $\hat{K}(\bar{\kappa}) = \kappa$.

Hence we claim that $J \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\kappa)$. To see this it is enough to claim that for all $K \in \mathcal{E}_{\kappa'+\omega}$, if $\beta < \kappa$, $K \in \mathcal{E}_{\kappa'+\omega+\beta}$, and $K(\bar{a}, \bar{b}) = (a, b)$, then $K \in \mathcal{E}^{\kappa}_{\bar{\lambda},\bar{\kappa}}(\beta)$. But, for all $\beta \leq \kappa$, $\kappa' + \omega + \beta$ is a good ordinal, and hence this follows immediately from what we remarked in the last paragraph together with the proof of Lemma 7. □

Unfortunately, it is unclear at present how to show the local version of Theorem 46. That is, reducing the hypothesis to, say, the existence of an elementary embedding

$$j : L_{\kappa\cdot 2+\omega}(V_{\lambda+1}) \to L_{\kappa\cdot 2+\omega}(V_{\lambda+1})$$

and proving the result without using Σ_1-reflection. Such a theorem would be more in line with the results in this paper.

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