Inverse Limit Reflection and the Structure of $L(V_{\lambda+1})$

by

Scott Stefan Cramer

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor W. Hugh Woodin, Chair
Professor John Steel
Professor Sherrilyn Roush

Fall 2012

Inverse Limit Reflection and the Structure of $L(V_{\lambda+1})$

Abstract

Inverse Limit Reflection and the Structure of $L(V_{\lambda+1})$

by

Scott Stefan Cramer

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor W. Hugh Woodin, Chair

We explore the technique of inverse limits, a tool first introduced by Laver in the context of rank-into-rank embeddings. We extend reflection results of Laver using inverse limits up to and including embeddings of the form $L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Isolating this technique as 'inverse limit reflection,' we use it to prove structural results of $L(V_{\lambda+1})$ very similar to properties of $L(\mathbb{R})$ under $AD^{L(\mathbb{R})}$. We then define a representation, first introduced by Woodin, for subsets of $V_{\lambda+1}$, and prove basic closure properties of this representation. Employing the technique of inverse limit reflection we then prove an important property called the Tower Condition and analyze certain measures generated from fixed points of embeddings to broaden the extent of these representations in $L(V_{\lambda+1})$.

To Lynn

Contents

List of Figures

Acknowledgments

I would most of all like to thank my advisor Hugh Woodin for sharing with me his deep knowledge of and insight into mathematics, for showing me the profound project which is set theory, and for teaching me what it is to be a mathematician. I am indebted to Vincenzo Dimonte, Peter Koellner, Richard Laver, Everett Piper, Xianghui Shi, John Steel, and Nam Trang for numerous discussions, comments, suggestions, and corrections. I would also like to thank my family, Mom, Dad, Brent, and Megan, for their support, and Lynn for her companionship and understanding.

Chapter 1

Introduction

The study of $L(V_{\lambda+1})$ is motivated primarily by the two goals of uncovering the structure of large cardinal axioms just below the limitation of Kunen's Theorem, and understanding the relationship between $L(V_{\lambda+1})$ and $L(\mathbb{R})$ (see [Lav02] for a survey of the subject). In this manuscript, we develop techniques for solving questions which arise in the first area, and then we use these techniques to further our understanding in the second area.

Large cardinal hierarchy serves as a fundamental measurement of consistency strength in set theory, and the computation of equiconsistency with large cardinals is a fundamental goal in set theory. At the highest reaches of the large cardinal hierarchy, however, there are more basic questions about the structure of large cardinals: first, large cardinal axioms must be identified; second, the relationship between these axioms must be understood; and third, basic justification for their consistency must be given.

For the first task, there is an analogy with models of determinacy which has proven very useful in discovering new axioms. This analogy stems from the similarity between the two structures $L(V_{\lambda+1})$ and $L(\mathbb{R})$, and is potentially a source of deep connections between the two theories. The second task can involve understanding the reflection properties of large cardinals, a subject which we will spend considerable time considering below. The third task might involve showing interesting structural results follow from the axioms. In our case, we will be concerned with how large cardinals affect the structure of $L(V_{\lambda+1})$.

Large cardinal axioms, for our purposes, are generally of the form: there exists a non-trivial elementary embedding $j : V \to M$, where M has a certain amount of agreement with V. Kunen's Theorem gives an upper bound for such axioms; it states that under ZFC there is no elementary embedding $V \to V$ (see Section 1.1.1 for more details; we assume all embeddings are non-trivial). Below this axiom, we are concerned with the following general divisions in the large cardinal hierarchy (see Sections 1.1.1 and 5.1.1 for definitions). On the left we indicate whether or not the axioms are potentially consistent with the Axiom of Choice.

$\neg$AC	$V \to V$ embeddings
	E^0_α-hierarchy
AC	$L(V_{\lambda+1}) \to L(V_{\lambda+1})$ embeddings
	rank-into-rank embeddings

Clearly, whether or not it is consistent for such choiceless $V \to V$ embeddings to exist is a fundamental question at this level, and in order to tackle this question, understanding the structure of the axioms just below this division is likely of paramount importance. Here we take the first step towards such an analysis by studying the structure of $L(V_{\lambda+1})$.

We begin by studying the reflection properties of $L(V_{\lambda+1}) \to L(V_{\lambda+1})$ embeddings. The main tool we use is the inverse limit of a sequence of elementary embeddings. Laver ([Lav97], [Lav01]) first introduced inverse limits in the study of reflecting rank into rank embeddings. An inverse limit is an embedding $V_{\bar{\lambda}+1} \to V_{\lambda+1}$ for some $\bar{\lambda} < \lambda$ which is built out of an ω-sequence of embeddings $V_{\lambda+1} \to V_{\lambda+1}$ (see Section 2.1). The basic question of inverse limits is to what extent for all α there exists an $\bar{\alpha}$ such that the inverse limit extends to an elementary embedding $L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$; inverse limit reflection is the statement that an inverse limit does have such an extension as long as the embeddings that make up the inverse limit are sufficiently strong. In Sections 2.2-2.5 we show through several different proofs that inverse limit reflection holds up through the hierarchy of axioms below $L(V_{\lambda+1}) \to L(V_{\lambda+1})$ embeddings. As a culmination to this analysis, we show that the existence of an elementary embedding

$$L(V^{\#}_{\lambda+1}) \to L(V^{\#}_{\lambda+1})$$

with critical point less than λ implies that there is some $\bar{\lambda} < \lambda$ such that there is an elementary embedding

$$L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$$

with critical point less than $\bar{\lambda}$.

In Chapter 3, using inverse limit reflection at level α we show a number of structural properties of $L(V_{\lambda+1})$. Let Θ be the sup of β such that there exists a surjection $V_{\lambda+1} \to \beta$ in $L(V_{\lambda+1})$. In $L(V_{\lambda+1})$, let $\kappa < \Theta$ be a cardinal with cofinality bigger than λ, let $\alpha < \lambda$ be an infinite cardinal, and let

$$S_\alpha = \{\beta < \kappa | \operatorname{cof}(\beta) = \alpha\}.$$

Assume there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Woodin showed that in $L(V_{\lambda+1})$, κ is measurable, and this is witnessed by the club filter restricted to a stationary set. However, he also showed that if $\alpha > \omega$ then it is consistent that the club filter restricted to S_α is not an ultrafilter. This theorem leaves open the case of $\alpha = \omega$, and we show, under the above assumptions, that S_ω cannot be partitioned into two stationary sets which are in $L(V_{\lambda+1})$. Woodin showed a similar result follows from the existence of $U(j)$-representations (see Chapter 4), but it is unclear at present if all subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ have $U(j)$-representations (see Section 4.4).

Along these same lines, Xianghui Shi and Woodin showed that the Perfect Set Property in $L(V_{\lambda+1})$ follows from a forcing argument and the generic absoluteness of Theorem 4.5.2. In Section 3.2 we prove an analogous result using inverse limit reflection.

This relationship between inverse limit reflection and the structure of $L(V_{\lambda+1})$ has an interesting consequence (see Corollary 2.5.11). Suppose that $X \subseteq V_{\lambda+1}$ and there exists an elementary embedding $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$. Then one might expect the analysis of $L(V_{\lambda+1})$ to carry over to $L(X, V_{\lambda+1})$, and that this more general situation is really the appropriate area to study. We show, however, that inverse limit X-reflection cannot hold in general, and that the set of $X \subseteq V_{\lambda+1}$ such that inverse limit X-reflection holds is very restricted. As inverse limit reflection is a very natural property one would expect of these structures, this fact highlights $L(V_{\lambda+1})$ and its extensions satisfying inverse limit reflection as the most natural objects to study at this level.

Furthermore, while it seems as though the existence of an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is an analogous assumption for $L(V_{\lambda+1})$ as assuming the axiom of determinacy holds in $L(\mathbb{R})$ is for $L(\mathbb{R})$, we argue that inverse limit reflection is actually a more appropriate analog of determinacy. Indeed, we will see that the structures $L(X, V_{\lambda+1})$ which have analogous embeddings $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ do not generally have structural properties which are analogous to the structural properties of models of determinacy, while those structures satisfying inverse limit reflection do have these properties.

Some of the above structural results which we obtain from inverse limit reflection were shown by Woodin to follow from $U(j)$-representations. In fact he showed that even stronger reflection properties follow from these representations (see Theorem 4.5.2). The similarity in the structural consequences of inverse limit reflection and $U(j)$-representations suggests that there might be some connection between the two. We explore this connection in Sections 4.2 and 4.3 and as a result are able to significantly expand the collection of $U(j)$-representable sets (see Section 4.4).

Considering these results, it seems likely that all sets in $V_{\lambda+2}^{L(V_{\lambda+1})}$ are $U(j)$-representable in $L(V_{\lambda+1})$, a conjecture which has been shown to have many interesting consequences (see Section 4.5). Proving this conjecture therefore appears to be the most natural extension of our work here.

While we concentrate here on studying $L(V_{\lambda+1})$, the above results likely extend to stronger axioms. We define the E^0_α hierarchy in Section 5.1.1, which was first introduced by Woodin in analogy with the definition of the minimal model of $AD_{\mathbb{R}}$ in the context of determinacy. Extending the above results on inverse limits and $U(j)$-representation to this hierarchy is a natural next step in the above analysis, and it might be the key to understanding the even stronger $V \to V$ axioms (see Section 5.1.2).

1.1 The structure $L(V_{\lambda+1})$

1.1.1 Very large cardinals

Large cardinals are the fundamental measure of consistency strength in set theory. At the lowest level, for instance, there are strongly inaccessible cardinals: a cardinal κ is *strongly inaccessible* if κ is regular and is a strong limit, so for all $\alpha < \kappa$, $2^\alpha < \kappa$. So if there exists a strongly inaccessible cardinal, Con(ZFC) holds.

For our purposes, large cardinals are of the following general form: κ is the critical point of $j : V \to M$ where M has a certain amount of agreement with V. κ is *measurable* if it is the critical point of such an embedding $j : V \to M$ where no agreement between M and V is required. In general, the more agreement M has with V, the stronger the large cardinal axiom. So for instance we have the following:

Fact 1.1.1. *Suppose that $\kappa = crit(j)$ where $j : V \to M$ is elementary. Suppose that $V_{\kappa+2} \subseteq M$. Then κ is a limit of measurable cardinals.*

Proof. Let $U \subseteq P(\kappa)$ be the normal ultrafilter induced by j. So $A \in U$ iff $\kappa \in j(A)$. We have that $U \in V_{\kappa+2}$ and hence κ is measurable in M. But by the elementarity of j, κ must be a limit of measurables in V. □

This phenomenon that stronger large cardinals reflect weaker large cardinals is a fundamental property of these axioms, although it becomes much more difficult to show for some of the strongest axioms in the large cardinal hierarchy.

A question which immediately arises in the study of large cardinals from elementary embeddings is: how much agreement is possible between M and V? Kunen's Theorem gives an upper bound under ZFC.

Theorem 1.1.2 (Kunen [Kun71])**.** *(ZFC) There is no (non-trivial) elementary embedding $j : V \to V$.*

In fact the proof gives the following upper bound for rank into rank embeddings.

Theorem 1.1.3 (Kunen (see [Kan94]))**.** *(ZFC) Suppose that α is such that there exists an elementary embedding $j : V_\alpha \to V_\alpha$. Then for $\lambda = \sup_{i<\omega} \kappa_i$ where $\kappa_0 = crit\, j$ and for $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$, we have*

1. *Either $\lambda = \alpha$ or $\lambda + 1 = \alpha$.*
2. *For all β such that $crit\, j \leq \beta < \lambda$, $j(\beta) > \beta$.*

Figure 1.1.1 gives a picture of rank-into-rank embeddings which we obtain from this theorem.

A fundamental question is whether it is possible to have an elementary embedding $j : V \to V$ under ZF.

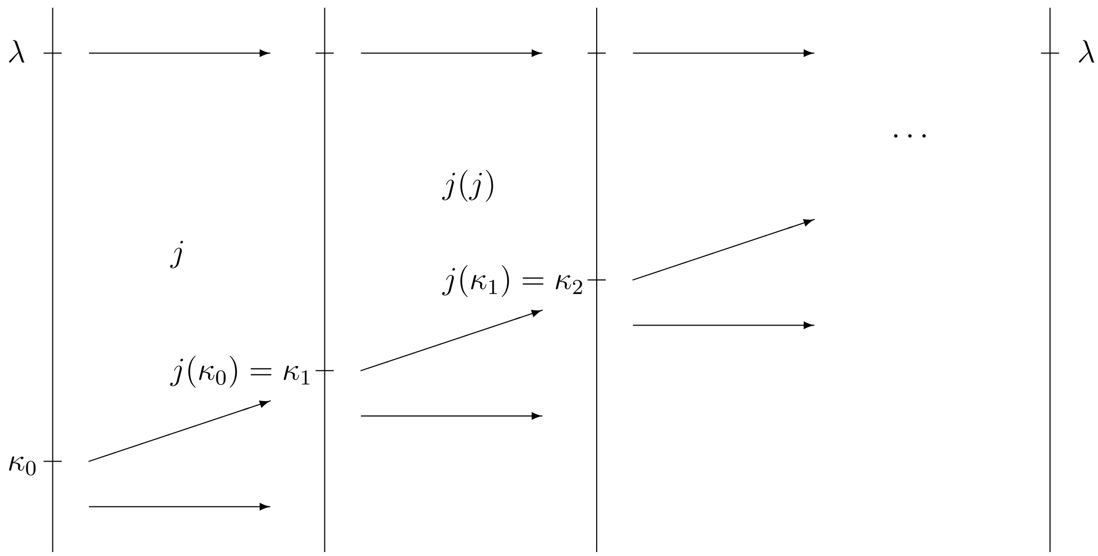

Figure 1.1: Rank-into-rank embeddings.

The following is a list of large cardinal axioms, ordered in decreasing strength, which we will be considering in this paper (λ is always the sup of the critical sequence of j, so in particular $\operatorname{crit}(j) < \lambda$; see Section 1.2 for basic definitions).

Name	Definition
Reinhardt Cardinal	$\exists j : V \to V$
	$\exists j : V_{\lambda+2} \to V_{\lambda+2}$
$I_0^{\#}$	$\exists j : L(V^{\#}_{\lambda+1}) \to L(V^{\#}_{\lambda+1})$
I_0	$\exists j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$
I_1	$\exists j : V_{\lambda+1} \to V_{\lambda+1}$
I_3	$\exists j : V_\lambda \to V_\lambda$

There is further subdivision between I_0 and I_1, for instance, where for a fixed α we demand the existence of an elementary embedding $j : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$. A similar type of subdivision exists between $I_0^{\#}$ and I_0, and so on.

1.1.2 Comparison with $L(\mathbb{R})$

Besides arising in the study of very large cardinals, $L(V_{\lambda+1})$ is an example of a structure $L(V_{\alpha+1})$ for some singular strong limit α. The case $\operatorname{cof}(\alpha) = \omega$ is special, however, by the following theorem of Shelah.

Theorem 1.1.4 (Shelah)**.** *Suppose λ is a singular strong limit of uncountable cofinality. Then $L(P(\lambda)) \models ZFC$.*

If I_0 holds at λ we have $L(V_{\lambda+1}) \models \neg\mathrm{AC}$, and so we have a similarity between $L(V_{\lambda+1})$ assuming I_0 at λ and $L(\mathbb{R})$ assuming $AD^{L(\mathbb{R})}$. Before exploring this connection further, we fix some notation and note other obvious similarities in a lemma.

Definition 1.1.5. Fix λ. Call an ordinal α *good* if every member of $L_\alpha(V_{\lambda+1})$ is definable over $L_\alpha(V_{\lambda+1})$ from a member of $V_{\lambda+1}$. Define

$$\Theta = \Theta_\lambda := \sup\{\alpha|\, (\exists\sigma(\sigma : V_{\lambda+1} \to \alpha \text{ is a surjection}))^{L(V_{\lambda+1})}\}.$$

Lemma 1.1.6. *Fix λ a strong limit such that cof$(\lambda) = \omega$. Then the following hold:*

1. *$L(V_{\lambda+1}) \models ZF + \lambda$-$DC$.*
2. *The good ordinals are cofinal in Θ_λ.*
3. *Θ_λ is regular in $L(V_{\lambda+1})$.*
4. *$L_{\Theta_\lambda}(V_{\lambda+1}) \models ZF^-$.*
5. *Suppose that $j : L_\alpha(V_{\lambda+1}) \to L_\beta(V_{\lambda+1})$ is elementary for α good. Then j is induced by $j \restriction V_\lambda$.*

Proof. 1, 3, and 4 are as in the $L(\mathbb{R})$ case. For 2 and 5, see [Lav01]. $\square$

The following is a selection of results which show a significant similarity between the two structures $L(\mathbb{R})$ and $L(V_{\lambda+1})$.

Theorem 1.1.7 (Woodin [Woo11])**.** *Fix λ such that there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Then the following hold in $L(V_{\lambda+1})$:*

1. *For cofinally many $\kappa < \Theta_\lambda$, κ is measurable, and this is witnessed by the club filter restricted to a stationary subset of κ.*
2. *If $\alpha < \Theta_\lambda$ then $P(\alpha) \in L_{\Theta_\lambda}(V_{\lambda+1})$.*

There are important differences between $L(\mathbb{R})$ and $L(V_{\lambda+1})$ however. As an example we consider the case of the club filter on λ^+. Solovay showed that if $AD^{L(\mathbb{R})}$ holds then the club filter is an ultrafilter on ω_1 in $L(\mathbb{R})$. One might expect such a result to hold in $L(V_{\lambda+1})$, but there are two reasons it cannot. First of all, if $S_\alpha = \{\kappa < \lambda^+ |\, \mathrm{cof}(\kappa) = \alpha\}$ then the set of S_α for $\alpha < \lambda$ regular is a collection of disjoint stationary subsets of λ^+. This problem however appears to be fairly trivial as we could simply modify our question by asking instead if the club filter restricted to each S_α is an ultrafilter for each regular $\alpha < \lambda$. This leads to a second more serious problem, which is displayed by the following theorem.

Theorem 1.1.8 (Woodin). *Fix λ and let $\kappa < \lambda$ be an uncountable regular cardinal. Let $S_\kappa = \{\alpha < \lambda^+ |\, cof(\alpha) = \kappa\}$ and let $\mathcal{F}$ be the club filter on λ^+. If $G \subseteq Coll(\kappa, \kappa^+)$ is V-generic then*

$$L(V[G]_{\lambda+1}) \models \mathcal{F} \text{ restricted to } S_\kappa \text{ is not an ultrafilter.}$$

In fact for any $\beta < \lambda$, there exists a poset $\mathbb{P}$ such that if $G \subseteq \mathbb{P}$ is V-generic then in $L(V[G]_{\lambda+1})$ there is a partition $\langle T_\alpha |\, \alpha < \gamma \rangle$ of S_κ into stationary sets for some $\gamma \geq \beta$, such that for all $\alpha < \gamma$, $\mathcal{F}$ restricted to T_α is an ultrafilter.

This second problem is part of a larger issue which we call the 'right V problem': the theory of V_λ can be changed by small forcing, while the theory of V_ω cannot. Hence a property of $L(V_{\lambda+1})$ might depend on the theory of V_λ, and thus not be provable from the existence of the elementary embedding alone.

There are several ways one might try to get around the right V problem. One might restrict the question to those cases which are not generically fragile. In the case of the club filter on λ^+, whether the club filter restricted to S_ω is an ultrafilter in $L(V_{\lambda+1})$ is left open by the above theorem. We will give evidence below towards proving that this is indeed the case (see Theorems 3.1.8 and 3.1.11).

Secondly, one could ask the question instead in the $L(V_{\lambda+1})$ of some canonical L-like inner model. Such an analysis would surely be possible and enlightening if such an inner model exists. We do not attempt such an analysis here.

Thirdly, one could modify the question in order to circumvent the right V problem. Such a modification would presumably restrict attention to generically stable objects. An example of such a modification is the generic absoluteness given by $U(j)$-representations (see Theorem 4.5.2).

The following table lists some differences and similarities between $L(\mathbb{R})$ and $L(V_{\lambda+1})$.

$L(\mathbb{R})$ assuming $AD^{L(\mathbb{R})}$	$L(V_{\lambda+1})$ assuming I_0 at λ
Θ is a limit of measurable cardinals (Kechris and Woodin, see Kechris [Kec85])	Θ is a limit of measurable cardinals (Woodin [Woo11])
$\forall \alpha < \Theta$, $P(\alpha) \in L_\Theta(\mathbb{R})$ (Moschovakis [Mos80])	$\forall \alpha < \Theta$, $P(\alpha) \in L_\Theta(V_{\lambda+1})$ (Woodin [Woo11])
the club filter on ω_1 is an ultrafilter (Solovay, see [Kan94])	for all $\beta < \lambda$ regular, there exists a partition $\langle T_\alpha |\, \alpha < \kappa \rangle$ for some $\kappa < \lambda$ of S_β into stationary subsets, such that for each $\alpha < \kappa$, the club filter restricted to T_α is an ultrafilter (Woodin [Woo11]; also see Corollary 3.1.9)
the perfect set property holds (Davis [Dav64])	the λ-splitting perfect set property holds (see Theorem 3.2.3)

1.2 Set theory definitions and conventions

We denote by V_α for α an ordinal the stratification of V according to rank. So $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$ and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for λ a limit. For a transitive set A, $L(A)$ is the constructible hierarchy built on top of A. So $L_0(A) = A$, $L_{\alpha+1}(A) = \mathrm{Def}(L_\alpha(A))$ and $L_\lambda(A) = \bigcup_{\alpha<\lambda} L_\alpha(A)$ for λ a limit. By $L(X, A)$ we mean the constructible hierarchy built on top of A, with X as a predicate. So $L_0(X, A) = A$, $L_{\alpha+1}(X, A) = \mathrm{Def}(L_\alpha(A), X \cap L_\alpha(A))$ and $L_\lambda(X, A) = \bigcup_{\alpha<\lambda} L_\alpha(X, A)$ for λ a limit.

Suppose that M and N are models of a fragment of set theory. Then $j : M \to N$ is an elementary embedding if for all $\phi[x_1, \dots, x_n]$ and $a_1, \dots, a_n \in M$ we have that

$$M \models \phi(a_1, \dots, a_n) \Rightarrow N \models \phi(j(a_1), \dots, j(a_n)).$$

We use the convention that all elementary embeddings are nontrivial, so $j \neq \mathrm{id}$. So $\mathrm{crit}\,(j)$ is the least ordinal α such that $j(\alpha) > \alpha$.

Suppose that $\langle a_i | \, i < \omega \rangle$ is a sequence of sets. Then by $a = \lim_{i\to\omega} a_i$ we mean that

$$a = \{b | \, \exists n \, \forall i \geq n (b \in a_i)\}.$$

So for instance, if each a_i is an ordinal and the sequence converges to an ordinal β, then $\lim_{i\to\omega} a_i = \beta$.

When we say that $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is an elementary embedding we will always be assuming that $\mathrm{crit}\,(j) < \lambda$ unless we specifically say otherwise. The same thing holds for embeddings $V_{\lambda+1} \to V_{\lambda+1}$, $L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ and so on.

When referring to a vector $\vec{j}$, we will almost always be assuming that it is of the form

$$\vec{j} = \langle j_i | \, i < \omega \rangle \, .$$

We will many times use this shorthand without explicitly specifying the indices.

Many times we will say, let $a_n = (j_0 \circ \cdots \circ j_{n-1})(a)$ for $n < \omega$. By this we mean that $a_0 = a$, $a_1 = j_0(a), \dots$.

When referring to elementary embeddings $j, k : V_{\lambda+1} \to V_{\lambda+1}$ we will many times apply $j(k)$, although this strictly does not make sense. By this we mean that we apply $j(k \restriction V_\lambda)$ and then look at the natural extension to an embedding on $V_{\lambda+1}$.

Also note that when referring to an elementary embedding j, it is many times standard in the literature to refer to the iterates of j as j_n. We will globally defy this convention, as we are most concerned with sequences of embeddings which we label as j_n. When we wish to refer to the iterate of j, we will use the notation $j_{(n)}$.

Chapter 2

Inverse Limits

2.1 Basic properties

In this section we introduce the theory of inverse limits. These structures are most readily used for reflecting large cardinal hypotheses of the form: there exists an elementary embedding $L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$. The use of inverse limits in reflecting such large cardinals is originally due to Laver (see Laver [Lav97], [Lav01]).

Suppose there exists an elementary embedding $j : V_\lambda \to V_\lambda$. Then if j extends to an elementary embedding $j^* : V_{\lambda+1} \to V_{\lambda+1}$ we have $j^*(A) = \bigcup_i j(A \cap V_{\lambda_i})$ for $\langle \lambda_i | \, i < \omega \rangle$ any cofinal sequence in λ, as λ is a continuity point. Hence any elementary embedding $V_{\lambda+1} \to V_{\lambda+1}$ can be coded as an element of $V_{\lambda+1}$.

Suppose that $\langle j_i | \, i < \omega \rangle$ is a sequence of elementary embeddings such that the following hold:

1. For all i, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.
2. There exists $\bar{\lambda} < \lambda$ such that $\operatorname{crit} j_0 < \operatorname{crit} j_1 < \cdots < \bar{\lambda}$ and $\lim_{i<\omega} \operatorname{crit} j_i = \bar{\lambda}$.

Then we can form the inverse limit

$$J = j_0 \circ j_1 \circ \cdots : V_{\bar{\lambda}} \to V_\lambda$$

by setting

$$J(a) = \lim_{i \to \omega} (j_0 \circ \cdots \circ j_i)(a)$$

for any $a \in V_{\bar{\lambda}}$.

Claim 2.1.1 (Laver)**.** *$J : V_{\bar{\lambda}} \to V_\lambda$ is elementary.*

Proof. It is enough to see that $\sup_{\alpha<\bar{\lambda}} J(\alpha) = \lambda$. To see this let $J_i = j_i \circ j_{i+1} \circ \cdots$. Set $\alpha_i = \sup_{\alpha<\bar{\lambda}} J_i(\alpha)$. Then we have

$$\alpha_0 \geq \alpha_1 \geq \cdots$$

as $j_i(\alpha_{i+1}) = \alpha_i$ for all $i < \omega$. Let n be large enough so that for all $i \geq n$, $\alpha_n = \alpha_i$. Then we have that $j_n(\alpha_n) = \alpha_n$. But since $\lambda \geq \alpha_n > \operatorname{crit} j_n$, we must have that $\alpha_n = \lambda$. But then $\alpha_0 = \lambda$ since $\alpha_0 = (j_0 \circ \cdots \circ j_{n-1})(\alpha_n)$. $\square$

So since $J : V_{\bar{\lambda}} \to V_\lambda$ is elementary, it can be extended to a Σ_0-embedding

$$J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

by $J(A) = \bigcup_i J(A \cap V_{\bar{\lambda}_i})$ for $\langle \bar{\lambda}_i | \, i < \omega \rangle$ any cofinal sequence in $\bar{\lambda}$. Furthermore by a theorem of Laver [Lav97], if for all i, j_i extends to an elementary embedding $V_{\lambda+1} \to V_{\lambda+1}$, then J^* is elementary. We defer the proof of this theorem to Section 2.2. In fact we will always assume that $J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ is elementary and define the inverse limit of $\langle j_i | \, i < \omega \rangle$ to be $J = J^* : V_{\bar{\lambda}+1} \to V_{\lambda+1}$. But we will sometimes treat J as if it were an element of $V_{\lambda+1}$. We write $\bar{\lambda}_J$ for the unique $\bar{\lambda}$ such that $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$. We will often drop reference to the sequence $\langle j_i | \, i < \omega \rangle$ in our notation when talking about the inverse limit J, though the sequence is not unique for a given inverse limit J (for instance, by simply grouping the embeddings as, say, $J = (j_0 \circ j_1) \circ j_2 \circ \cdots$); it will always be clear from context which embeddings we mean when referring to $\langle j_i | \, i < \omega \rangle$.

Suppose $J = j_0 \circ j_1 \circ \cdots$ is an inverse limit. Then for $i < \omega$ we write $J_i := j_i \circ j_{i+1} \circ \cdots$, the inverse limit obtained by 'chopping off' the first i embeddings. For $i < \omega$ we write

$$J^{(i)} := (j_0 \circ \cdots \circ j_i)(J)$$

and for $n < \omega$,

$$j_n^{(i)} := (j_0 \circ \cdots \circ j_i)(j_n).$$

Then we can rewrite J in the following useful ways:

$$\begin{aligned} J = j_0 \circ j_1 \circ \cdots &= \cdots (j_0 \circ j_1)(j_2) \circ j_0(j_1) \circ j_0 \\ &= \cdots j_2^{(1)} \circ j_1^{(0)} \circ j_0 \end{aligned}$$

and

$$\begin{aligned} J &= j_0 \circ J_1 = j_0(J_1) \circ j_0 = J_1^{(0)} \circ j_0 \\ &= (j_0 \circ \cdots \circ j_{i-1})(J_i) \circ j_0 \circ \cdots \circ j_{i-1} = J_i^{(i-1)} \circ j_0 \circ \cdots \circ j_{i-1} \end{aligned}$$

for any $i > 0$. Hence we can view an inverse limit J as a direct limit (see Figure 2.1), though both perspectives are useful in different situations. We let $\mathcal{E}$ be the set of inverse limits. So

$$\mathcal{E} = \{(J, \langle j_i | \, i < \omega \rangle) | \, J = j_0 \circ j_1 \circ \cdots : V_{\bar{\lambda}_J+1} \to V_{\lambda+1}\}.$$

Lemma 2.1.2. *If $(K, \vec{k}) \in \mathcal{E}$ and $A \in V_{\lambda+1}$ are such that $A \in rng\, K$, then for all i, $A \in rng\,(k_0 \circ \cdots \circ k_i)$.*

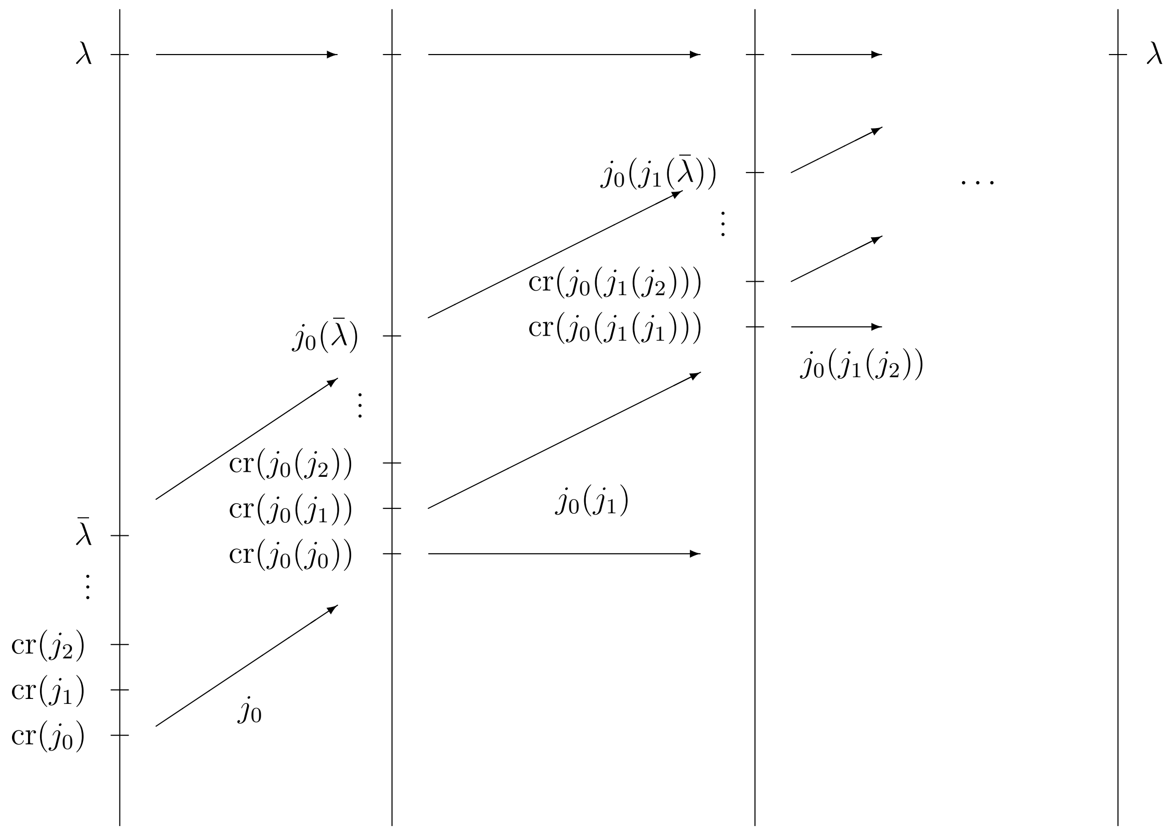

Figure 2.1: Direct limit decomposition of an inverse limit.

Proof. It is enough to see this for any $A \in V_\lambda$. But then there is an $\bar{A}$ and an n such that

$$K(\bar{A}) = (k_0 \circ \cdots \circ k_n)(\bar{A}) = A,$$

and for all $i > n$, $k_i(\bar{A}) = \bar{A}$. Hence for all i we have that $A \in \mathrm{rng}\,(k_0 \circ \cdots \circ k_i)$. □

Suppose $j, k : V_{\lambda+1} \to V_{\lambda+1}$. Then we say k is a *square root of* j if $k(k) = j$ (thinking of k and j as elements of $V_{\lambda+1}$, so actually $k(k \restriction V_\lambda) = j \restriction V_\lambda$). We use the same terminology for $j, k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ where α is good. We have the following 'square root lemma' which says that strength of the embedding gives a large number of square roots. This is the key lemma which takes advantage of the strength of our embeddings, and we will use many variations of it below.

Lemma 2.1.3 (Martin)**.** *Suppose α is good. If $j : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$ is elementary then for all $A, B \in V_{\lambda+1}$ and $\beta < crit(j)$ there exists a $k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that k is a square root of j, $k(A) = j(A)$, $B \in rng\, k$ and $\beta < crit(k) < crit(j)$.*

Proof. Given α, j, A, B, and β as in the hypothesis, we want to show that $L_{\alpha+1}(V_{\lambda+1}) \models \exists k : V_\lambda \to V_\lambda$ which induces $\hat{k} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that

$$\beta < \mathrm{crit}\, k < \mathrm{crit}\, j,\ j(A) = k(A) \text{ and } B \in \mathrm{rng}\,(k).$$

Note that since α is good, an elementary embedding $k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is induced by $k \restriction V_\lambda$. Applying j, this is equivalent to $L_{\alpha+1}(V_{\lambda+1}) \models \exists k : V_\lambda \to V_\lambda$ which induces $\hat{k} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ such that $j(\beta) < \mathrm{crit}\, k < \mathrm{crit}\, j(j)$, $j(j)(j(A)) = \hat{k}(j(A))$ and $j(B) \in \mathrm{rng}\,(\hat{k})$. But $j \restriction V_\lambda$ satisfies this second statement. So we are done by elementarity of j. □

Note that we can replace A and B with any sequence of length less than $\mathrm{crit}\, j$ by coding. We will do so below without any comment.

Define

$$\mathcal{E}_\alpha = \{(J, \vec{j}) \in \mathcal{E} \,|\, \forall i < \omega\, (j_i \text{ extends to an elementary embedding } L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

Lemma 2.1.4 (Laver)**.** *Suppose there exists an elementary embedding*

$$j : L_{\alpha+1}(V_{\lambda+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

where α is good. Then $\mathcal{E}_\alpha \neq \emptyset$.

Proof. Inductively define j_i as follows, repeatedly using Lemma 2.1.3. Let j_0 be such that $\mathrm{crit}\, j_0 < \mathrm{crit}\, j$ and $j_0 : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is elementary. Having chosen

$$j_0, \ldots, j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$$

such that

$$\mathrm{crit}\, j_0 < \mathrm{crit}\, j_1 < \cdots < \mathrm{crit}\, j_i < \mathrm{crit}\, j,$$

let j_{i+1} be such that

$$\operatorname{crit} j_i < \operatorname{crit} j_{i+1} < \operatorname{crit} j$$

and j_{i+1} extends to $j_{i+1} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$.

Then clearly we have that

$$\operatorname{crit} j_0 < \operatorname{crit} j_1 < \cdots < \operatorname{crit} j$$

and hence $\lim_{i\to\omega} \operatorname{crit} j_i = \bar{\lambda} < \lambda$ for some $\bar{\lambda}$. Let $J = j_0 \circ j_1 \circ \cdots$. $\square$

Remark 2.1.5. While the motivation for using inverse limits for reflection is fairly clear, we will find below that inverse limits have a wide array of uses which go far beyond their initial use as reflection embeddings. In fact, for some applications (for instance Theorem 4.2.3) we do not even use inverse limits as embeddings, but rather as a sort of operator. This might seem somewhat bizarre at first, but it is perhaps somewhat clarified by the following alternative definition of a restricted class of inverse limits:

Suppose $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary and $\langle k_i | \, i < \omega \rangle$ is a sequence such that the following hold for all $i < \omega$.

1. $k_i : V_{\lambda+1} \to V_{\lambda+1}$ is elementary.

2. k_i is a square root of j.

3. $k_i \restriction V_\lambda \in \operatorname{rng} k_{i+1}$.

Then we have

$$\operatorname{crit} k_0 < \operatorname{crit} k_1 < \cdots < \operatorname{crit} j < \lambda,$$

and hence for $K = k_0 \circ k_1 \circ \cdots$, $(K, \langle k_i | \, i < \omega \rangle)$ is an inverse limit.

To see this, note for instance that since $k_0 \restriction V_\lambda \in \operatorname{rng} k_1$, that $\operatorname{crit} k_0 \in \operatorname{rng} k_1$. But if $\operatorname{crit} k_1 \leq \operatorname{crit} k_0$ then, since

$$k_1(\operatorname{crit}(k_1)) = \operatorname{crit} j > \operatorname{crit} k_0,$$

we must have $\operatorname{crit} k_0 \notin \operatorname{rng} k_1$, a contradiction. Hence $\operatorname{crit} k_0 < \operatorname{crit} k_1 < \operatorname{crit} j$. And we have $\operatorname{crit} k_0 < \operatorname{crit} k_1 < \cdots < \operatorname{crit} j < \lambda$ similarly by induction.

So an inverse limit is a natural outcome of repeated applications of the square root lemma. In fact this restricted class of inverse limits has many useful properties which we will make use of in Section 4. In light of this fact, we make the following definition:

Definition 2.1.6. Suppose $(K, \vec{k}) \in \mathcal{E}$ and $j : V_{\lambda+1} \to V_{\lambda+1}$ are such that the following hold:

1. For all i, $k_i(k_i \restriction V_\lambda) = j \restriction V_\lambda$.

2. For all i, $k_0 \restriction V_\lambda, \ldots, k_i \restriction V_\lambda \in \operatorname{rng} k_{i+1}$.

Then we say that $(K, \vec{k})$ is an *inverse limit root of* j.

There is a corresponding square root lemma for inverse limits. Suppose

$$(J, \langle j_i \rangle), (K, \langle k_i \rangle) \in \mathcal{E}.$$

Then we say that K *is a limit root of* J if there is $n < \omega$ such that $\bar{\lambda}_J = \bar{\lambda}_K$ and

$$\forall i < n\, (k_i = j_i) \text{ and } \forall i \geq n\, (k_i(k_i) = j_i).$$

We say K *is an (n-close) limit root of* J if n witnesses that K is a limit root of J.

Lemma 2.1.7 (Laver [Lav97])**.** *Suppose α is good. If $(J, \vec{j}) \in \mathcal{E}_{\alpha+1}$ then for all $\bar{A} \in V_{\bar{\lambda}+1}$ and $B \in V_{\lambda+1}$ there exists a $(K, \vec{k}) \in \mathcal{E}_\alpha$ such that K is a limit root of J, $K(\bar{A}) = J(\bar{A})$ and $B \in rng\, K$.*

While Laver's original statement did not include the notion of being a limit root, the proof is identical.

Proof. We use Lemma 2.1.3 ω-many times to $j_0, j_1, \ldots$ in succession. Define $k_0, k_1, \ldots$ by induction as follows. Let $k_0 : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ be given by Lemma 2.1.3 such that $B \in \operatorname{rng} k_0$ and for all i

$$j_0((j_1 \circ \cdots \circ j_i)(\bar{A})) = k_0((j_1 \circ \cdots \circ j_i)(\bar{A})).$$

After defining $k_0, \ldots, k_n$ let $k_{n+1} : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ be given by Lemma 2.1.3 such that

$$(k_0 \circ \cdots \circ k_n)^{-1}(B) \in \operatorname{rng} k_{n+1},$$

$\operatorname{crit} j_n < \operatorname{crit} k_{n+1} < \operatorname{crit} j_{n+1}$ and for all i

$$j_n((j_{n+1} \circ \cdots \circ j_{n+i})(\bar{A})) = k_n((j_{n+1} \circ \cdots \circ j_{n+i})(\bar{A})).$$

A calculation shows that $\operatorname{crit} k_0 < \operatorname{crit} k_1 < \cdots < \bar{\lambda}$, $\lim_{i \to \omega} \operatorname{crit}(k_i) = \bar{\lambda}$, and for

$$K := k_0 \circ k_1 \circ \cdots$$

we have $K(\bar{A}) = J(\bar{A})$ and $B \in \operatorname{rng} K$:

To see that $K(\bar{A}) = J(\bar{A})$, note that it is enough to see that for all $\beta < \bar{\lambda}$, if $\bar{A}' = \bar{A} \cap V_\beta$, then $K(\bar{A}') = J(\bar{A}')$. Let n be large enough so that $\operatorname{crit}(k_n) > \beta$. Then we have that

$$\begin{aligned} J(\bar{A}') &= (j_0 \circ \cdots \circ j_{n-1})(\bar{A}') \\ &= (j_0 \circ \cdots \circ j_{n-2})(k_{n-1}(\bar{A}')) \\ &= (j_0 \circ \cdots \circ j_{n-3})((k_{n-2} \circ k_{n-1})(\bar{A}')) = \cdots \\ &= (k_0 \circ \cdots \circ k_{n-1})(\bar{A}') = K(\bar{A}') \end{aligned}$$

To see that $B \in \operatorname{rng} K$, let $\bar{\kappa}_i = \operatorname{crit} k_i$ and set $\kappa_i = K(\bar{\kappa}_i)$. It is enough to see that for all $i < \omega$, if $B' = B \cap V_{\kappa_i}$, then $B' \in \operatorname{rng} K$. Let $i < \omega$. Then we have that $(k_0 \circ \cdots \circ k_i)^{-1}(B')$ is defined since

$$K(\bar{\kappa}_i) = (k_0 \circ \cdots \circ k_i)(\bar{\kappa}_i).$$

But then we have that

$$K((k_0 \circ \cdots \circ k_i)^{-1}(B')) = B',$$

which is what we wanted. $\square$

A key difference between embeddings for square roots and being a limit root for inverse limits is that if $k(k) = j$ then $\operatorname{crit} k < \operatorname{crit} j$ whereas if K is a limit root of J then $\operatorname{crit} K \leq \operatorname{crit} J$. So while there is no sequence $k_0, k_1, \dots$ such that for all $i < \omega$, $k_{i+1}(k_{i+1}) = k_i$, we have the following lemma for limit roots.

Lemma 2.1.8. *Suppose that α is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Then there exists a sequence $\left\langle (K^i, \vec{k}^i) | \, i < \omega \right\rangle$ such that the following hold:*

1. *$K^0 = J$.*
2. *For all i, $(K^i, \vec{k}^i) \in \mathcal{E}_\alpha$.*
3. *For all i, K^{i+1} is a limit root of K^i.*

Proof. Let $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Set $K^0 = J$, and choose $(K^{m+1}, \vec{k}^{m+1})$ by induction as follows. Suppose that $(K^0, \vec{k}^0), \dots, (K^m, \vec{k}^m)$ have been chosen so that $(K^m, \vec{k}^m) \in \mathcal{E}_\alpha$ and there exists $\langle n_i^m | \, i < \omega \rangle$ such that for all $i < \omega$, $n_i^m < \omega$, k_i^m extends to

$$\hat{k}_i^m : L_{\alpha+n_i^m}(V_{\lambda+1}) \to L_{\alpha+n_i^m}(V_{\lambda+1}),$$

and $\lim_{i\to\omega} n_i^m = \infty$. Let i be large enough so that for all $i' \geq i$, $n_{i'}^m > 0$. Then by the proof of Lemma 2.1.7, there is K^{m+1} which is an i-close limit root of K^m such that for all $i' \geq i$, $k_{i'}^{m+1}$ extends to

$$\hat{k}_{i'}^{m+1} : L_{\alpha+n_{i'}^m-1}(V_{\lambda+1}) \to L_{\alpha+n_{i'}^m-1}(V_{\lambda+1}).$$

We have that

$$\lim_{i\to\omega}(n_i^m - 1) = \infty,$$

and hence we can continue the induction. The sequence we produce $\left\langle (K^i, \vec{k}^i) | \, i < \omega \right\rangle$ clearly satisfies the lemma. $\square$

Of course, if we considered the more restrictive notion of being a 0-close limit root, then such sequences as in Lemma 2.1.8 would indeed be impossible. We will see though that the added benefit afforded by Lemma 2.1.8 will be very useful. As a first example, we obtain sets of inverse limits which are in a sense closed under the square root lemma.

Definition 2.1.9. Suppose $E \subseteq \mathcal{E}$. Then we say that E is *saturated* if for all $(J, \vec{j}) \in E$ there exists an $i < \omega$ such that for all $A \in V_{\bar{\lambda}_J+1}$, and $B \in V_{\lambda+1}$, there exists $(K, \vec{k}) \in E$ such that K is an i-close limit root of J, $K_i(A) = J_i(A)$ and $B \in \operatorname{rng} K_i$. We set $i(E, J) =$ the least such i.

Note that if K is an i-close limit root of J and $K_i(A) = J_i(A)$ then $K(A) = J(A)$. However, we cannot conclude that $B \in \operatorname{rng} K$ if $B \in \operatorname{rng} K_i$. For instance if $i = 1$ then we always have that $\operatorname{crit}(J) = \operatorname{crit}(K) \notin \operatorname{rng} K$.

We will use the same terminology of being saturated for E such that there is α good such that for all $(J, \vec{j}) \in E$ and $i < \omega$, $j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is elementary.

As a corollary to the proof of Lemma 2.1.8 we have:

Corollary 2.1.10. *Suppose that α is good and $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$. Then there exists a saturated set $E \subseteq \mathcal{E}_\alpha$ such that $(J, \vec{j}) \in E$.*

Proof. Let E be the set of all $(K, \vec{k}) \in \mathcal{E}_\alpha$ such that there exists a sequence $\langle n_i | \, i < \omega \rangle$ such that $\lim_{i \to \omega} n_i = \infty$ and for all $i < \omega$, $n_i < \omega$ and k_i extends to

$$\hat{k}_i : L_{\alpha+n_i}(V_{\lambda+1}) \to L_{\alpha+n_i}(V_{\lambda+1}).$$

Since $(J, \vec{j}) \in \mathcal{E}_{\alpha+\omega}$ we must have that $(J, \vec{j}) \in E$. So the lemma follows by the proofs of Lemmas 2.1.7 and 2.1.8. □

Lemma 2.1.11. *Suppose $E \subseteq \mathcal{E}$ is saturated. Let $(J, \vec{j}) \in E$, $\bar{A} \in V_{\bar{\lambda}_J+1}$, and suppose*

$$J(\bar{A}) = A \in V_{\lambda+1}.$$

Set

$$E(\bar{A}, A) = \{(K, \vec{k}) \in E | \, K(\bar{A}) = A\}.$$

Then $E(\bar{A}, A)$ is saturated.

Proof. Suppose $(K, \vec{k}) \in E(\bar{A}, A)$. Then $(K, \vec{k}) \in E$, so there is $i < \omega$ such that for all $C \in V_{\bar{\lambda}_J+1}$ and $B \in V_{\lambda+1}$ there exists $(K', \vec{k}') \in E$, an i-close limit root of K such that $K_i(C) = K'_i(C)$ and $B \in \operatorname{rng} K'_i$. But then i is such that for all $C \in V_{\bar{\lambda}_J+1}$ and $B \in V_{\lambda+1}$ there exists $K' \in E$, an i-close limit root of K such that $K_i(C) = K'_i(C)$, $K'_i(\bar{A}) = K_i(\bar{A}) = A$ and $B \in \operatorname{rng} K'_i$. So $K'(\bar{A}) = K(\bar{A}) = A$ and hence $(K', \vec{k}') \in E(\bar{A}, A)$. Hence $E(\bar{A}, A)$ is saturated. □

Finally note that if $k(k) = j$ and $A \in \operatorname{rng} k$, then $k(A) = j(A)$. To see this suppose $k(B) = A$, and notice

$$k(A) = k(k(B)) = k(k)(k(B)) = j(k(B)) = j(A).$$

We can show a similar property for inverse limits:

Lemma 2.1.12. *Suppose that $(K, \vec{k}), (J, \vec{j}) \in \mathcal{E}$ and K is a limit root of J. Let $\bar{\lambda} = \bar{\lambda}_J$. Suppose $\bar{A} \in V_{\bar{\lambda}}$ and $A = J(\bar{A})$. Then if $A \in rng\, K$, we have*

$$K(\bar{A}) = A = J(\bar{A}).$$

Proof. Let A_n for $n < \omega$ be defined by induction as

$$A_0 = (j_0)^{-1}(A) \text{ and for } n \geq 0,\ A_{n+1} = (j_{n+1})^{-1}(A_n).$$

Then we have (case 1)

$$\begin{aligned}&k_0 \text{ is a squareroot of } j_0 \text{ and } A \in \operatorname{rng} k_0 \cap \operatorname{rng} j_0 \\ &\qquad \Rightarrow A_0 = j_0^{-1}(A) \in \operatorname{rng} k_0 \Rightarrow k_0(A_0) = j_0(A_0) \Rightarrow A_0 \in \operatorname{rng} K_1\end{aligned}$$

and (case 2)

$$k_0 = j_0 \Rightarrow k_0(A_0) = j_0(A_0) \Rightarrow A_0 \in \operatorname{rng} K_1.$$

Similarly, for $n \geq 0$, (case 1)

$$\begin{aligned}&k_{n+1} \text{ is a squareroot of } j_{n+1} \text{ and } A_n \in \operatorname{rng} k_{n+1} \cap \operatorname{rng} j_{n+1} \\ &\qquad \Rightarrow A_{n+1} = j_{n+1}^{-1}(A_n) \in \operatorname{rng} k_{n+1} \Rightarrow k_{n+1}(A_{n+1}) = j_{n+1}(A_{n+1}) \Rightarrow A_{n+1} \in \operatorname{rng} K_{n+2}\end{aligned}$$

and (case 2)

$$k_{n+1} = j_{n+1} \Rightarrow k_{n+1}(A_{n+1}) = j_{n+1}(A_{n+1}) \Rightarrow A_{n+1} \in \operatorname{rng} K_{n+2}.$$

Hence we have that

$$K(\bar{A}) = A = J(\bar{A})$$

as in the proof of Lemma 2.1.7 □

We note the following two lemmas for completeness, although we will not use them. Lemma 2.1.14 provides a slight generalization of Lemma 2.1.3, and it is in fact implied by that lemma in the case that $J \in \operatorname{rng} K$. Lemmas 2.1.12-2.1.14 provide a somewhat complete picture of the agreement between an inverse limit and its limit root. Note that Lemma 2.1.13 displays a strict limitation on the agreement of Lemma 2.1.14. And hence the agreement of Lemma 2.1.12 is in some sense much stronger.

Lemma 2.1.13. *Suppose $(J, \vec{j}) \in \mathcal{E}$ and $A \in V_\lambda$. Then there exists an i such that $A \in rng\, J_i^{(i-1)}$.*

Proof. Let $\alpha < \lambda$ be such that $A \in V_\alpha$. Then since $\left\langle \operatorname{crit} J_i^{(i-1)} \mid i < \omega \right\rangle$ is cofinal in λ, there is an i such that $\alpha < \operatorname{crit} J_i^{(i-1)}$. Clearly then we have that $A \in \operatorname{rng} J_i^{(i-1)}$. □

Lemma 2.1.14. *Suppose that $(K,\vec{k}),(J,\vec{j}) \in \mathcal{E}$, K is a limit root of J and for all i,*

$$k_0 \upharpoonright V_\lambda, \ldots, k_i \upharpoonright V_\lambda \in rng\, k_{i+1}.$$

Let $\bar\lambda = \bar\lambda_0 = \bar\lambda_J$ and

$$\lambda_i = (j_0 \circ \cdots \circ j_{i-1})(\bar\lambda).$$

Suppose $\bar A \in V_{\lambda_0+1}$ and $A = J(\bar A)$. Then if i is such that $\bar A \in rng\, K_i^{(i-1)}$, then

$$K_i^{(i-1)}((j_0 \circ \cdots j_{i-1})(\bar A)) = A = J(\bar A).$$

Proof. Without loss of generality we assume $i = 1$. Then we have that $\bar A \in \operatorname{rng} k_1^{(0)}$. But since $k_0 \in \operatorname{rng} k_1$, we have $j_0 \in \operatorname{rng} k_1^{(0)}$. Hence $j_0(\bar A) \in \operatorname{rng} k_1^{(0)}$. And so since $k_1^{(0)}$ is a square root of $j_1^{(0)}$, we have that

$$k_1^{(0)}(j_0(\bar A)) = j_1^{(0)}(j_0(\bar A)) = (j_0 \circ j_1)(\bar A).$$

And since

$$(k_1^{(0)})^{-1}(\bar A), k_1^{(0)} \in \operatorname{rng} k_2^{(0)}$$

we have $\bar A \in \operatorname{rng} k_2^{(0)}$. Furthermore $k_0 \in \operatorname{rng} k_2$ implies that $j_0 \in \operatorname{rng} k_2^{(0)}$, so we have that $j_0(\bar A) \in \operatorname{rng} k_2^{(0)}$. And hence that

$$k_1^{(0)}(j_0(\bar A)) \in \operatorname{rng} k_1^{(0)}(k_2^{(0)}) = k_2^{(1)}.$$

But this shows that

$$k_2^{(1)}(k_1^{(0)}(j_0(\bar A))) = j_2^{(1)}(k_1^{(0)}(j_0(\bar A))) = j_2^{(1)}(j_1^{(0)}(j_0(\bar A))) = (j_0 \circ j_1 \circ j_2)(\bar A)$$

since $k_2^{(1)}$ is a square root of $j_2^{(1)}$.

Continuing this way we have that

$$(j_0 \circ \cdots \circ j_{i-1})(\bar A) = (k_1^{(0)} \circ \cdots \circ k_{i-1}^{(0)})(j_0(\bar A))$$

for all $i > 0$, which proves the lemma. □

2.1.1 Sequences of inverse limits

We will show in this section that sequences of inverse limit roots have a powerful continuity property. We will use this property many times below. As usual, we often write $\langle K^i |\, i < \omega \rangle$ instead of $\left\langle (K^i, \vec{k}^i) |\, i < \omega \right\rangle$ for a sequence of inverse limits, with the underlying embeddings being understood.

Lemma 2.1.15. *Suppose $\langle K^i |\, i<\omega\rangle$ is such that for all i, K^{i+1} is a limit root of K^i. Then there exists an increasing sequence $\langle i_n |\, n<\omega\rangle$ such that for all $n<\omega$ and $s \geq i_n$, we have that $k_n^s = k_n^{i_n}$.*

Proof. Suppose the lemma does not hold and n is least such that there is no i_n such that for all $s > i_n$, $k_n^s = k_n^{i_n}$. Then there is a sequence $\langle s_i |\, i<\omega\rangle$ such that $\langle k_n^{s_i} |\, i<\omega\rangle$ is such that for all $i>0$ there exists an m such that

$$(k_n^{s_i})_{(m)} = k_n^{s_{i-1}}$$

where we write $j_{(m)}$ for the m-th iterate of an embedding j. But there can be no such sequence since for all $i>0$, $\operatorname{crit}(k_n^{s_i}) < \operatorname{crit}(k_n^{s_{i-1}})$. So the lemma follows. □

For $\langle K^i |\, i<\omega\rangle$ such that there exists $\langle i_n |\, n<\omega\rangle$, an increasing sequence satisfying that for all $n<\omega$ and $s\geq i_n$, $k_n^s = k_n^{i_n}$, we call

$$K = k_0^{i_0} \circ k_1^{i_1} \circ \cdots$$

the *common part of* $\langle K^i |\, i<\omega\rangle$, and

$$\langle i_n |\, n<\omega\rangle$$

a *common part index sequence for* $\langle K^i |\, i<\omega\rangle$.

The following is a key continuity property of inverse limit sequences.

Lemma 2.1.16. *Suppose that for $i<\omega$, $(J^i, \vec{j}^i) \in \mathcal{E}$. And suppose the common part of $\langle J^i |\, i<\omega\rangle$ is K and $\bar\lambda_{J^0} = \bar\lambda_K = \bar\lambda$. Then for all $\bar A \in V_{\bar\lambda+1}$ such that for all i, $J^0(\bar A) = J^i(\bar A)$, we have $K(\bar A) = J^0(\bar A)$.*

Proof. Let $J^0(\bar A) = A$ and let $\langle i_n |\, n<\omega\rangle$ be a common part index sequence for $\langle J^i |\, i<\omega\rangle$. It is enough to show that for cofinally many $\bar\kappa < \bar\lambda$, $K(\bar A \cap V_{\bar\kappa}) = A \cap V_\kappa$, where $\kappa = K(\bar\kappa)$. Let $\bar\kappa<\bar\lambda$, and let $n<\omega$ be least such that $\operatorname{crit}(k_n) > \bar\kappa$. Then we have that

$$K(\bar A \cap V_{\bar\kappa}) = (k_0 \circ \cdots \circ k_{n-1})(\bar A \cap V_{\bar\kappa}).$$

On the other hand, for some $\kappa^* < \lambda$,

$$A \cap V_{\kappa^*} = J^{i_n}(\bar A \cap V_{\bar\kappa}) = (j_0^{i_n} \circ \cdots \circ j_{n-1}^{i_n})(\bar A \cap V_{\bar\kappa}) = (k_0 \circ \cdots \circ k_{n-1})(\bar A \cap V_{\bar\kappa}).$$

And hence $\kappa^* = K(\bar\kappa)$, and $K(\bar A \cap V_{\bar\kappa}) = A \cap V_{\kappa^*}$, as desired. □

It is possible that if K is the common part of $\langle J^i |\, i<\omega\rangle$ then $\bar\lambda_K < \bar\lambda_{J^0}$. To avoid this possibility, we can fix a sequence $\left\langle \bar\lambda_n |\, n<\omega\right\rangle$ cofinal in $\bar\lambda_{J_0}$. Then we add to our requirement on J^{i+1} that for all $m<\omega$ if n is largest such that $\operatorname{crit} j_m^i > \bar\lambda_n$, then $\operatorname{crit} j_m^{i+1} > \bar\lambda_n$. In this case we say that J^{i+1} is *a limit root of J^i, supported by $\left\langle \bar\lambda_n |\, n<\omega\right\rangle$*.

Definition 2.1.17. Suppose $E \subseteq \mathcal{E}$ is a set of inverse limits. Then we let $CL(E)$ be the set

$$CL(E) = \{(K, \vec{k}) \in \mathcal{E} |\, \exists \vec{K}\, (K \text{ is the common part of } \vec{K}, \bar\lambda_K = \bar\lambda_{K^0}, \text{ and}$$
$$\forall i<\omega\, ((K^i, \vec{k}^i) \in E))\}.$$

2.2 Reflecting I_1

In the next several sections we will use inverse limits to prove a number of reflection results. These will culminate in Section 2.5 where we will show that $I_0^{\#}$ reflects I_0. That proof is in fact independent of the proofs in the preceding sections, and the reader can skip to that proof if desired. However the following progression of proofs provides a more gentle introduction to using inverse limits and therefore can help motivate the methods of the final proof.

In this section, as an introduction to the use of inverse limits for reflecting large cardinals, we prove the following theorem of Laver.

Theorem 2.2.1 (Laver). *Suppose $(J, \vec{j})$ is an inverse limit such that for all i,*

$$j_i : V_{\lambda+1} \to V_{\lambda+1}$$

is elementary. Then $J : V_{\bar{\lambda}_J+1} \to V_{\lambda+1}$ is elementary.

The key tool is the following square root lemma.

Lemma 2.2.2 (Martin). *Fix $n > 0$ odd. Suppose that $j : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_n-elementary. Let $a, b \in V_{\lambda+1}$ and $\alpha < crit(j)$. Then there is $k : V_{\lambda+1} \to V_{\lambda+1}$ which is Σ_{n-2}-elementary if $n > 1$ and Σ_0-elementary if $n = 1$ such that $k(a) = j(a)$, $b \in rng\, k$ and $\alpha < crit(k) < crit(j)$.*

Proof. First suppose that $n = 1$. We first note that the elementary embeddings $V_\lambda \to V_\lambda$ are the branches of a tree on V_λ. Furthermore, the elementary embeddings $k : V_\lambda \to V_\lambda$ such that $k(a) = j(a)$, $b \in \operatorname{rng} k$, and $\alpha < \operatorname{crit}(k) < \operatorname{crit}(j)$ are also the branches of a tree T on V_λ. So we just need to show that this tree is not well founded. But we must have that j is an infinite branch of $j(T)$. And hence by the Σ_1-elementarity of j, we have that T must have an infinite branch.

Now suppose that $n > 1$. The main point is that the statement '$k : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_{n-2}' is Π_{n-1} over $V_{\lambda+1}$. Hence the existence of such an embedding k such that $k(a) = j(a)$, $b \in \operatorname{rng} k$ and $\alpha < \operatorname{crit}(k) < \operatorname{crit}(j)$ is Σ_n. So applying j to this statement, we want an embedding k such that

$$k(j(a)) = j(j)(j(a)) = j(j(a)),$$

$j(b) \in \operatorname{rng} k$ and

$$j(\alpha) = \alpha < \operatorname{crit}(k) < \operatorname{crit} j(j).$$

But j is such a witness, so pulling back the statement by j, we are done. □

Lemma 2.2.3 (Martin). *Suppose that $j : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_n elementary for n odd. Then j is Σ_{n+1} elementary.*

Proof. We prove this by induction on n. First suppose $n = 1$. Suppose that $V_{\lambda+1} \models \exists x\, \phi[x, a]$ where ϕ is Π_1. Then we want to show that $V_{\lambda+1} \models \exists x\, \phi[x, j(a)]$. But if x_0 is a witness then we must have that $V_{\lambda+1} \models \phi[x_0, a]$ implies $V_{\lambda+1} \models \phi[j(x_0), j(a)]$ as j is Σ_1-elementary.

For the converse, suppose that

$$V_{\lambda+1} \models \exists x\, \phi[x, j(a)]$$

where ϕ is Π_1. Let x_0 be a witness to this. We have that by Lemma 2.2.2 there is a $k : V_{\lambda+1} \to V_{\lambda+1}$ such that $x_0 \in \operatorname{rng} k$ and $j(a) = k(a)$ and k is Σ_0-elementary. So since ϕ is Π_1,

$$V_{\lambda+1} \models \neg\phi[k^{-1}(x_0), a] \Rightarrow V_{\lambda+1} \models \neg\phi[x_0, j(a)].$$

Hence

$$V_{\lambda+1} \models \phi[x_0, j(a)] \Rightarrow V_{\lambda+1} \models \phi[k^{-1}(x_0), a],$$

and so $V_{\lambda+1} \models \exists x\, \phi[x, a]$ as we wanted.

Assume $n > 1$. Suppose that $V_{\lambda+1} \models \exists x\, \phi[x, a]$ where ϕ is Π_n. Then we want to show that $V_{\lambda+1} \models \exists x\, \phi[x, j(a)]$. But if x_0 is a witness then we must have that $V_{\lambda+1} \models \phi[x_0, a]$ implies $V_{\lambda+1} \models \phi[j(x_0), j(a)]$ as j is Σ_n-elementary.

For the converse, suppose that $V_{\lambda+1} \models \exists x\, \phi[x, j(a)]$ where ϕ is Π_n. Let x_0 be a witness to this. We have that there is a $k : V_{\lambda+1} \to V_{\lambda+1}$ such that $x_0 \in \operatorname{rng} k$ and $j(a) = k(a)$ and k is Σ_{n-2}-elementary. So by induction k is Σ_{n-1}-elementary. So since ϕ is Π_n,

$$V_{\lambda+1} \models \phi[x_0, j(a)] \Rightarrow V_{\lambda+1} \models \phi[k^{-1}(x_0), a].$$

So $V_{\lambda+1} \models \exists x\, \phi[x, a]$ as we wanted. □

We now show that inverse limits have a corresponding square root property.

Lemma 2.2.4 (Laver)**.** *Suppose $(J, \vec{j})$ is an inverse limit such that for all i, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_{n+2}-elementary. Then for all $a \in V_{\bar{\lambda}_J+1}$ and $b \in V_{\lambda+1}$ there is a $(K, \vec{k})$ such that for all i, $k_i : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_n-elementary, $\bar{\lambda}_J = \bar{\lambda}_K$, $J(a) = K(a)$ and $b \in rng\, K$.*

Proof. Choose a sequence k_i, b_i for $i < \omega$ such that the following hold:

1. $k_0(b_0) = b$.

2. For all $i < \omega$, $k_{i+1}(b_{i+1}) = b_i$.

3. For all $i < n < \omega$, $k_i((j_{i+1} \circ \cdots \circ j_{n-1})(a)) = j_i((j_{i+1} \circ \cdots \circ j_{n-1})(a))$.

4. For all $i < \omega$, $\operatorname{crit}(j_i) < \operatorname{crit}(k_{i+1}) < \operatorname{crit}(j_{i+1})$.

5. For all $i < \omega$, $k_i : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_n-elementary.

We claim that $(K, \vec{k})$ witnesses the lemma holds. We have by construction that $(K, \vec{k})$ is an inverse limit and $\bar{\lambda}_J = \bar{\lambda}_K$. We want to show that $K(a) = J(a)$ and $b \in \operatorname{rng} K$. To see that $K(a) = J(a)$, it is enough to show that for all $\kappa < \bar{\lambda}$, that $K(a \cap V_\kappa) = J(a \cap V_\kappa)$. Let $i < \omega$ be large enough so that $\operatorname{crit}(j_i) > \operatorname{crit}(k_i) > \kappa$. Let $a^* = a \cap V_\kappa$. Then we have that

$$\begin{aligned} J(a^*) &= (j_0 \circ \cdots \circ j_{i-1})(a^*) = k_0((j_1 \circ \cdots \circ j_{i-1})(a^*)) = k_0(k_1((j_2 \circ \cdots \circ j_{i-1})(a^*))) = \cdots \\ &= (k_0 \circ \cdots \circ k_{i-1})(a^*) = K(a^*). \end{aligned}$$

Now we want to see that $b \in \operatorname{rng} K$. In fact, let $\bar{b} = \lim_{i \to \omega} b_i$. We show that $K(\bar{b}) = b$. It is enough to show that for all $\kappa < \bar{\lambda}$ that $K(\bar{b} \cap V_\kappa) = b \cap V_{K(\kappa)}$. Let $b^* = b \cap V_\kappa$. Let $i < \omega$ be large enough so that $\operatorname{crit}(k_i) > \kappa$. Then we have that

$$K(b^*) = (k_0 \circ \cdots \circ k_{i-1})(b^*) = (k_0 \circ \cdots \circ k_{i-1})(b_{i-1} \cap V_\kappa) = b \cap V_{K(\kappa)}$$

which is what we wanted. Hence $(K, \vec{k})$ witnesses the lemma. $\square$

We finally prove the following by induction, from which Theorem 2.2.1 follows immediately.

Theorem 2.2.5 (Laver)**.** *Fix $n < \omega$. Suppose $(J, \vec{j})$ is an inverse limit such that for all i, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_n-elementary. Then $J : V_{\bar{\lambda}_J+1} \to V_{\lambda+1}$ is Σ_n-elementary.*

Proof. The case $n = 0$ is immediate from the fact that $\sup_{\alpha<\bar{\lambda}} J(\alpha) = \lambda$. Assuming the theorem for $n-1$ we prove it for n.

So suppose $(J, \vec{j})$ is an inverse limit such that for all i, $j_i : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_n-elementary. Then $J : V_{\bar{\lambda}_J+1} \to V_{\lambda+1}$ is Σ_{n-1}-elementary by induction and preserves Σ_n statements upwards. We show that it preserves Σ_n statements downwards.

Suppose that $V_{\lambda+1} \models \exists x\, \phi(x, J(a))$ where ϕ is Π_{n-1}. Let x_0 be a witness, and let $(K, \vec{k})$ be such that $K(a) = J(a)$, $x_0 \in \operatorname{rng} K$, $\bar{\lambda}_J = \bar{\lambda}_K$, and for all i, $k_i : V_{\lambda+1} \to V_{\lambda+1}$ is Σ_{n-2} (or Σ_0 if $n = 1$). Then K preserves Π_{n-1} statements downwards, and hence

$$V_{\lambda+1} \models \phi(x_0, J(a)) \Rightarrow V_{\lambda+1} \models \phi(K^{-1}(x_0), a).$$

So we have that J is Σ_n as desired. $\square$

We can now use the proof of Theorem 2.2.1 to reflect the axiom I_1.

Theorem 2.2.6 (Laver)**.** *Suppose there exists an elementary embedding $j : L_1(V_{\lambda+1}) \to L_1(V_{\lambda+1})$. Then there exists a $\bar{\lambda} < \lambda$ and an elementary embedding $k : V_{\bar{\lambda}+1} \to V_{\bar{\lambda}+1}$.*

Proof. By the proofs of Lemmas 2.1.4 and 2.1.7 there exists $(J, \vec{j}) \in \mathcal{E}$ such that $J(\bar{j}) = j \upharpoonright V_\lambda$. Furthermore by Theorem 2.2.1, $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ is elementary. By elementarity, we have that $\bar{j} : V_{\bar{\lambda}} \to V_{\bar{\lambda}}$ is elementary and that $\bar{j}$ extends to a Σ_0-elementary embedding $\bar{j}^* : V_{\bar{\lambda}+1} \to V_{\bar{\lambda}+1}$. We show that $\bar{j}^*$ is fully elementary.

To see this this suppose that ϕ is a Σ_n-formula and $\bar{a} \in V_{\bar{\lambda}+1}$. We wish to show that $V_{\bar{\lambda}+1} \models \phi[\bar{a}] \iff V_{\bar{\lambda}+1} \models \phi[\bar{j}^*(\bar{a})]$. But we have that

$$V_{\lambda+1} \models \phi[J(\bar{a})] \iff V_{\lambda+1} \models \phi[j(J(\bar{a}))] \iff V_{\lambda+1} \models \phi[J(\bar{j}^*(\bar{a}))].$$

So by elementarity of J we have

$$V_{\bar{\lambda}+1} \models \phi[\bar{a}] \iff V_{\bar{\lambda}+1} \models \phi[\bar{j}^*(\bar{a})].$$

And hence $\bar{j}^*$ is elementary as desired. □

2.3 Reflecting below the least admissible

Laver in fact extended Theorem 2.2.1 significantly beyond reflecting I_1. For instance he showed the following.

Theorem 2.3.1 (Laver)**.** *Suppose that $(J, \vec{j}) \in \mathcal{E}_{\lambda^+ + \omega}$. Then $J : V_{\bar{\lambda}+1} \to V_{\lambda+1}$ extends to an elementary embedding*

$$\hat{J} : L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1}) \to L_{\lambda^+}(V_{\lambda+1}).$$

The proof uses projective prewellorderings to code how the extension should map ordinals, and so the result extends in fact to the sup of the projective prewellorderings of $V_{\lambda+1}$. We do not present this proof, and instead we prove a similar result up to the least admissible, a point which is strictly beyond the sup of the projective prewellorderings. Instead of using prewellorderings to guide the extension on ordinals, we use the failure of admissibility and the corresponding local surjections of $V_{\lambda+1}$ onto the levels of $L(V_{\lambda+1})$ as a guide.

The following is an obvious generalization of the technique of Martin used above.

Lemma 2.3.2. *Suppose that $\bar{M}$ and M are transitive sets, $\bar{X} \subseteq \bar{M}$ and $X \subseteq M$. Suppose $\langle E_i | \, i < \omega \rangle$ is such that*

1. *For $i < \omega$, $E_i \supseteq E_{i+1}$.*

2. *For all i, for all $j \in E_i$, $j : (\bar{M}, \bar{X}) \to (M, X)$ is a Σ_0-elementary embedding.*

3. *For all i and n, $j \in E_{i+1}$, $\bar{a}_1, \ldots, \bar{a}_n \in \bar{M}$, $b_1, \ldots, b_n \in M$, there exists $k \in E_i$ such that for all m, $1 \leq m \leq n$,*

$$j(\bar{a}_m) = k(\bar{a}_m) \text{ and } b_m \in rng(k).$$

Then for all i, if $j \in E_i$, $j : (\bar{M}, \bar{X}) \to (M, X)$ is Σ_i elementary. And hence for $j \in \bigcap_i E_i$, j is elementary.

Proof. We prove the lemma by induction. The case $i = 0$ is by assumption. Assume it true for i. We prove it for $i+1$. Let $j \in E_{i+1}$. We want to show that j is Σ_{i+1}-elementary. Suppose that $M \models \exists x\, \phi(x, j(\vec{a}), X)$ for $\vec{a} \in \bar{M}$ and ϕ a Π_i-formula. Let x_0 be a witness, and let $k \in E_i$ be such that $j(\vec{a}) = k(\vec{a})$ and $x_0 \in \operatorname{rng}(k)$, say $k(\bar{x}_0) = x_0$. Then we have

$$(M, X) \models \phi(x_0, j(\vec{a}), X) \Rightarrow (\bar{M}, \bar{X}) \models \phi(\bar{x}_0, \vec{a}, \bar{X}),$$

using that by induction k is Σ_i-elementary. Hence $(\bar{M}, \bar{X}) \models \exists x\, \phi(x, \vec{a}, \bar{X})$.

Now for the opposite direction assume that $(\bar{M}, \bar{X}) \models \exists x\, \phi(x, \vec{a}, \bar{X})$. Then if $\bar{x}_0 \in \bar{M}$ is a witness, we have that since j is Σ_i,

$$(\bar{M}, \bar{X}) \models \phi(\bar{x}_0, \vec{a}, \bar{X}) \Rightarrow (M, X) \models \phi(j(\bar{x}_0), j(\vec{a}), X).$$

So $(M, X) \models \exists x\, \phi(x, j(\vec{a}), X)$.

Hence j is Σ_{i+1} elementary, and by induction we are done. □

The following is a slight generalization of the technique of Laver used above.

Lemma 2.3.3. *Suppose that $\bar{M}$ and M are transitive sets, $\bar{X} \subseteq \bar{M}$ and $X \subseteq M$. Suppose $\langle E_i |\, i < \omega \rangle$ is such that*

1. *For all i, for all $(J, \langle j_i |\, i < \omega \rangle) \in E_i$, $J : (\bar{M}, \bar{X}) \to (M, X)$ is a Σ_0-elementary embedding, and for all i, $j_i : (M, X_{i+1}) \to (M, X_i)$ is an elementary embedding, where for $i < \omega$, $X_i \subseteq M$ and $X_0 = X$. And for all i there exists $J_i : (\bar{M}, \bar{X}) \to (M, X_i)$ such that*
$$J = j_0 \circ \cdots \circ j_i \circ J_{i+1}.$$
2. *For all i and n_0, $(J, \langle j_i |\, i < \omega \rangle) \in E_{i+1}$, $\bar{a}_0, \ldots, \bar{a}_{n_0} \in \bar{M}$, $b = \langle b_n |\, n < \omega \rangle$ such that $b_n \in M$ for $n < \omega$, there exists $(K, \langle k_i |\, i < \omega \rangle) \in E_i$ and $i_0 < \omega$ such that for all m, $1 \leq m \leq n_0$,*
$$J_{i_0}(\bar{a}_m) = K_{i_0}(\bar{a}_m) \text{ and } \forall n < \omega\, (b_n \in rng\,(K_{i_0})).$$

Then for all i, if $(J, \langle j_i |\, i < \omega \rangle) \in E_i$, then $J : (\bar{M}, \bar{X}) \to (M, X)$ is Σ_i elementary. And hence for $(J, \langle j_i |\, i < \omega \rangle) \in \bigcap_i E_i$, J is elementary.

Proof. We prove the lemma by induction in a similar fashion as the proof of Lemma 2.3.2. The case $i = 0$ is by assumption. Assume it true for i. We prove it for $i+1$. Let $(J, \langle j_i \rangle) \in E_{i+1}$. We want to show that J is Σ_{i+1}-elementary. Suppose that $M \models \exists x\, \phi(x, J(\vec{a}), X)$ for $\vec{a} \in \bar{M}$ and ϕ a Π_i formula. Then for all i, we have that

$$M \models \exists x\, \phi(x, J_i(\vec{a}), X_i).$$

For $i < \omega$, let x_i be a witness to the formula with parameters $J_i(\vec{a})$ and X_i. Let $(K, \langle k_i \rangle) \in E_i$ be such that for some $i_0 < \omega$, $J_{i_0}(\vec{a}) = K_{i_0}(\vec{a})$ and $x_i \in \operatorname{rng}(K_{i_0})$ for all $i < \omega$, say $K_{i_0}(\bar{x}_i) = x_i$. Then we have

$$(M, X) \models \phi(x_{i_0}, J_{i_0}(\vec{a}), X_{i_0}) \Rightarrow (\bar{M}, \bar{X}) \models \phi(\bar{x}_{i_0}, \vec{a}, \bar{X}),$$

using that by induction K is Σ_i. Hence $(\bar{M}, \bar{X}) \models \exists x\, \phi(x, \vec{a}, \bar{X})$.

The opposite direction is exactly same is in the proof of Lemma 2.3.2. Hence J is Σ_{i+1} elementary, and by induction we are done. □

We will mostly be using Jensen's J-hierarchy to stratify $L(V_{\lambda+1})$. The above results hold for the J-hierarchy with the same proofs, and we will use these analogous results without comment. We will also use the following stronger notion of goodness.

Definition 2.3.4. Call an ordinal α *strongly good* if it is *not* the case that

$$J_\alpha(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1} \cup \{V_{\lambda+1}\})} J_{\alpha+1}(V_{\lambda+1}).$$

For α strongly good, $A \in V_{\lambda+1}$ and ϕ a Σ_1 formula, say that (A, ϕ) λ-*tags* α if α is least such that

$$J_{\alpha+1}(V_{\lambda+1}) \models \phi[A, V_{\lambda+1}].$$

Much of the time it will be easier to work with structures of the form $(V_{\lambda+1}, X)$ for $X \subseteq V_{\lambda+1}$. Along these lines, we make the following definition.

Definition 2.3.5. Suppose that $X \subseteq V_{\lambda+1}$. Let $\mathcal{E}(X)$ be the set

$$\begin{aligned}\mathcal{E}(X) := \{(J, \vec{j}) |\, & \forall i < \omega\, (j_i : (V_{\lambda+1}, X) \to (V_{\lambda+1}, X) \text{ is elementary}),\\ & \exists \bar{\lambda} < \lambda, \bar{X} \in V_{\bar{\lambda}+1}\, (\text{crit}\,(j_0) < \text{crit}\,(j_1) < \cdots < \bar{\lambda},\\ & \lim_{i \to \omega} \text{crit}\,(j_i) = \bar{\lambda},\ J = j_0 \circ j_1 \circ \cdots,\ \text{and}\\ & J : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X) \text{ is } \Sigma_0\text{-elementary})\}.\end{aligned}$$

We use the terminology *limit root*, *saturated*, etc. with the obvious definitions for $\mathcal{E}(X)$.

Lemma 2.3.6. *Suppose $X \subseteq V_{\lambda+1}$ and $E \subseteq \mathcal{E}(X)$ is saturated. Suppose that $\bar{X} \subseteq V_{\bar{\lambda}+1}, X \subseteq V_{\lambda+1}$, $A \in V_{\lambda+1}$ and $\bar{A} \in V_{\bar{\lambda}+1}$ are such that for all $(J, \vec{j}) \in E$ with $J(\bar{A}) = A$ we have*

$$J : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is Σ_0-elementary. Furthermore assume that for all $(J, \vec{j}) \in E$ and $i < \omega$, j_i extends to an elementary embedding $J_1(V_{\lambda+1}, X) \to (V_{\lambda+1}, X)$. Then there is a unique extension $\hat{J} : J_1(V_{\bar{\lambda}+1}, \bar{X}) \to J_1(V_{\lambda+1}, X)$ which is elementary. And hence if $\kappa < \Theta_\lambda$, $\alpha < \kappa, \bar{\alpha}$ and ϕ are such that (A, ϕ) λ-tags α, and such that X and $\bar{X}$ code $J_\alpha(V_{\lambda+1})$ and $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ respectively, then $(\bar{A}, \phi)$ $\bar{\lambda}$-tags $\bar{\alpha}$.

Proof. This follows from Lemma 2.3.3 with $E_i = E$ for all i. The Lemma immediately gives us elementarity of the maps $J : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$ for $J \in E$. Hence we have that $J \in E$ extends uniquely to $\hat{J} : J_1(V_{\bar{\lambda}+1}, \bar{X}) \to J_1(V_{\lambda+1}, X)$ is Σ_0. We similarly get by induction that $J \in E$ extends to $J_n(V_{\bar{\lambda}+1}, \bar{X}) \to J_n(V_{\lambda+1}, X)$ for any $n < \omega$. So the rest of the lemma follows. □

Remark 2.3.7. At first glance it might seem as though Lemma 2.3.6 gives a straightforward generalization of Theorem 2.2.1 to any $X \subseteq V_{\lambda+1}$, showing that for any $(J, \vec{j}) \in \mathcal{E}(X)$, if there exists a saturated $E \subseteq \mathcal{E}(X)$ with $(J, \vec{j}) \in E$, then there exists an $\bar{X} \subseteq V_{\bar{\lambda}_J+1}$ such that

$$J : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is elementary. This is not the case, however, with the key point being that the hypothesis of Lemma 2.3.6 requires a *fixed* $\bar{X}$ for all elements of E. In fact almost all of the reflection proofs below boil down to fixing this $\bar{X}$. Furthermore, we will see that such a general X-reflection result does not hold (see Corollary 2.5.11).

Recall an ordinal α is *admissible relative to* $V_{\lambda+1}$ if α is a limit and $J_\alpha(V_{\lambda+1})$ satisfies Σ_0-collection. We use the following standard property of admissibility:

Lemma 2.3.8. *Suppose κ is the least admissible relative to $V_{\lambda+1}$ and that $\alpha < \kappa$. Then there exists a surjection $\sigma : a \to J_\alpha(V_{\lambda+1})$ such that $a \subseteq V_{\lambda+1}$, $a \in J_\alpha(V_{\lambda+1})$, and σ is Σ_1-definable over $J_\alpha(V_{\lambda+1})$ from some parameter $B \in V_{\lambda+1}$.*

Sketch. We prove this by induction on α. Assume that for all $\beta < \alpha$ there is a surjection $\sigma : b \to J_\beta(V_{\lambda+1})$ such that $b \in J_\beta(V_{\lambda+1})$, $b \subseteq V_{\lambda+1}$, and σ is Σ_1-definable over $J_\beta(V_{\lambda+1})$ from some parameter $B \in V_{\lambda+1}$. Since α is not admissible there is a cofinal map $\sigma : a \to \alpha$ Σ_0-definable over $J_\alpha(V_{\lambda+1})$ where $a \in J_\alpha(V_{\lambda+1})$. We can assume that $a \subseteq V_{\lambda+1}$ by induction. But then we can define a surjection onto $J_\alpha(V_{\lambda+1})$ from

$$\bigcup_{x \in a, B \in V_{\lambda+1}} \{x\} \times \{B\} \times a_{\sigma(x),B} \to J_\alpha(V_{\lambda+1})$$

by (x, B, C) maps to $\tau_{B,\sigma(x)}(C)$ where $\tau_{B,\beta}$ is the least Σ_1-definable from B surjection of an element $a_{\beta,B} \in J_\beta(V_{\lambda+1})$ onto $J_\beta(V_{\lambda+1})$ such that $a_{\beta,B} \subseteq V_{\lambda+1}$. □

For $A \in V_{\lambda+1}$ and ϕ a Σ_1 formula, we say that (A, ϕ) λ-tags α if α is least such that $J_{\alpha+1}(V_{\lambda+1}) \models \phi(A)$.

Theorem 2.3.9. *Let κ be the least admissible relative to $V_{\lambda+1}$. Suppose that there exists an elementary embedding*

$$j : J_{\kappa+1}(V_{\lambda+1}) \to J_{\kappa+1}(V_{\lambda+1}).$$

Then there exists $\bar{\lambda} < \lambda$ such that for all $\alpha < \kappa$ there exists $\bar{\alpha} < \lambda$ and an elementary embedding

$$J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

Proof. First using the fact that there is an elementary embedding

$$j : J_{\kappa+1}(V_{\lambda+1}) \to J_{\kappa+1}(V_{\lambda+1}),$$

there must be a sequence of elementary embeddings $\langle j_i : i < \omega\rangle$ such that for all i,

$$j_i : J_\kappa(V_{\lambda+1}) \to J_\kappa(V_{\lambda+1}),$$

and such that

$$\operatorname{crit}(j_0) < \operatorname{crit}(j_1) < \cdots < \operatorname{crit}(j) < \lambda.$$

Hence letting $\bar\lambda = \sup_{i<\omega} \operatorname{crit}(j_i)$, we have that $\bar\lambda < \lambda$. We fix this $\bar\lambda$. Let

$$\begin{aligned}\mathcal{E}^e = \{(K, \langle k_i | \, i < \omega\rangle) | \, \forall i \, \exists \alpha_i \, (\alpha_i \text{ is good and } k_i : J_{\alpha_i}(V_{\lambda+1}) \to J_{\alpha_i}(V_{\lambda+1})),\\ \operatorname{crit}(k_0) < \operatorname{crit}(k_1) < \cdots \uparrow \bar\lambda, \text{ and } K = k_0 \circ k_1 \circ \cdots\}.\end{aligned}$$

And let for β an ordinal,

$$\mathcal{E}^e_{\geq\beta} = \{(K, \langle k_i | \, i < \omega\rangle) \in \mathcal{E}^e | \, \forall i \, \exists \gamma_i \geq \beta \, (\gamma_i \text{ is good and } k_i : J_{\gamma_i}(V_{\lambda+1}) \to J_{\gamma_i}(V_{\lambda+1}))\},$$

and

$$\mathcal{E}^e_{\beta} = \{(K, \langle k_i | \, i < \omega\rangle) \in \mathcal{E}^e | \, \forall i \, (j_i(\beta) = \beta \text{ or } k_i : J_\beta(V_{\lambda+1}) \to J_\beta(V_{\lambda+1}))\}.$$

We define by induction for each $(J, \langle j_i | \, i < \omega\rangle) \in \mathcal{E}^e$, a function π_J on ordinals (we omit the $\langle j_i | \, i < \omega\rangle$ from the notation, though it does depend on $\langle j_i\rangle$).

Let $\Phi(\alpha)$ be the statement that for all $\alpha' < \alpha$ we have:

> For all $(J, \langle j_i | \, i < \omega\rangle) \in \mathcal{E}^e_{\geq\omega\cdot\alpha'+\omega+1}$ if (A, ϕ) λ-tags α' and $J(\bar A) = A$ for some $\bar A \in V_{\bar\lambda+1}$, then there exists $\bar\alpha$ such that $(\bar A, \phi)$ $\bar\lambda$-tags $\bar\alpha$, $\pi_J(\bar\alpha) = \alpha'$, and $\pi_J \restriction \bar\alpha$ extends to
> $$\hat J : J_{\bar\alpha}(V_{\bar\lambda+1}) \to J_{\alpha'}(V_{\lambda+1})$$
> which is elementary.

Now suppose that $\Phi(\alpha)$ holds and $\alpha < \kappa$ is a limit. Then we extend the definition of π and prove $\Phi(\alpha+1)$. Suppose that $(J, \langle j_i\rangle) \in \mathcal{E}^e_{\geq\omega\cdot\alpha+\omega+1}$, (A, ϕ) λ-tags α and $J(\bar A) = A$ for some $\bar A \in V_{\bar\lambda+1}$. There are two cases: either for all i, $j_i(\alpha) = \alpha$, or not:

First assume that this is not the case, so there exists an i such that $j_i(\alpha) > \alpha$. Then we have for

$$A_i := (j_0 \circ \cdots \circ j_i)^{-1}(A), \; \alpha_i := (j_0 \circ \cdots \circ j_i)^{-1}(\alpha)$$

that for all i, (A_i, ϕ) λ-tags α_i (note that $\alpha \in \operatorname{rng}(j_0 \circ \cdots \circ j_i)$ since it is definable from A_i in $J_\beta(V_{\lambda+1})$ for any $\beta > \alpha$). But then by assumption there is some i such that $\alpha_i < \alpha$. So by our induction hypothesis applied to J_{i+1} we have that $\pi_{J_{i+1}}(\bar\alpha) = \alpha_i$ for some $\bar\alpha$ such that $(\bar A, \phi)$ $\bar\lambda$-tags $\bar\alpha$. And so setting $\pi_J(\bar\alpha) = (j_0 \circ \cdots \circ j_i \circ \pi_{J_{i+1}})(\bar\alpha)$, we are done by the elementarity of $j_0, \ldots, j_i$.

Now assume that for all i, $j_i(\alpha) = \alpha$. Then $(J, \langle j_i | \, i < \omega\rangle) \in \mathcal{E}^e_{\omega\cdot\alpha+\omega+1}$. Recall that (A, ϕ) λ-tags α. Let $p \in J_{\alpha+1}(V_{\lambda+1})$ be a witness to ϕ with parameter A. Let $s \in J_\alpha(V_{\lambda+1})$ be such that p is definable over $J_\alpha(V_{\lambda+1})$ from s.

Since $\alpha < \kappa$ there is a surjection $\sigma : a \to J_\alpha(V_{\lambda+1})$ such that $a \subseteq V_{\lambda+1}$, $a \in J_\alpha(V_{\lambda+1})$ and σ is $\Sigma_1(J_\alpha(V_{\lambda+1}))$ from some parameter $b \in V_{\lambda+1}$. Let $\alpha' < \alpha$ be such that a, b, and s are definable over $J_{\alpha'}(V_{\lambda+1})$ from formulas ψ_1, ψ_2, ψ_3 and parameters $B_1, B_2, B_3 \in V_{\lambda+1}$, respectively. Let (A', ϕ') λ-tag α'. By Lemma 2.1.7 there is $K \in \mathcal{E}^e_{\omega\cdot\alpha+\omega}$ such that

$$K(\bar{A}) = J(\bar{A}),\ K(\bar{B}_1) = B_1,\ K(\bar{B}_2) = B_2,\ K(\bar{B}_3) = B_3,\ K(\bar{A}') = A',$$

for some $\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{A}' \in V_{\bar{\lambda}+1}$. Let $\bar{\alpha}'$ be the ordinal which (by induction) is $\bar{\lambda}$-tagged by $(\bar{A}', \phi')$. Let $\bar{a}$, $\bar{b}$, and $\bar{s}$ be defined the same as a, b, and s but with parameters $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ and over $J_{\bar{\alpha}'}(V_{\bar{\lambda}+1})$. Let $\bar{\sigma}$ have the same definition (over $L(V_{\bar{\lambda}+1})$) as σ (over $L(V_{\lambda+1})$) but with parameter $\bar{b}$. Then (we will prove) there is an $\bar{\alpha}$ such that $\bar{\sigma}$ is a surjection of $\bar{a}$ onto $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. Define $\pi_J(\bar{\alpha}) = \alpha$.

Let $\bar{C} \in V_{\bar{\lambda}+1}$ code $(\bar{A}, \bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{A}')$.

Claim 2.3.10. *There is some $\bar{\alpha}$ such that $\bar{\sigma}$ is a surjection of $\bar{a}$ onto $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$.*

Proof. Now let $\bar{\alpha}$ be least such that $\bar{\sigma}''\bar{a} \subseteq J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. We claim that $\bar{\sigma}$ is a surjection onto $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. Suppose that $\bar{s} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. We want to show that $\bar{s} \in \operatorname{rng}\bar{\sigma}$. Let $\bar{\delta} < \bar{\alpha}$ be large enough so that $\bar{s} \in J_{\bar{\delta}}(V_{\bar{\lambda}+1})$, and let $\bar{t} \in J_{\bar{\delta}}(V_{\bar{\lambda}+1})$ be such that for some $\bar{c} \in V_{\bar{\lambda}+1}$, $\bar{\sigma}(\bar{c}) = \bar{t}$ and assume $\bar{\delta}$ is least such that this is true. Set $t = \sigma(K(\bar{c}))$, and let $\beta < \alpha$ be big enough so that $(\sigma)^{J_\beta(V_{\lambda+1})}(K(\bar{c})) = t$. Suppose (B', ψ') λ-tags β. By Lemma 2.1.7 there is a $K' \in \mathcal{E}^e_{\omega\cdot\alpha+1}$ such that

$$K'(\bar{C}) = K(\bar{C}),\ K'(\bar{c}) = K(\bar{c}),\ \hat{K}'(\bar{t}', \bar{B}') = (t, B')$$

for some $\bar{t}'$ and $\bar{B}'$, where by induction

$$\hat{K}' : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_\beta(V_{\lambda+1})$$

is elementary (to get t in the range of $\hat{K}'$, we use Lemma 2.1.7 to get a parameter in $V_{\lambda+1}$ from which s is definable over $J_\beta(V_{\lambda+1})$ into the range, and then define $\bar{s}'$ analogously over $J_{\bar{\beta}}(V_{\bar{\lambda}+1})$ and use the elementarity of $\hat{K}'$ to show that $\hat{K}'(\bar{s}') = s$. We will omit this type of argument in the future without much comment). And by elementarity we must have that $\bar{t}' = \bar{t}$, since $(\sigma)^{J_\beta(V_{\lambda+1})}(K(\bar{c})) = t$ implies

$$\hat{K}'(\bar{t}') = t = (\sigma(K'(\bar{c})))^{J_\beta(V_{\lambda+1})} = \hat{K}'((\bar{\sigma}(\bar{c}))^{J_{\bar{\beta}}(V_{\bar{\lambda}+1})}) = \hat{K}'(\bar{t}),$$

the last equality following from the fact that since $\bar{\sigma}(\bar{c})$ is defined in $J_{\bar{\beta}}(V_{\bar{\lambda}+1})$ (by elementarity), it must be $\bar{t}$, since $\bar{\sigma}$ is Σ_1-definable. But then by elementarity we have that

$\bar{\delta} \in \mathrm{dom}(\hat{K}')$, and hence that $\bar{s} \in \mathrm{dom}(\hat{K}')$. Now let $\beta_1 < \alpha$ be such that $\beta_1 \geq \beta$ and $\hat{K}'(\bar{s}) \in \mathrm{rng}\,(\sigma)^{J_{\beta_1}(V_{\lambda+1})}$. Then as above we can get $K'' \in \mathcal{E}^e_{\omega\cdot\alpha}$ such that

$$\hat{K}'' : J_{\bar{\beta}_1}(V_{\bar{\lambda}+1}) \to J_{\beta_1}(V_{\lambda+1})$$

is elementary,

$$\hat{K}''(\bar{s}) = \hat{K}'(\bar{s}) \text{ and } K''(\bar{C},\bar{c}) = K(\bar{C},\bar{c}).$$

But since $\hat{K}'(\bar{s}) \in \mathrm{rng}\,(\sigma)^{J_{\beta_1}(V_{\lambda+1})}$ we have that $\bar{s} \in \mathrm{rng}\,(\bar{\sigma})^{J_{\bar{\beta}_1}(V_{\bar{\lambda}+1})}$, which is what we wanted.

Now, for any $\bar{c} \in \bar{a}$, we have by a very similar argument that there is a $\bar{\beta}$ such that $\bar{c} \in (\mathrm{dom}(\bar{\sigma}))^{J_{\bar{\beta}}(V_{\bar{\lambda}+1})}$. Namely, let $c = K(\bar{c})$, and suppose $\beta < \alpha$ is such that $c \in \mathrm{dom}(\sigma)^{J_\beta(V_{\lambda+1})}$. Suppose (B',ψ') λ-tags β. By Lemma 2.1.7 there is a $K' \in \mathcal{E}^e_{\geq\omega\cdot\alpha}$ such that

$$K'(\bar{C}) = K(\bar{C}),\ K'(\bar{c}) = K(\bar{c}),\ K'(\bar{B}') = B'$$

for some $\bar{B}' \in V_{\bar{\lambda}+1}$, and by induction

$$\hat{K}' : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_\beta(V_{\lambda+1})$$

is elementary for some $\bar{\beta}$. So by elementarity since $c \in (\mathrm{dom}(\sigma))^{J_\beta(V_{\lambda+1})}$ we have $\bar{c} \in (\mathrm{dom}(\bar{\sigma}))^{J_{\bar{\beta}}(V_{\bar{\lambda}+1})}$.

Hence, combining what we have shown, we have that $\bar{\sigma}$ is total function with domain $\bar{a}$ which is a surjection onto $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. □

We let $\bar{\alpha}$ be as in the claim.

Claim 2.3.11. *For all $K' \in \mathcal{E}^e_{\omega\cdot\alpha+1}$ such that*

$$K'(\bar{C}) = K(\bar{C})$$

we have that for all $\bar{\beta}$,

$$\bar{\beta} < \bar{\alpha} \iff \pi_{K'}(\bar{\beta}) < \alpha.$$

Proof. Both directions of this claim are proved in a similar manner to the previous claim. Suppose $K' \in \mathcal{E}^e_{\omega\cdot\alpha+1}$ is such that

$$K'(\bar{C}) = K(\bar{C}).$$

We prove the left to right direction first. Suppose $\bar{\beta} < \bar{\alpha}$. Let $(\bar{B},\varphi)$ $\bar{\lambda}$-tag $\bar{\beta}$. Then there exists $\bar{c} \in \bar{a}$ such that $\bar{\sigma}(\bar{c}) = \bar{\beta}$. Let $\beta = \sigma(K'(\bar{c}))$, and suppose $\gamma < \alpha$ is such that $\beta \in J_\gamma(V_{\lambda+1})$ and $(\sigma)^{J_\gamma(V_{\lambda+1})}(K'(\bar{c})) = \beta$. Then for $K'' \in \mathcal{E}^e_{\omega\cdot\alpha}$ such that

$$\hat{K}''(\bar{C},\bar{c},\bar{B}) = \hat{K}'(\bar{C},\bar{c},\bar{B}),\text{ and } \hat{K}''(\bar{\gamma}) = \gamma$$

for some $\bar{\gamma}$, where by induction

$$\hat{K}'' : J_{\bar{\gamma}}(V_{\bar{\lambda}+1}) \to J_\gamma(V_{\lambda+1})$$

is elementary. And we have

$$\beta = (\sigma(K'(\bar{c})))^{J_\gamma(V_{\lambda+1})} = \hat{K}''((\bar{\sigma}(\bar{c}))^{J_{\bar{\gamma}}(V_{\bar{\lambda}+1})}) = \hat{K}''(\bar{\beta}).$$

Here we use the fact that $K'(\bar{C}) = K(\bar{C})$. So $\pi_{K''}(\bar{\beta}) < \alpha$. It is enough to show that $\pi_{K'}(\bar{\beta}) = \pi_{K''}(\bar{\beta})$. But by elementarity we must have that $(K''(\bar{B}), \varphi)$ λ-tags β. And hence by induction, since $K''(\bar{B}) = K'(\bar{B})$, we have $\pi_{K'}(\bar{\beta}) = \pi_{K''}(\bar{\beta}) = \beta$.

Now we prove the right to left direction. Suppose that $\beta := \pi_{K'}(\bar{\beta}) < \alpha$. Then by induction $\hat{K}' : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_\beta(V_{\lambda+1})$ is elementary. Let $(\bar{B}, \phi)$ $\bar{\lambda}$-tag $\bar{\beta}$, let $c \in a$ and $\gamma < \alpha$ be such that $(\sigma)^{J_\gamma(V_{\lambda+1})}(c) = \beta$. Then there is a $K'' \in \mathcal{E}^e_{\omega\cdot\alpha}$ such that

$$K''(\bar{C}, \bar{B}) = K'(\bar{C}, \bar{B}), K''(\bar{c}) = c, \text{ and } \hat{K}''(\bar{\gamma}) = \gamma,$$

for some $\bar{c}$ and $\bar{\gamma}$. Then by elementarity we have that

$$(\bar{\sigma})^{J_{\bar{\gamma}}(V_{\bar{\lambda}+1})}(\bar{c}) = \bar{\beta},$$

which shows that $\bar{\beta} < \bar{\alpha}$. $\square$

So we have that for any $K' \in \mathcal{E}^e_{\omega\cdot\alpha+1}$ such that $K'(\bar{C}) = K(\bar{C})$, if $\bar{X} \subset V_{\bar{\lambda}+1}$ and $X \subset V_{\lambda+1}$ are canonical codings of $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ and $J_\alpha(V_{\lambda+1})$, respectively, then

$$K' : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is a Σ_0-elementary embedding. But then by Lemma 2.3.2 if $K' \in \mathcal{E}^e_{\omega\cdot\alpha+n+1}$ then

$$K' : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is Σ_n-elementary. And so since $K \in \mathcal{E}^e_{\omega\cdot\alpha+\omega}$,

$$K : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is elementary. Hence K extends to a Σ_0 embedding

$$\hat{K} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to J_{\alpha+1}(V_{\lambda+1}).$$

Recall that $p \in J_{\alpha+1}(V_{\lambda+1})$ was a witness to ϕ with parameter A. If $\bar{p}$ is defined the same as p but with parameter $\bar{s}$, then we must have $\hat{K}(\bar{p}) = p$. And hence by Σ_0 elementarity, since p witnesses ϕ over $J_{\alpha+1}(V_{\lambda+1})$ with parameter A, we have that $\bar{p}$ witnesses ϕ over $J_{\bar{\alpha}+1}(V_{\bar{\lambda}+1})$ with parameter $\bar{A}$. Hence by definition $(\bar{A}, \phi)$ $\bar{\lambda}$-tags $\bar{\alpha}$. So $\bar{\alpha}$ does not depend of the choice of K above.

Now we want to show that J satisfies that for all $\bar{\beta}$,

$$\bar{\beta} < \bar{\alpha} \iff \pi_J(\bar{\beta}) < \alpha.$$

But for any $\bar{\beta} < \bar{\alpha}$, if $(\bar{B}_4, \psi_4)$ $\bar{\lambda}$-tags $\bar{\beta}$, then we could have required above that $K(\bar{B}_4) = J(\bar{B}_4)$. But the above claim showed that $\pi_K(\bar{\beta}) < \alpha$, and hence by elementarity, that $(K(\bar{B}_4), \psi_4) = (J(\bar{B}_4), \psi_4)$ λ-tags $\pi_K(\bar{\beta})$. But then by induction we have that $\pi_J(\bar{\beta}) = \pi_K(\bar{\beta}) < \alpha$. So we have proved the left to right direction. The right to left direction follows immediately by induction.

The argument we just gave actually proves the following claim:

Claim 2.3.12. *For all $K' \in \mathcal{E}^e_{\omega\cdot\alpha+2}$ such that $K'(\bar{A}) = A$ we have that for all $\bar{\beta}$,*

$$\bar{\beta} < \bar{\alpha} \iff \pi_{K'}(\bar{\beta}) < \alpha.$$

Hence as above by Lemma 2.3.2 if $K' \in \mathcal{E}^e_{\omega\cdot\alpha+n+2}$ and $K'(\bar{A}) = A$ then

$$K' : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is Σ_n-elementary. And so since $J \in \mathcal{E}^e_{\omega\cdot\alpha+\omega}$,

$$J : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is elementary.

Hence we have proved $\Phi(\alpha+1)$ for the case that α is a limit. For the case that $\alpha = \beta+1$ is a successor, we set $\pi_J(\bar{\beta}+1) = \pi_J(\bar{\beta})+1$ where $\pi_J(\bar{\beta}) = \beta$. It is straightforward to prove $\Phi(\alpha+1)$ using Lemma 2.3.2.

So by induction we have $\Phi(\alpha)$ holds for all $\alpha < \kappa$. So since for all $A \in V_{\lambda+1}$ there exists $(J, \langle j_i | \, i < \omega\rangle) \in \mathcal{E}^e_\kappa$ such that $A \in \operatorname{rng} J$ (by varying the $\bar{\lambda}$ we fixed, if needed), the theorem follows. $\square$

As a corollary of the previous proof we have the following:

Corollary 2.3.13. *Let κ be the least admissible. Then for all $\alpha < \kappa$, if $J \in \mathcal{E}^e_{\geq\omega\cdot\alpha+\omega+1}$ and $J(\bar{A}) = A$ where $A \in V_{\lambda+1}$, $\bar{A} \in V_{\bar{\lambda}+1}$, and ϕ are such that (A, ϕ) λ-tags α, then J extends to*

$$\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$$

where $\bar{\alpha}$ is $\bar{\lambda}$-tagged by $(\bar{A}, \phi)$.

2.4 Reflecting below the first Σ_1-gap

We now extend our reflection results up to the first Σ_1-gap. The proof again uses simply definable surjections to guide the extensions, but the verification that our extensions are correct is more difficult. Here we use saturated sets to accomplish this step. While our definition above of saturated sets are subsets of $\mathcal{E}$, we use the obvious extension of this definition to $\mathcal{E}^e_\kappa$ for some κ good.

Theorem 2.4.1. *Let κ be least such that $J_\kappa(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1}\cup\{V_{\lambda+1}\})} J_{\kappa+1}(V_{\lambda+1})$. Suppose there exists an elementary embedding $j : J_{\kappa+\omega+1}(V_{\lambda+1}) \to J_{\kappa+\omega+1}(V_{\lambda+1})$. Then there is $\bar\lambda < \lambda$ such that for all $\alpha < \kappa$, there exists $\bar\alpha$ and an elementary embedding*

$$\hat{J} : J_{\bar\alpha}(V_{\bar\lambda+1}) \to J_\alpha(V_{\lambda+1}).$$

Proof. We prove this by induction on α. Specifically, for $\mathcal{F}^\kappa \subseteq \mathcal{E}^e_\kappa$ a fixed saturated set, we will inductively define for every $J \in \mathcal{F}^\kappa$ a function π_J on ordinals. Our induction hypothesis for α will be that for all $\alpha' < \alpha$ if (A, ϕ) λ-tags α' and $J \in \mathcal{F}^\kappa$ is such that $J(\bar{A}) = A$, then there exists $\bar\alpha$ such that $\pi_J(\bar\alpha) = \alpha'$, and $(\bar{A}, \phi)$ $\bar\lambda$-tags $\bar\alpha$. Furthermore,

1. $\pi_J \restriction \bar\alpha$ extends to an elementary embedding
$$\hat{J} : J_{\bar\alpha}(V_{\bar\lambda+1}) \to J_{\alpha'}(V_{\lambda+1}).$$

2. For all i,
$$\pi_J \restriction \bar\alpha + 1 = j_0 \circ \cdots \circ j_i \circ \pi_{J_{i+1}} \restriction \bar\alpha + 1.$$

Let τ be Σ_1 definable and such that for all $\alpha < \kappa$, $(\tau)^{J_\alpha(V_{\lambda+1})}$ is a partial function $V_{\lambda+1} \to J_\alpha(V_{\lambda+1})$ which is a surjection (see Steel [Ste83]). Let $\bar\tau$ be defined similarly for $\bar\lambda$. By τ and $\bar\tau$ we mean $(\tau)^{L(V_{\lambda+1})}$ and $\bar\tau^{L(V_{\bar\lambda+1})}$.

Now assuming the induction hypothesis at α, we prove it for $\alpha + 1$. Hence, we need to find for every $J \in \mathcal{F}^\kappa$ such that $J(\bar{A}) = A$ for some $\bar{A}$ and (A, ϕ) which λ-tags α, an appropriate $\bar\alpha$. We can again reduce to the case that for all i, $j_i(\alpha) = \alpha$ just as in the proof of Theorem 2.3.9. Let $\bar\alpha$ be least such that either:

1. There exists $\bar{a} \in V_{\bar\lambda+1}$ such that $\bar\tau(\bar{a}) = \bar\alpha$, but either
$$J(\bar{a}) \notin \operatorname{dom}(\tau) \text{ or } \tau(J(\bar{a})) \notin J_\alpha(V_{\lambda+1}),$$
or

2. $\bar\alpha \notin \operatorname{rng}\bar\tau$.

Such an $\bar\alpha$ clearly exists. Set $\pi_J(\bar\alpha) = \alpha$. We need to see that this is an appropriate definition in the sense that:

1. $\pi_J(\bar\alpha)$ is not already defined.

2. For $\bar\alpha' < \bar\alpha$, $\pi_J(\bar\alpha')$ is defined and is less than α.

To see 2, suppose that $\bar\alpha' < \bar\alpha$, and suppose $(\bar{A}', \phi')$ $\bar\lambda$-tags $\bar\alpha'$. Then by our definition of $\bar\alpha$, it must be the case that there exists $\bar{a}' \in V_{\bar\lambda+1}$ such that $\bar\tau(\bar{a}') = \bar\alpha'$, and $\tau(J(\bar{a}')) \in J_\alpha(V_{\lambda+1})$.

Suppose that $\beta < \alpha$ is such that $\tau(J(\bar{a}')) \in J_\beta(V_{\lambda+1})$. Let (B, ψ) λ-tag β, and consider $K \in \mathcal{F}^\kappa$ such that for some $i_0 < \omega$,

$$\left\langle K_{i_0}(\bar{A}), K_{i_0}(\bar{A}'), K_{i_0}(\bar{a}), K_{i_0}(\bar{a}')\right\rangle = \left\langle J_{i_0}(\bar{A}), J_{i_0}(\bar{A}'), J_{i_0}(\bar{a}), J_{i_0}(\bar{a}')\right\rangle \text{ and } K_{i_0}(\bar{B}) = B$$

for some $\bar{B} \in V_{\bar{\lambda}+1}$. By induction there exists $\bar{\beta}$ such that

$$\hat{K}_{i_0} : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_\beta(V_{\lambda+1})$$

is elementary. And since $J_{i_0}(\bar{a}') = K_{i_0}(\bar{a}')$ and $(\tau)^{J_\beta(V_{\lambda+1})}(K_{i_0}(\bar{a}')) = \tau(J_{i_0}(\bar{a}'))$, we have that $\tau(J_{i_0}(\bar{a}')) \in \operatorname{rng} \hat{K}_{i_0}$. And then by elementarity we must have that

$$\hat{K}_{i_0}^{-1}(\tau(J_{i_0}(\bar{a}'))) = \hat{K}_{i_0}^{-1}((\tau)^{J_\beta(V_{\lambda+1})}(K_{i_0}(\bar{a}'))) = (\bar{\tau})^{J_{\bar{\beta}}(V_{\bar{\lambda}+1})}(\bar{a}') = \bar{\alpha}'.$$

Hence $\alpha' := \hat{K}_{i_0}(\bar{\alpha}')$ is an ordinal by elementarity and it is less than α. Furthermore by elementarity, we have that since $(\bar{A}', \phi')$ $\bar{\lambda}$-tags $\bar{\alpha}'$, that $(K_{i_0}(\bar{A}'), \phi') = (J_{i_0}(\bar{A}'), \phi')$ λ-tags α'. But by induction this implies that $\pi_{J_{i_0}}(\bar{\alpha}') = \alpha' < \alpha$. Furthermore we have that $\pi_J(\bar{\alpha}') = (j_0 \circ \cdots j_{i_0-1} \circ \pi_{J_{i_0}})(\bar{\alpha}') < \alpha$, since $j_i(\alpha) = \alpha$ for all i.

To see 1, if $\pi_J(\bar{\alpha})$ is already defined then $\pi_J(\bar{\alpha}) < \alpha$. Let $\pi_J(\bar{\alpha}) = \alpha' < \alpha$. Then $\hat{J} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to J_{\alpha'+1}(V_{\lambda+1})$ is elementary by induction. Since $(\tau)^{J_{\alpha'}(V_{\lambda+1})}$ is a partial function $V_{\lambda+1} \to J_{\alpha'}(V_{\lambda+1})$ which is a surjection, by elementarity of $\hat{J}$, $(\bar{\tau})^{J_{\bar{\alpha}'}(V_{\bar{\lambda}+1})}$ is a partial function $V_{\bar{\lambda}+1} \to J_{\bar{\alpha}'}(V_{\bar{\lambda}+1})$ which is a surjection. But this contradicts the definition of $\bar{\alpha}$.

Now let $\bar{\alpha}_0$ be least such that there exists $K \in \mathcal{F}^\kappa$ such that $K(\bar{A}) = A$ and $\pi_K(\bar{\alpha}_0) = \alpha$. We will show that $(\bar{A}, \phi)$ $\bar{\lambda}$-tags $\bar{\alpha}_0$. Suppose that we are in case 1 of the definition of $\pi_J(\bar{\alpha})$, and suppose $\bar{\tau}(\bar{a}) = \bar{\alpha}_0$. The key point is that for any $K' \in \mathcal{F}^\kappa$ such that $K'(\bar{A}) = K(\bar{A})$ and $K'(\bar{a}) = K(\bar{a})$, we must have that $\pi_{K'}(\bar{\alpha}_0) = \alpha$. This follows by minimality of $\bar{\alpha}_0$, and the fact that either $\tau(K'(\bar{a})) = \tau(K(\bar{a})) \notin J_\alpha(V_{\lambda+1})$ or $K'(\bar{a}) \notin \operatorname{dom}(\tau)$. And hence for any such K', if $\bar{X} \subset V_{\bar{\lambda}+1}$ codes $J_{\bar{\alpha}_0}(V_{\bar{\lambda}+1})$ and $X \subset V_{\lambda+1}$ codes $J_\alpha(V_{\lambda+1})$, then

$$K' : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$$

is Σ_0-elementary. But then by Lemma 2.3.6, $(\bar{A}, \phi)$ $\bar{\lambda}$-tags $\bar{\alpha}_0$, which is what we wanted.

If we are in case 2 of the definition of $\pi_J(\bar{\alpha})$, then we immediately have that $\pi_{K'}(\bar{\alpha}_0) = \pi_K(\bar{\alpha}_0)$ for any $K' \in \mathcal{F}^\kappa$ such that $K'(\bar{A}) = K(\bar{A})$, by minimality of $\bar{\alpha}_0$. Then to show that $(\bar{A}, \phi)$ $\bar{\lambda}$-tags $\bar{\alpha}_0$ is exactly the same is in case 1.

Now since $(\bar{A}, \phi)$ $\bar{\lambda}$-tags $\bar{\alpha}_0$, and $J(\bar{A}) = A$, if $\pi_J(\bar{\alpha}) = \alpha$ where $\bar{\alpha} > \bar{\alpha}_0$, then $\pi_J(\bar{\alpha}_0)$ is already defined and is less than α. But then for $\alpha_0 := \pi_J(\bar{\alpha}_0)$, $\hat{J} : J_{\bar{\alpha}_0+1}(V_{\bar{\lambda}+1}) \to J_{\alpha_0+1}(V_{\lambda+1})$ is elementary, and hence (A, ϕ) λ-tags α_0 by elementarity. But this is a contradiction since (A, ϕ) λ-tags $\alpha > \alpha_0$. Hence $\pi_J(\bar{\alpha}_0) = \alpha$, and the proof above shows that π_J extends to $\hat{J} : J_{\bar{\alpha}_0}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$.

Hence we have shown that the induction hypothesis holds for $\alpha + 1$ for the case of α a limit. For α a successor, the induction hypothesis follows immediately from Lemma 2.3.6. Hence by induction the theorem is proved. $\square$

We can use a similar technique to push the reflection a bit past κ:

Theorem 2.4.2. *Let κ be least such that $J_\kappa(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1}\cup\{V_{\lambda+1}\})} J_{\kappa+1}(V_{\lambda+1})$. Suppose there exists an elementary embedding $j : J_{\kappa+\omega+1}(V_{\lambda+1}) \to J_{\kappa+\omega+1}(V_{\lambda+1})$. Then there is $\bar\lambda < \lambda$ such that there is an elementary embedding*

$$\hat{J} : J_{\bar\kappa}(V_{\bar\lambda+1}) \to J_\kappa(V_{\lambda+1}),$$

where $\bar\kappa$ is defined similarly to κ for $\bar\lambda$ instead of λ.

Proof. Define π as in the previous proof. Let $\mathcal{F}^{\kappa+2} \subseteq \mathcal{E}^e_{\kappa+2}$ be saturated. We prove that for every $J \in \mathcal{F}^{\kappa+2}$, if $\bar\alpha_J = \sup\{\bar\beta |\, \pi_J(\bar\beta) \text{ is defined }\}$, then $\bar\alpha_J = \bar\kappa$. Suppose not, and let $\bar\alpha_0 < \bar\kappa$ be least such that there exists $J \in \mathcal{F}^{\kappa+2}$ with $\bar\alpha_J = \bar\alpha_0$. Since $\bar\alpha_0 < \bar\kappa$, there exists $\bar A \in V_{\bar\lambda+1}$ and ϕ such that $(\bar A, \phi)$ $\bar\lambda$-tags $\bar\alpha_0$. Fix J such that $\bar\alpha_J = \bar\alpha_0$.

Claim 2.4.3. *For all $K \in \mathcal{F}^{\kappa+2}$ such that $K(\bar A) = J(\bar A)$, we have $\bar\alpha_K = \bar\alpha_0$.*

Proof. We must have that there is no $\alpha < \kappa$ such that $(J(\bar A), \phi)$ λ-tags α, since otherwise by the arguments of the previous proof, we would have $\pi_J(\bar\alpha_0) = \alpha$. And hence if $K \in \mathcal{F}^{\kappa+2}$ is such that $K(\bar A) = J(\bar A)$ and $\bar\alpha_K > \bar\alpha_0$, then $\alpha_0 := \pi_K(\bar\alpha_0)$ is defined and less than κ. Furthermore, $\hat K : J_{\bar\alpha_0}(V_{\bar\lambda+1}) \to J_{\alpha_0}(V_{\lambda+1})$ is elementary, and hence we must have that $(K(\bar A), \phi)$ λ-tags $\alpha_0 < \kappa$. But $K(\bar A) = J(\bar A)$, which is a contradiction. □

So for $\bar X \subseteq V_{\bar\lambda+1}$ and $X \subseteq V_{\lambda+1}$ such that $\bar X$ codes $J_{\bar\alpha_0}(V_{\bar\lambda+1})$ and X codes $J_\kappa(V_{\lambda+1})$, we have that $K : (V_{\bar\lambda+1}, \bar X) \to (V_{\lambda+1}, X)$ is Σ_0 for any $K \in \mathcal{F}^{\kappa+2}$ such that $K(\bar A) = J(\bar A)$. But then by Lemma 2.3.6, we have that any such embedding is actually elementary. Hence we have that $\hat K : J_{\bar\alpha_0}(V_{\bar\lambda+1}) \to J_\kappa(V_{\lambda+1})$ is elementary, for any such K, and hence $\hat K : J_{\bar\alpha_0+1}(V_{\bar\lambda+1}) \to J_{\kappa+1}(V_{\lambda+1})$ is elementary. But then it must be that $J_{\bar\alpha_0}(V_{\bar\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1}\cup\{V_{\lambda+1}\})} J_{\bar\alpha_0+1}(V_{\bar\lambda+1})$, which is a contradiction to the fact that $\bar\alpha_0 < \bar\kappa$. Hence we must have that $\bar\alpha_J = \bar\kappa$ for all $J \in \mathcal{F}^{\kappa+2}$. And arguing as we just did, we have that

$$\hat J : J_{\bar\kappa}(V_{\bar\lambda+1}) \to J_\kappa(V_{\lambda+1})$$

is elementary. □

Corollary 2.4.4. *Let κ be least such that*

$$J_\kappa(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1}\cup\{V_{\lambda+1}\})} J_{\kappa+1}(V_{\lambda+1}).$$

Suppose there exists an elementary embedding

$$j : J_{\kappa+\omega+1}(V_{\lambda+1}) \to J_{\kappa+\omega+1}(V_{\lambda+1}).$$

Then there is $\bar\lambda < \lambda$ such that there is an elementary embedding

$$\bar j : J_{\bar\kappa}(V_{\bar\lambda+1}) \to J_{\bar\kappa}(V_{\bar\lambda+1}).$$

Theorem 2.4.5. *Let κ_0 be least such that*

$$J_{\kappa_0}(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1}\cup\{V_{\lambda+1}\})} J_{\kappa_0+1}(V_{\lambda+1}).$$

Let κ_1 be least such that

$$J_{\kappa_1}(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1}\cup\{V_{\lambda+1},\kappa_0\})} J_{\kappa_1+1}(V_{\lambda+1}).$$

Suppose there exists an elementary embedding

$$j : J_{\kappa_1+\omega+1}(V_{\lambda+1}) \to J_{\kappa_1+\omega+1}(V_{\lambda+1}).$$

Then there is $\bar{\lambda} < \lambda$ and $\bar{\kappa}_1$ such that there is an elementary embedding

$$\hat{J} : J_{\bar{\kappa}_1}(V_{\bar{\lambda}+1}) \to J_{\kappa_1}(V_{\lambda+1}).$$

Proof. The proof is almost exactly the same as the proof of Theorem 2.4.1. □

There are similar reflection results provable by the same method for κ_n defined analogously for $n < \omega$. The method here appears to break down as the Σ_1-gaps get larger, however. To get past these larger gaps we need a new method, which we present in the next section.

2.5 Reflecting I_0

In this section we prove our main reflection result. First we introduce some terminology which identifies the general form of reflection as obtained by inverse limits. In Section 3 we will find that this form of reflection is even more useful than simply having reflection embeddings.

Definition 2.5.1. We define *inverse limit reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in E$, J extends to $\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$ which is elementary.

We define *strong inverse limit reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$ and a saturated set $E \subseteq \mathcal{E}$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to $\hat{J} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$ which is elementary.

We will also need the notion of inverse limit X-reflection where $X \subseteq V_{\lambda+1}$. As before we let

$$\begin{aligned}\mathcal{E}(X) = \{(J, \langle j_i|\, i < \omega\rangle)|\, \forall i\, (j_i : (V_{\lambda+1}, X) \to (V_{\lambda+1}, X)) \text{ and}\\ J = j_0 \circ j_1 \circ \cdots : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X) \text{ is } \Sigma_0\}.\end{aligned}$$

Here we let $\bar{X} = J^{-1}[X]$. We modify the definition of saturated to X-saturated, requiring in addition that $J^{-1}[X] = K^{-1}[X]$.

Definition 2.5.2. Suppose $X \subseteq V_{\lambda+1}$. We define *inverse limit X-reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X-saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in E$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \to L_\alpha(X, V_{\lambda+1})$ which is elementary.

We define *strong inverse limit X-reflection at α* to mean the following: There exists $\bar{\lambda}, \bar{\alpha} < \lambda$, $\bar{X} \subseteq V_{\bar{\lambda}+1}$ and an X-saturated set $E \subseteq \mathcal{E}(X)$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to $\hat{J} : L_{\bar{\alpha}}(\bar{X}, V_{\bar{\lambda}+1}) \to L_\alpha(X, V_{\lambda+1})$ which is elementary.

Note that we cannot immediately conclude elementarity of $J : (V_{\bar{\lambda}+1}, \bar{X}) \to (V_{\lambda+1}, X)$ as $\bar{X}$ depends on J in general. And in fact we will show that inverse limit X-reflection does not hold in general.

Theorem 2.5.3. *Suppose there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Then there exists $\bar{\lambda} < \lambda$ such that for all $\alpha < \Theta_\lambda$, there exists $\bar{\alpha}$ such that*

$$L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \equiv L_\alpha(V_{\lambda+1}).$$

Proof. Suppose that $\alpha < \Theta_\lambda$ is good, $\rho : V_{\lambda+1} \to L_\alpha(V_{\lambda+1})$ is a surjection definable over $L_{\alpha+1}(V_{\lambda+1})$, with $X \subseteq V_{\lambda+1}$ the preimage. Let $G \subseteq \mathrm{Coll}(\omega, \lambda)$ be V-generic.

Let $E \subseteq \mathcal{E}_{\alpha+1}$ be saturated and $(J, \vec{j}) \in E$. Let $\vec{\lambda}$ be cofinal in $\bar{\lambda}_J = \bar{\lambda}$. In $V[G]$, let $\langle a_i | \, i < \omega \rangle$ be an enumeration of $V_{\bar{\lambda}+1}$, and let $\langle \phi^i | \, i < \omega \rangle$ be an enumeration of all formulas in the language $(\in)$. We define sequences $\langle J^i | \, i < \omega \rangle$, $\langle n_i | \, i < \omega \rangle$ in $V[G]$ with the following properties:

1. $J^0 = J$. For all $i < \omega$, $J^i \in E$ and J^{i+1} is a limit root of J^i, supported by $\vec{\lambda}$.
2. $\langle n_i | \, i < \omega \rangle$ is increasing, and for all $i < \omega$, for all $n \leq n_i$, $J^{i+1}(a_n) = J^i(a_n)$.
3. For all $i_0 < \omega$, suppose that $L_\alpha(V_{\lambda+1}) \models \exists x \, \phi(x, \vec{B})$ where

   $$\vec{B} = \left\langle \rho(J^{i_0}(a_{s_1})), \ldots, \rho(J^{i_0}(a_{s_m})) \right\rangle$$

   and for all $i < m$, $s_i \leq i_0$ and $\exists x \phi(x, \vec{X})$ is the formula ϕ^i for some $i < i_0$. Then for some b which is a witness to ϕ with parameter $\vec{B}$, we have

   $$\rho(J^{i_0+1}(a_{\bar{t}})) = b$$

   and $n_{i_0+1} \geq \bar{t}$.

Note that we can arrange (3) as follows. Suppose that $i_0 < \omega$ and

$$L_\alpha(V_{\lambda+1}) \models \exists x \, \phi(x, \vec{B}),$$

where

$$\vec{B} = \left\langle \rho(J^{i_0}(a_{s_1})), \ldots, \rho(J^{i_0}(a_{s_m})) \right\rangle.$$

Let i be such that for all $A \in V_{\bar\lambda+1}$ and $B \in V_{\lambda+1}$, there exists $(K, \vec{k}) \in E$, with K an i-close limit root of J^{i_0}, $K_i(A) = J_i^{i_0}(A)$ and $B \in \operatorname{rng} K_i$. Pulling back by $j_0^{i_0} \circ \cdots \circ j_{i-1}^{i_0}$, we have

$$L_\alpha(V_{\lambda+1}) \models \exists x\, \phi(x, \vec{B}_i),$$

where

$$\vec{B}_i = \left\langle \rho(J_i^{i_0}(a_{s_1})), \ldots, \rho(J_i^{i_0}(a_{s_m})) \right\rangle .$$

Let b be a witness to ϕ with parameter $\vec{B}_i$. Then if $(K, \vec{k}) \in E$ is an i-close limit root of J^{i_0}, satisfies (2), and for some $\bar{t}$, $\rho(K_i(a_{\bar{t}})) = b$ then

$$L_\alpha(V_{\lambda+1}) \models \phi(\rho((j_0^{i_0} \circ \cdots \circ j_{i-1}^{i_0})(b)), \vec{B}).$$

To arrange (3), we simply work with the finitely many $\vec{B}$ and ϕ required by (3) simultaneously.

Let J^* be the common part of $\langle J^i |\, i < \omega\rangle$. Then by (2) and Lemma 2.1.16 we have that $J^* : V_{\bar\lambda+1} \to V_{\lambda+1}$ since for all $a \in V_{\bar\lambda+1}$, there is an n such that $a_n = a$. And hence for i large enough, we have that $J^*(a_n) = J^i(a_n) \in V_{\lambda+1}$.

Let $M = \rho[J^*[V_{\bar\lambda+1}]]$. We claim that $M \prec L_\alpha(V_{\lambda+1})$. But this follows immediately from condition (3). Furthermore, M is wellfounded. Let $\bar{M}$ be the transitive collapse of M and let π be the inverse of the transitive collapse. We have that $V_{\bar\lambda+1} = \pi^{-1}[V_{\lambda+1}]$, and hence by condensation, we have that $\bar{M} = L_{\bar\alpha}(V_{\bar\lambda+1})$ for some $\bar\alpha$. So $L_{\bar\alpha}(V_{\bar\lambda+1}) \equiv L_\alpha(V_{\lambda+1})$. But, by absoluteness, in V we have that $L_{\bar\alpha}(V_{\bar\lambda+1}) \equiv L_\alpha(V_{\lambda+1})$, which is what we wanted. $\square$

Based on the proof of Theorem 2.5.3, we fix some terminology which will be useful in the following theorems.

Definition 2.5.4. Fix $E \subseteq \mathcal{E}$ saturated, α good, and $J \in E$. Set $\bar\lambda = \bar\lambda_J$ and let $\vec{\lambda}$ be cofinal in $\bar\lambda$. Fix $\langle \phi^i |\, i < \omega\rangle$, an enumeration of all formulas in the language $(\in)$. We define a forcing $\mathbb{P}(E, \alpha, J)$. Conditions are elements $(\langle J^i, n_i |\, i < m\rangle, \langle a_i |\, i < n_{m-1}\rangle)$ where $m \geq 1$ and the following hold.

1. $J^0 = J$. For all $i < m - 1$, J^{i+1} is a limit root of J^i supported by $\vec{\lambda}$, and $J^{i+1} \in E$.
2. $\langle n_i |\, i < m\rangle \in \omega^m$ is an increasing sequence.
3. For all $i < n_{m-1}$, $a_i \in V_{\bar\lambda_J+1}$.
4. For all $1 \leq m' < m$, and $i < n_{m'-1}$, $J^{m'-1}(a_i) = J^{m'}(a_i)$.
5. For all $m' < m - 1$, suppose that $L_\alpha(V_{\lambda+1}) \models \exists x\, \phi(x, \vec{B})$ where

$$\vec{B} = \left\langle \rho(J^{m'}(a_{s_1})), \ldots, \rho(J^{m'}(a_{s_n})) \right\rangle$$

and for all $i < n$, $s_i \leq m'$ and $\exists x\phi(x, \vec{X})$ is the formula ϕ^i for some $i < m'$. Then for some b which is a witness to ϕ with parameter $\vec{B}$, we have

$$\rho(J^{m'+1}(a_{\bar{t}})) = b$$

for some $\bar{t} < n_{m'+1}$.

For $(\langle J^i, n_i |\, i < m\rangle, \langle a_i |\, i < n_{m-1}\rangle), (\langle K^i, n'_i |\, i < m'\rangle, \langle a'_i |\, i < n'_{m'-1}\rangle) \in \mathbb{P}(E, \alpha, J)$ we put

$$(\left\langle J^i, n_i |\, i < m\right\rangle, \langle a_i |\, i < n_{m-1}\rangle) \geq_{\mathbb{P}(E,\alpha,J)} (\left\langle K^i, n'_i |\, i < m'\right\rangle, \left\langle a'_i |\, i < n'_{m'-1}\right\rangle)$$

iff

1. $m \leq m'$,
2. for all $i < m$, $J^i = K^i$, $n_i \leq n'_i$, and for all $s < n_{m-1}$, $a_s = a'_s$.

Suppose that $g \subseteq \mathbb{P}(E, \alpha, J)$ is $L(V_{\lambda+1})$-generic. Then clearly in $L(V_{\lambda+1})[g]$ we obtain a unique sequence $\langle J^i |\, i < \omega\rangle$ from g such that for all i, J^{i+1} is a limit root of J^i. We set J^g to be the common part of $\langle J^i |\, i < \omega\rangle$.

Lemma 2.5.5. *Assume we are in the situation of Definition 2.5.4. Suppose that*

$$g \subseteq \mathbb{P}(E, \alpha, J)$$

is $L(V_{\lambda+1})$-generic. Then J^g maps $V_{\bar{\lambda}+1} \to V_{\lambda+1}$, and there exists an $\bar{\alpha}$ such that J^g extends to an elementary embedding

$$\hat{J}^g : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

Proof. This follows exactly as in the proof of Theorem 2.5.3. □

Lemma 2.5.6. *Assume we are in the situation of Definition 2.5.4. Suppose that*

$$(\left\langle J^i, n_i |\, i < m\right\rangle, \langle a_i |\, i < n_{m-1}\rangle) \in \mathbb{P}(E, \alpha, J)$$

and there exists $\bar{\alpha}$ such that

$$(\left\langle J^i, n_i |\, i < m\right\rangle, \langle a_i |\, i < n_{m-1}\rangle) \Vdash J^{\dot{g}} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}) \text{ is elementary.}$$

Then J^{m-1} extends to an elementary embedding $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$.

Proof. We assume for simplicity of notation that $m = 1$. So we have

$$p := (\langle J, n_0\rangle, \langle a_i |\, i < n_0\rangle) \in \mathbb{P}(E, \alpha, J)$$

and there exists $\bar{\alpha}$ such that

$$(\langle J, n_0\rangle, \langle a_i | \, i < n_0\rangle) \Vdash \dot{J^g} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}) \text{ is elementary.}$$

We extend J to a map $\hat{J}$ as follows. Suppose that $\bar{B} \in V_{\bar{\lambda}+1}$, $B \in V_{\lambda+1}$, $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$, $b \in J_\alpha(V_{\lambda+1})$, and ϕ are such that $J(\bar{B}) = B$, and b is the unique element of $J_\alpha(V_{\lambda+1})$ such that $J_\alpha(V_{\lambda+1}) \models \phi(b, B)$ and $\bar{b}$ is the unique element of $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ such that $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \phi(\bar{b}, \bar{B})$. Then set $\hat{J}(\bar{b}) = b$.

We need to check that $\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$ is well-defined, total, and elementary. The proofs of each of these facts are very similar. First we check that $\hat{J}$ is well-defined. Suppose that $\bar{B}_1, B_1, \phi_1$ witness that $\hat{J}(\bar{b}) = b_1$ and $\bar{B}_2, B_2, \phi_2$ witness that $\hat{J}(\bar{b}) = b_2$. Let $p' \leq_{\mathbb{P}(E,\alpha,J)} p$ be the condition

$$p' = (\langle J, n_0 + 2\rangle, \langle a_i | \, i < n_0\rangle^\frown \langle \bar{B}_1, \bar{B}_2\rangle).$$

Then

$$p' \Vdash \dot{J^g}(\bar{B}_1) = B_1 \wedge \dot{J^g}(\bar{B}_2) = B_2,$$

and hence

$$p' \Vdash b_1 = \dot{\hat{J}^g}(\bar{b}) = b_2.$$

So $b_1 = b_2$ by absoluteness, which is what we wanted.

Now we check that $\hat{J}$ is total. We first show that $\bar{\alpha}$ is ($\bar{\lambda}$-)good. Let $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. Suppose $p \in g \subseteq \mathbb{P}(E, \alpha, J)$ is $L(V_{\lambda+1})$-generic and $\hat{J}^g(\bar{b}) = b \in J_\alpha(V_{\lambda+1})$. Then since α is good there exists a $B \in V_{\lambda+1}$ such that $J_\alpha(V_{\lambda+1}) \models b$ is the unique element such that $\phi(b, B)$. Hence

$$J_\alpha(V_{\lambda+1}) \models \exists B' \in V_{\lambda+1}(b \text{ is the unique element such that } \phi(b, B')).$$

But then by elementarity of J^g,

$$J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \exists \bar{B}' \in V_{\bar{\lambda}+1}(\bar{b} \text{ is the unique element such that } \phi(\bar{b}, \bar{B}')).$$

So this shows that $\bar{\alpha}$ is good.

To see that $\hat{J}$ is total, let $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ and let $\bar{B}$ and ϕ be such that $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \bar{b}$ is the unique element such that $\phi(\bar{b}, \bar{B})$. Set $B = J(\bar{B})$. Let $p' \leq_{\mathbb{P}(E,\alpha,J)} p$ be the condition

$$p' = (\langle J, n_0 + 1\rangle, \langle a_i | \, i < n_0\rangle^\frown \langle \bar{B}\rangle).$$

Then

$$p' \Vdash \dot{J^g}(\bar{B}) = B,$$

and hence

$$p' \Vdash J_\alpha(V_{\lambda+1}) \models \dot{J^g}(\bar{b}) \text{ is the unique element such that } \phi(\dot{J^g}(\bar{b}), B).$$

Let $p'' \leq_{\mathbb{P}(E,\alpha,J)} p'$ be such that for some $b \in J_\alpha(V_{\lambda+1})$, $p'' \Vdash \dot{J^g}(\bar{b}) = b$. But then by absoluteness we have that

$$J_\alpha(V_{\lambda+1}) \models b \text{ is the unique element such that } \phi(b, B).$$

So we must have that $\hat{J}(\bar{b}) = b$.

To see that $\hat{J}$ is elementary, suppose that $\bar{b} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ and ψ is a formula. Let $\bar{B}$ and ϕ be such that $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \bar{b}$ is the unique element such that $\phi(\bar{b}, \bar{B})$. Set $b = \hat{J}(\bar{b})$ and $B = J(\bar{B})$. Let $p' \leq_{\mathbb{P}(E,\alpha,J)} p$ be the condition

$$p' = (\langle J, n_0 + 1\rangle, \langle a_i | \, i < n_0\rangle^\frown \langle \bar{B}\rangle).$$

Then

$$p' \Vdash \dot{J^g}(\bar{B}) = B \wedge \dot{\hat{J}^g}(\bar{b}) = b,$$

and hence

$$p' \Vdash J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \psi(\bar{b}) \iff J_\alpha(V_{\lambda+1}) \models \psi(b).$$

But by absoluteness $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \models \psi(\bar{b}) \iff J_\alpha(V_{\lambda+1}) \models \psi(b)$, which is what we wanted.

So $\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$ is an elementary embedding, as desired. □

Theorem 2.5.7. *Suppose that there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Then inverse limit reflection holds at α for all $\alpha < \Theta$.*

Proof. It is enough to show that for all $\alpha < \Theta$ good, inverse limit reflection holds at α, since if inverse limit reflection holds at α good then it holds at all $\beta \leq \alpha$. So assume that $\alpha < \Theta$ is good. Since there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$, there must exist a saturated set $E \subseteq \mathcal{E}_\alpha$. Fix $J \in E$.

Let

$$p = (\langle J^i, n_i | \, i < m\rangle, \langle a_i | \, i < n_{m-1}\rangle) \in \mathbb{P}(E, \alpha, J)$$

be a condition such that for some $\bar{\alpha}$

$$p \Vdash \dot{J^g} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}) \text{ is elementary.}$$

Then we have that J^{m-1} extends to an elementary embedding $J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$. Let E_p be the set of inverse limits $K \in E$ such that for some $q \leq_{\mathbb{P}(E,\alpha,J)} p$ if

$$q = (\langle K^i, n'_i | \, i < m'\rangle, \langle a'_i | \, i < n_{m'-1}\rangle)$$

then $K = K^{m'-1}$.

Clearly by definition of $\mathbb{P}(E, \alpha, J)$ we have that E_p is saturated as well. Furthermore by Lemma 2.5.6 we have that for all $K \in E_p$ that K extends to an elementary embedding

$$\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

Hence inverse limit reflection holds at α. □

In Section 2.6 we will show, using a more detailed analysis, that strong inverse limit reflection holds all the way up to Θ. The following theorem, which shows that it holds up to the least stable of $L(V_{\lambda+1})$, follows from the same type of argument as above, however.

Theorem 2.5.8. *Suppose that there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Let δ be least such that*

$$J_\delta(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1}\cup\{V_{\lambda+1}\})} L(V_{\lambda+1}).$$

Then strong inverse limit reflection holds at α for all $\alpha < \delta$.

Proof. Suppose $\alpha < \delta$, $A \in V_{\lambda+1}$ and (A, ϕ) is a tag for α (such α are cofinal in δ). Let $E \subseteq \mathcal{E}_{\alpha+1}$ be a saturated set of inverse limits such that for some $\bar{A} \in V_{\bar{\lambda}+1}$, for all $(J, \vec{j}) \in E$, $J(\bar{A}) = A$.

Let $J \in E$. We claim that for some $\bar{\alpha}$,

$$\emptyset \Vdash_{\mathbb{P}(E,\alpha+1,J)} J^{\dot{g}} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to J_{\alpha+1}(V_{\lambda+1}) \text{ is elementary.}$$

But this is clear since

$$\emptyset \Vdash_{\mathbb{P}(E,\alpha+1,J)} \exists\bar{\alpha}' \,(J^{\dot{g}} : J_{\bar{\alpha}'+1}(V_{\bar{\lambda}+1}) \to J_{\alpha+1}(V_{\lambda+1}) \text{ is elementary } \wedge J^{\dot{g}}(\bar{A}) = A \wedge (\bar{A}, \phi) \text{ tags } \bar{\alpha}').$$

And hence by absoluteness there is an $\bar{\alpha}$ which is tagged by $(\bar{A}, \phi)$, and this $\bar{\alpha}$ is as desired.

Hence we have by Lemma 2.5.6 that for all $K \in E$, that K extends to an elementary embedding $\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$.

We also have that for any $K \in \mathcal{E}_{\alpha+\omega}$ such that $K(\bar{A}) = A$ that there exists a saturated set $E_K \subseteq \mathcal{E}_{\alpha+1}$ such that $K \in E_K$ and for all $K' \in E_K$, $K'(\bar{A}) = A$. Hence this shows that for any $K \in \mathcal{E}_{\alpha+\omega}$ such that $K(\bar{A}) = A$ that K extends to an elementary embedding

$$\hat{K} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

To complete the proof we consider a saturated set $E \subseteq \mathcal{E}_{\alpha+\omega}$ such that for all $J \in E$, $J(\bar{A}) = A$ for some (A, ϕ) a tag for α. Such an E must exist since there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Let $\bar{\alpha}$ be as above. Then for all $K \in CL(E)$ we have that $K(\bar{A}) = A$ and $K \in \mathcal{E}_{\alpha+\omega}$. Hence by what we proved above we have that K extends to an elementary embedding $\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$. Hence this E witnesses strong inverse limit reflection at α. $\square$

Theorem 2.5.9. *Suppose that there exists an elementary embedding*

$$j : L_{\omega\cdot 2+1}(V^{\#}_{\lambda+1}, V_{\lambda+1}) \to L_{\omega\cdot 2+1}(V^{\#}_{\lambda+1}, V_{\lambda+1}).$$

Then there exists $\bar{\lambda} < \lambda$ and a $V^{\#}_{\lambda+1}$-saturated set $E \subseteq \mathcal{E}(V^{\#}_{\lambda+1})$ of inverse limits such that for all $(J, \vec{j}) \in E$, J is an elementary embedding

$$J : (V^{\#}_{\bar{\lambda}+1}, V_{\bar{\lambda}+1}) \to (V^{\#}_{\lambda+1}, V_{\lambda+1}).$$

And hence there exists an elementary embedding $\bar{j} : L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$. Furthermore strong inverse limit $V^{\#}_{\lambda+1}$-reflection holds at 0.

Proof. We describe how to modify the proof of Theorem 2.5.3. Let $E \subseteq \mathcal{E}(V^{\#}_{\lambda+1})$ be saturated (but not necessarily $V^{\#}_{\lambda+1}$-saturated). Then proceeding exactly as in the proof of Theorem 2.5.3, replacing $L_\alpha(V_{\lambda+1})$ with $(V^{\#}_{\lambda+1}, V_{\lambda+1})$, the argument is exactly the same until the point that we defined M.

Let $M = J[V_{\bar{\lambda}+1}]$. Then $M \prec V_{\lambda+1}$ and for $\bar{M}$ the transitive collapse of M we have $\bar{M} = V_{\bar{\lambda}+1}$. Let π be the inverse of the transitive collapse. Let $\bar{X} = \pi^{-1}[V^{\#}_{\lambda+1}]$. Then we have that $\pi : (\bar{X}, V_{\bar{\lambda}+1}) \to (V^{\#}_{\lambda+1}, V_{\lambda+1})$ is elementary. But by definability of the sharp, we must have $\bar{X} = V^{\#}_{\bar{\lambda}+1}$. So we have that $(V^{\#}_{\bar{\lambda}+1}, V_{\bar{\lambda}+1}) \equiv (V^{\#}_{\lambda+1}, V_{\lambda+1})$. But by absoluteness this is true in V.

The rest of the proof proceeds exactly as in the proof of Theorem 2.5.8.

To see that there is an elementary embedding

$$L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1}),$$

we have that $(V_{\lambda+1}, V^{\#}_{\lambda+1})$ satisfies that there is a Σ_1-elementary embedding

$$(V_{\lambda+1}, V^{\#}_{\lambda+1}) \to (V_{\lambda+1}, V^{\#}_{\lambda+1}).$$

And hence $(V_{\bar{\lambda}+1}, V^{\#}_{\bar{\lambda}+1})$ satisfies that there is a Σ_1-elementary embedding

$$\bar{j} : (V_{\bar{\lambda}+1}, V^{\#}_{\bar{\lambda}+1}) \to (V_{\bar{\lambda}+1}, V^{\#}_{\bar{\lambda}+1}).$$

So $\bar{j} \upharpoonright V_{\bar{\lambda}+1}$ extends to an elementary embedding

$$\bar{j}^* : L_{\bar{\Theta}}(V_{\bar{\lambda}+1}) \to L_{\bar{\Theta}}(V_{\bar{\lambda}+1}).$$

Here we are using that every subset of $V_{\bar{\lambda}+1}$ in $L(V_{\bar{\lambda}+1})$ is Σ_1-definable over $(V_{\bar{\lambda}+1}, V^{\#}_{\bar{\lambda}+1})$ with parameters in $V_{\bar{\lambda}+1}$. But as in [Woo11] we can define the following ultrafilter $U_{\bar{j}}$ from $\bar{j}$,

$$X \in U_{\bar{j}} \iff \bar{j} \upharpoonright V_{\bar{\lambda}} \in \bar{j}^*(X).$$

Taking the ultrapower by $U_{\bar{j}}$ yields an elementary embedding $L(V_{\bar{\lambda}+1}) \to L(V_{\bar{\lambda}+1})$ which extends $\bar{j} \upharpoonright V_{\bar{\lambda}+1}$ (see [Woo11]). □

Theorem 2.5.9 gives an example of an $X \subseteq V_{\lambda+1}$ such that inverse limit X-reflection holds. The set of such X is very restricted however, as inverse limit X-reflection gives structural properties of $L(X, V_{\lambda+1})$. Specifically, we will prove the following theorem in Section 3.1.

Theorem 2.5.10. *Suppose $X \subseteq V_{\lambda+1}$ and strong inverse limit X-reflection holds at α. Then there are no disjoint stationary subsets S_1 and S_2 of $\{\beta < \lambda^+ | \operatorname{cof}(\beta) = \omega\}$ such that*

$$S_1, S_2 \in L_\alpha(X, V_{\lambda+1}).$$

Corollary 2.5.11. *Assume there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Suppose that $G \subseteq Coll(\omega, \omega_1)$ is V-generic. Then in $V[G]$, (strong) inverse limit $V_{\lambda+1}$-reflection at 1 does not hold.

Proof. We work with $(H(\lambda^+), V_{\lambda+1})$ for ease of notation. We have that for

$$S_1 = \{\alpha < \lambda^+ | \, (\mathrm{cof}(\alpha) = \omega)^{L(V_{\lambda+1})}\} \text{ and } S_2 = \{\alpha < \lambda^+ | \, (\mathrm{cof}(\alpha) = \omega_1)^{L(V_{\lambda+1})}\},$$

that S_1 and S_2 are definable over $(H(\lambda^+)^{V[G]}, V_{\lambda+1})$. Furthermore, S_1 and S_2 are stationary in $V[G]$. And since

$$S_1, S_2 \in L_1(H(\lambda^+)^{V[G]}, V_{\lambda+1}),$$

we have that inverse limit $V_{\lambda+1}$-reflection at 1 does not hold by Theorem 2.5.10. $\square$

2.6 Strong inverse limit reflection

In this section we show that strong inverse limit reflection holds all the way up to Θ. If the reader is tired of reflection at this point, this section can be skipped, as it is only used to weaken the large cardinal assumptions of Corollary 3.1.9. We do however develop the theory of inverse limits significantly further to show, for instance, that inverse limit roots display the pointwise non-decreasing property of their embeddings.

We first show that if an inverse limit has a limit root which extends to an elementary embedding, then it extends as well and in fact factors through its limit root, in some sense.

Lemma 2.6.1. *Suppose $\alpha < \Theta$ is good and that $K \in \mathcal{E}_\alpha$, $J \in \mathcal{E}_{\alpha+1}$ and K is a limit root of J. Suppose that K extends to an elementary embedding*

$$\hat{K} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1})$$

and that for $\langle \bar{\alpha}_n | \, n < \omega \rangle$ defined by $\bar{\alpha}_n = (k_0 \circ \cdots \circ k_{n-1})(\bar{\alpha})$ we have for some $n < \omega$ that $crit(K_n^{(n-1)}) > \bar{\lambda}$ and $\bar{\alpha}_n \in rng(j_0 \circ \cdots \circ j_{n-1})$. Then for some $\bar{\beta} \geq \bar{\alpha}$, J extends to an elementary embedding

$$\hat{J} : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_\alpha(V_{\lambda+1}).$$

Proof. Define $M_J^\alpha \subseteq J_\alpha(V_{\lambda+1})$ as

$$M_J^\alpha = \{a \in J_\alpha(V_{\lambda+1}) | \, \exists A \in V_{\lambda+1}(a \text{ is definable from } A \text{ over } J_\alpha(V_{\lambda+1}) \text{ and } A \in \mathrm{rng}\, J)\}.$$

Then M_J^α is wellfounded, satisfies $ZF^- -$ Replacement since $J_\alpha(V_{\lambda+1})$ has these properties and M_J^α is closed under definability over $J_\alpha(V_{\lambda+1})$.

Similarly define M_K^α. Note that $M_K^\alpha = \hat{K}[J_{\bar\alpha}(V_{\bar\lambda+1})]$, and hence $\hat{K}$ is just the inverse of the transitive collapse of M_K^α. Also define $M^\alpha_{K_n^{(n-1)}}$ in the same way and let $\bar\alpha_n$, $\bar\lambda_n$, and $\hat{K}_n^{(n-1)}$ be such that for all $n < \omega$, $J_{\bar\alpha_n}(V_{\bar\lambda_n+1})$ is the transitive collapse of $M^\alpha_{K_n^{(n-1)}}$ and $\hat{K}_n^{(n-1)}$ is the inverse of the collapse.

Let $n < \omega$ be such that $\mathrm{crit}\,(K_n^{(n-1)}) > \bar\lambda$ and $\bar\alpha_n \in \mathrm{rng}\,(j_0 \circ \cdots \circ j_{n-1})$. Set $\bar\beta = (j_0 \circ \cdots \circ j_{n-1})^{-1}(\bar\alpha_n)$. Then we have that for all $\bar A \in V_{\bar\lambda+1}$,

$$K_n^{(n-1)}((j_0 \circ \cdots \circ j_{n-1})(\bar A)) = J_n^{(n-1)}((j_0 \circ \cdots \circ j_{n-1})(\bar A)).$$

Hence we must have $M_J^\alpha \subseteq M^\alpha_{K_n^{(n-1)}}$.

Furthermore we claim that

$$M_J^\alpha = (\hat{K}_n^{(n-1)} \circ j_0 \circ j_1 \circ \cdots \circ j_{n-1})[J_{\bar\beta}(V_{\bar\lambda+1})].$$

To see this, suppose that $a \in M_J^\alpha$ and that $A \in V_{\lambda+1} \cap \mathrm{rng}\, J$ is such that a is definable over $J_\alpha(V_{\lambda+1})$ from A by some formula ϕ. Let $\bar A \in V_{\bar\lambda+1}$ be such that $J(\bar A) = A$, and set $\bar A_n = (j_0 \circ \cdots \circ j_{n-1})(\bar A)$. Then we have that

$$K_n^{(n-1)}(\bar A_n) = J_n^{(n-1)}(\bar A_n) = A.$$

Hence by elementarity of $\hat{K}_n^{(n-1)}$ we have that there is $\bar a_n \in J_{\bar\alpha_n}(V_{\bar\lambda_n+1})$ such that $\bar a_n$ is defined by ϕ over $J_{\bar\alpha_n}(V_{\bar\lambda_n+1})$ with parameter $\bar A_n$. But then by elementarity of $j_0 \circ \cdots \circ j_{n-1}$ we have that there is $\bar a \in J_{\bar\alpha}(V_{\bar\lambda+1})$ such that $\bar a$ is defined by ϕ over $J_{\bar\alpha}(V_{\bar\lambda+1})$ with parameter $\bar A$. And hence we have that

$$(\hat{K}_n^{(n-1)} \circ j_0 \circ j_1 \circ \cdots \circ j_{n-1})(\bar a) = a.$$

And since a was arbitrary we have the claim.

Hence, putting everything together, we have that

$$\hat{K}_n^{(n-1)} \circ j_0 \circ j_1 \circ \cdots \circ j_{n-1} : J_{\bar\beta}(V_{\bar\lambda+1}) \to J_\alpha(V_{\lambda+1})$$

is the desired extension of J. □

The previous lemma required that some $\bar\alpha_n$ be in the range of the fragments of J. The next lemma shows that we can always find such a K where this occurs.

Lemma 2.6.2. *Suppose $\alpha < \Theta$ is good and $J \in \mathcal{E}_{\alpha+1}$. Suppose further that there is a $K \in \mathcal{E}_\alpha$, a limit root of J such that for some $\bar\alpha$, K extends to an elementary embedding*

$$\hat{K} : J_{\bar\alpha}(V_{\bar\lambda+1}) \to J_\alpha(V_{\lambda+1}).$$

Then there is a $K \in \mathcal{E}_\alpha$ a limit root of J such that for some $n < \omega$ we have $\mathrm{crit}\,(K_n^{(n-1)}) > \bar\lambda_J$, $K_n^{(n-1)}$ extends to an elementary embedding

$$\hat{K}_n^{(n-1)} : J_{\bar\alpha}(V_{\bar\lambda_n+1}) \to J_\alpha(V_{\lambda+1})$$

for some $\bar\alpha$ and $\bar\alpha \in \mathrm{rng}\,(j_0 \circ \cdots \circ j_{n-1})$.

Proof. Fix $\delta < \lambda$ such that $\delta > \bar{\lambda}_J$ and $\delta \in \operatorname{rng} J$ and fix n such that crit $(J_n^{(n-1)}) > \delta$. Let $\bar{\alpha}$ be least such that there is $K^n \in \mathcal{E}_{\alpha}$ a 0-close limit root of $J_n^{(n-1)}$ such that crit $(K^n) > \delta$ and K^n extends to an elementary embedding

$$\hat{K}^n : J_{\bar{\alpha}}(V_{\bar{\lambda}_n+1}) \to J_{\alpha}(V_{\lambda+1}).$$

Then since $J \in \mathcal{E}_{\alpha+1}$ and

$$J_n^{(n-1)}, \delta, \bar{\lambda}_n \in \operatorname{rng}(j_0 \circ \cdots \circ j_{n-1})$$

we have that $\bar{\alpha} \in \operatorname{rng}(j_0 \circ \cdots \circ j_{n-1})$. Now let $k_0, \ldots, k_{n-1}$ be square roots of $j_0, \ldots, j_{n-1}$ such that for some $\bar{K}^n$,

$$(k_0 \circ \cdots \circ k_{n-1})(\bar{K}^n) = K^n$$

and for $K = k_0 \circ \cdots k_{n-1} \circ \bar{K}^n \in \mathcal{E}_{\alpha}$, K is a limit root of J, and

$$(k_0 \circ \cdots \circ k_{n-1})^{-1}(\bar{\alpha}) = (j_0 \circ \cdots \circ j_{n-1})^{-1}(\bar{\alpha}).$$

Then clearly we have $\bar{K}_n^{(n-1)} = K^n$ and hence K is a witness to the lemma. □

Putting the previous two lemmas together, we obtain that a very large collection of inverse limits extend to elementary embeddings.

Lemma 2.6.3. *Suppose $\alpha < \Theta$ is good and $J \in \mathcal{E}_{\alpha+1}$. Also assume that there is a saturated set $E \subseteq \mathcal{E}_{\alpha+1}$ such that $J \in E$. Then for some $\bar{\alpha}$, J extends to an elementary embedding*

$$\hat{J} : J_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}).$$

Proof. Let J be as in the hypothesis. Then by the proof of Theorem 2.5.7 there is a sequence $\langle K^n | \, n < \omega \rangle$ such that the following hold:

1. $K^0 = J$ and for all $n < \omega$, $K^n \in \mathcal{E}_{\alpha+1}$,
2. for all $n < \omega$, K^{n+1} is a limit root of K^n,
3. there is a $\bar{\beta}$ and an n_0 such that for all $n \geq n_0$ K^n extends to an elementary embedding

$$\hat{K}^n : J_{\bar{\beta}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}).$$

By applying the previous two lemmas we have that there must be some $\bar{\alpha}_{n-1}$ such that K^{n-1} extends to an elementary embedding

$$\hat{K}^{n-1} : J_{\bar{\alpha}_{n-1}}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}).$$

And similarly by induction we have that there are $\bar{\alpha}_{n-1}, \ldots, \bar{\alpha}_0$ such that for all $i < n$, K^i extends to an elementary embedding

$$\hat{K}^i : J_{\bar{\alpha}_0}(V_{\bar{\lambda}+1}) \to J_{\alpha}(V_{\lambda+1}).$$

So considering $i = 0$ the lemma follows. □

We now show that if J extends to $\alpha+1$ for α good, then the extension is in some sense unique.

Lemma 2.6.4. *Suppose $\alpha < \Theta$ is good, that $J \in \mathcal{E}_{\alpha+1}$ and J extends to an elementary embedding*

$$\hat{J} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}_J+1}) \to J_{\alpha+1}(V_{\lambda+1})$$

for some $\bar{\alpha}$. Then for all $\beta \geq \alpha+1$, if J extends to an elementary embedding

$$\hat{J}^* : J_{\bar{\beta}}(V_{\bar{\lambda}_J+1}) \to J_{\beta}(V_{\lambda+1})$$

with $\alpha \in rng\,\hat{J}^*$, then $(\hat{J}^*)^{-1}(\alpha) = \bar{\alpha}$ and in fact

$$\hat{J}^* \restriction J_{\bar{\alpha}+1}(V_{\bar{\lambda}_J+1}) = \hat{J}.$$

Proof. The main point is that $\bar{\alpha}$ is $\bar{\lambda}_J$-good. And hence we claim that

$$\operatorname{rng} \hat{J} = M^J_\alpha = \operatorname{rng}(\hat{J}^*) \cap J_\alpha(V_{\lambda+1}),$$

where M^J_α is defined as in the proof of Lemma 2.6.1.

To see the first equality, suppose that $a \in \operatorname{rng} \hat{J}$. Let $\bar{a} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ be such that $\hat{J}(\bar{a}) = a$ and let $\bar{A}$ and ϕ be such that $\bar{a}$ is defined by ϕ with parameter $\bar{A}$ over $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. Let $A = J(\bar{A})$. By elementarity we have that a is defined by ϕ with parameter A over $J_\alpha(V_{\lambda+1})$. Hence $a \in M^J_\alpha$.

Now suppose that $a \in M^J_\alpha$ and let $A = J(\bar{A})$ and ϕ be such that a is defined by ϕ with parameter A over $J_\alpha(V_{\lambda+1})$. Then by elementarity, there is $\bar{a} \in J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$ such that $\bar{a}$ is defined by ϕ with parameter $\bar{A}$ over $J_{\bar{\alpha}}(V_{\bar{\lambda}+1})$. And hence by elementarity $\hat{J}(\bar{a}) = a$.

So we have that $\operatorname{rng} \hat{J} = M^J_\alpha$. Now it is enough to see that $M^J_\alpha = \operatorname{rng}(\hat{J}^*) \cap J_\alpha(V_{\lambda+1})$. But we have that $(\hat{J}^*)^{-1}(\alpha)$ is $\bar{\lambda}_J$-good, and hence the same argument will work with $\bar{\alpha}$ replaced by $(\hat{J}^*)^{-1}(\alpha)$. So the lemma follows. □

We define now an ordering on certain equivalence classes of elements of V_λ. This is a natural ordering generated by an inverse limit J, and it turns out to be a well-ordering if $J \in \mathcal{E}_1$.

Definition 2.6.5. Let $J \in \mathcal{E}$, and define the ordering $\leq_J$ on tuples (α, n) for $\alpha < \lambda$ and $n < \omega$ as follows:

1. $(\alpha, n) \leq_J (\beta, n)$ if $\alpha \leq \beta$.

2. $(\alpha, n) \leq_J (\beta, m)$ if $n \leq m$ and $(j_n^{(n-1)} \circ \cdots \circ j_{m-1}^{(n-1)})(\alpha) \leq \beta$.

3. $(\alpha, n) \leq_J (\beta, m)$ if $m \leq n$ and $(j_m^{(m-1)} \circ \cdots \circ j_{n-1}^{(m-1)})(\beta) \leq \alpha$.

We put $(\alpha, n) \sim_J (\beta, m)$ if $(\beta, m) \leq_J (\alpha, n)$ and $(\alpha, n) \leq_J (\beta, n)$. Let $[\alpha, n]_J$ be the equivalence class of (α, n) under the equivalence relation $\sim_J$. Let $\mathcal{I}^J$ be the set of equivalence classes $[\alpha, n]_J$ for $\alpha < \lambda$. Let $\mathcal{I}^J_{\leq(\gamma,m)}$ be the set of equivalence classes $[\alpha, n]_J$ such that $(\alpha, n) \leq_J (\gamma, m)$.

Lemma 2.6.6. *Suppose $J \in \mathcal{E}_1$. Then $(\mathcal{I}^J, \leq_J)$ is a well-ordering.*

We give two separate proofs of this lemma.

Proof 1. Suppose the lemma fails for J. For all $n < \omega$ let α_n be least such that $\leq_{J_n^{(n-1)}}$ restricted to those elements $\leq_{J_n^{(n-1)}}$-less than or equal to $[\alpha_n, n]_{J_n^{(n-1)}}$ is not well-founded. Then, since $J \in \mathcal{E}_1$, for all $n < \omega$, $\alpha_n \in \operatorname{rng}(j_0 \circ \cdots \circ j_{n-1})$. But by definition of the α_n we clearly must have $j_n^{(n-1)}(\alpha_n) = \alpha_{n+1}$. Hence $(\alpha_n, n) \in [\alpha_0, 0]_J$ for all n. Now, let $\langle [\beta_n, i_n]_J | \, n < \omega \rangle$ be a $\leq_J$-decreasing sequence below $[\alpha_0, 0]_J$. Then we have $\beta_0 < \alpha_{i_0}$. But by definition of α_{i_0}, $\leq_J$ is wellfounded below $[\beta_0, i_0]_J$, a contradiction. □

Proof 2. Suppose that $\langle [\alpha_i, n_i]_J | \, i < \omega \rangle$ are such that $(\alpha_i, n_i) >_J (\alpha_{i+1}, n_{i+1})$ for all $i < \omega$. Let $K \in \mathcal{E}_0$ have the following properties:

1. For all $i, n, m < \omega$, and $(\alpha, n') \in [\alpha_i, n_i]_J$,
$$(j_n^{(n-1)} \circ \cdots j_{m-1}^{(n-1)})(\alpha) = (k_n^{(n-1)} \circ \cdots k_{m-1}^{(n-1)})(\alpha).$$
2. For all $i < \omega$ and $(\alpha, n) \in [\alpha_i, n_i]_J$ we have that
$$\alpha \in \operatorname{rng}(k_0 \circ \cdots \circ k_{n-1}).$$

It is then easy to see that (1) implies that for all $i < \omega$,
$$[\alpha_i, n_i]_J \subseteq [\alpha_i, n_i]_K,$$
and (2) implies that for all $i < \omega$, there exists an α'_i such that
$$(\alpha'_i, 0) \in [\alpha_i, n_i]_K.$$
But then for all $i < \omega$, $(\alpha_i, n_i) >_K (\alpha_{i+1}, n_{i+1})$, and hence $\alpha'_i > \alpha'_{i+1}$, a contradiction. □

We need the iterated version of being a limit root for inverse limits.

Definition 2.6.7. For $\alpha < \omega_1$ we define an *α-limit root sequence* $\langle K^n | \, n < \alpha \rangle$ by induction as follows. A 1-limit root sequence is just $\langle K^0 \rangle$ such that $K^0 \in \mathcal{E}$. For $\alpha = \beta + 1$ a successor, $\langle K^n | \, n < \alpha \rangle$ is an α-limit root sequence if $\langle K^n | \, n < \beta \rangle$ is a β-limit root sequence and the following hold:

1. If β is a limit, then K^β is the common part of the sequence $\langle K^n | \, n < \beta \rangle$.

2. If β is a successor, then K^β is a limit root of $K^{\beta-1}$.

If α is a limit, then $\langle K^n | \, n < \alpha\rangle$ is an α-limit root if for all $\beta < \alpha$, $\langle K^n | \, n < \beta\rangle$ is a β-limit root.

We say that K is an *α-limit root* of J if there is an $\alpha+1$-limit root sequence $\langle K^n | \, n \leq \alpha\rangle$ such that $K^0 = J$ and $K^\alpha = K$. So K is a limit root of J iff K is a 1-limit root of J.

Suppose $\gamma < \Theta$ is good and suppose that $c : \alpha \to \omega$ is a function. Then we say that $\langle K^n | \, n < \alpha\rangle$ is an *α-limit root sequence following c at γ* if the following hold:

1. For all $n < \alpha$, $K^n \in \mathcal{E}_\gamma$.

2. Suppose that $\alpha = \beta + 1$ is a successor. Then K^α is a $c(\alpha)$-close limit root of K^β.

Lemma 2.6.8. *Suppose that $\gamma < \Theta$ is good, $\alpha < \omega_1$, and $c : \alpha \to \omega$ is an injection. Suppose that $K^0 \in \mathcal{E}_{\gamma+\omega}$. Then there is $\langle K^n | \, n < \alpha\rangle$ an α-limit root sequence following c at γ.*

Proof. First let K^1 be a 0-close limit root of K^0 such that for all $i < \omega$, k^1_i extends to an embedding

$$J_{\gamma+i+1}(V_{\lambda+1}) \to J_{\gamma+i+1}(V_{\lambda+1}).$$

For $\alpha' < \alpha$ such that $\alpha' = \beta + 1$, having defined the sequence below α', we choose $K^{\alpha'}$ to be a $c(\alpha')$-close limit root of K^β such that for all $i \in [c(\alpha'), \omega)$, if k^β_i extends to an embedding

$$J_{\gamma+s_i+1}(V_{\lambda+1}) \to J_{\gamma+s_i+1}(V_{\lambda+1})$$

then $k^{\alpha'}_i$ extends to an embedding

$$J_{\gamma+s_i}(V_{\lambda+1}) \to J_{\gamma+s_i}(V_{\lambda+1}).$$

For $\alpha' < \alpha$ a limit, we simply take $K^{\alpha'}$ to be the common part of $\langle K^n | \, n < \alpha'\rangle$.

Clearly this construction succeeds, as for all $i < \omega$, the set $\{\alpha' < \alpha | \, c(\alpha') \leq i\}$ has cardinality less than or equal to i, as c is injective. □

We note the following fact about square roots of elementary embeddings, which we will extend to inverse limits.

Lemma 2.6.9. *Let α be good. Suppose that $j, k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ and $k(k) = j$. Then for all $\beta < \alpha$, we have that $k(\beta) \geq j(\beta)$.*

Proof. We prove this by induction on β. If β is a successor or a continuity point of j then there is nothing to prove. So assume that β is a discontinuity point of j. Let $\gamma = \sup j``\beta$. We have by induction that $\gamma \leq \sup k``\beta$. Suppose for a contradiction that $k(\beta) < j(\beta)$. Then $j(\beta)$ is definable in $L_\alpha(V_{\lambda+1})$ from $j \restriction V_{\lambda+1}$ and $k(\beta)$, as the least image point of the unique extension of $j \restriction V_{\lambda+1}$ above $k(\beta)$. But then $j(\beta) \in \operatorname{rng} k$, and since $j \in \operatorname{rng} k$, we have that $\beta \in \operatorname{rng} k$. But then $k(\beta) = j(\beta)$, since it is a square root, a contradiction. □

Lemma 2.6.10. *Suppose $\alpha < \Theta$ is good, $\delta + 1 < \omega_1$, and $\langle J^\gamma | \gamma < \delta + 1\rangle$ is a $\delta + 1$-limit root sequence from $\mathcal{E}_1$. Then for all $[\beta, n]_{J^0}$, there is an $n_0 < \omega$ such that for all $\gamma < \delta + 1$ and $m, m' \geq n_0$, if $(\beta_1, m), (\beta_2, m') \in [\beta, n]_{J^0}$ then*

$$(\beta_1, m) \sim_{J^\gamma} (\beta_2, m')$$

and hence if J^γ extends to an embedding

$$\hat{J}^\gamma : J_{\bar{\alpha}_\gamma+1}(V_{\bar{\lambda}_{J^\gamma}+1}) \to J_{\alpha+1}(V_{\lambda+1})$$

then we have

$$\hat{J}^{\gamma,(m-1)}_m(\beta_1) = \hat{J}^{\gamma,(m'-1)}_{m'}(\beta_2).$$

Proof. Fixing $[\beta, n]_{J^0}$, we prove first that for each $\gamma < \delta$ there is such an n_0, the least which we call n_γ. The full lemma follows by noticing that for γ a limit, $n_\gamma \geq \sup_{\gamma' < \gamma} n_{\gamma'}$. The proofs of these two facts are basically the same.

Let $j_i = j^0_i$ and $k_i = j^\gamma_i$ for $i < \omega$. So $K = J^\gamma$. Then we have for all $\xi < \lambda$ and $i < \omega$ that $k_i(\xi) \geq j_i(\xi)$. Hence for all $n < \omega$, if $(\alpha_n, n), (\alpha_{n+1}, n+1) \in [\beta, n]_J$ and $(\alpha_n, n) \nsim_K (\alpha_{n+1}, n+1)$ then

$$k^{(n-1)}_n(\alpha_n) > \alpha_{n+1} = j^{(n-1)}_n(\alpha_n).$$

Hence, if there are infinitely many $n < \omega$ such that

$$(\alpha_n, n) \nsim_K (\alpha_{n+1}, n+1),$$

then $\langle [\alpha_n, n]_K | n < \omega\rangle$ contains an infinite decreasing subsequence in the $\leq_K$ ordering, which is a contradiction to the well-foundedness of $\leq_K$.

For the limit step, basically the same proof works, since if γ is such that

$$j^{\gamma,(n-1)}_n(\alpha_n) > j^{0,(n-1)}_n(\alpha_n),$$

then for all $\gamma' \in [\gamma, \delta]$,

$$j^{\gamma',(n-1)}_n(\alpha_n) > j^{0,(n-1)}_n(\alpha_n).$$

And hence the lemma follows. □

We need the following notation. Let $\sim$ be the equivalence relation defined as follows. Suppose $(K, \vec{k}), (K', \vec{k}')$ are such that for some n and m,

$$(k_0 \circ \cdots \circ k_{n-1})(\langle k_n, k_{n+1}, \ldots\rangle) = (k'_0 \circ \cdots \circ k'_{m-1})(\langle k'_m, k'_{m+1}, \ldots\rangle).$$

Then $\vec{k} \sim \vec{k}'$. We let $[\vec{k}]_\sim$ denote the equivalence class which $\vec{k}$ belongs to.

Lemma 2.6.11. *Suppose $\alpha < \Theta$ is good, $\delta < \omega_1$, $J \in \mathcal{E}_{\alpha+3}$, and J extends to an elementary embedding*

$$\hat{J} : J_{\bar{\alpha}+1}(V_{\bar{\lambda}_J+1}) \to J_{\alpha+1}(V_{\lambda+1})$$

for some $\bar{\alpha}$. Then if $K \in \mathcal{E}_{\alpha+2}$ is a δ-limit root of J, and K extends to an elementary embedding

$$\hat{K} : J_{\bar{\beta}+1}(V_{\bar{\lambda}_J+1}) \to J_{\alpha+1}(V_{\lambda+1})$$

for some $\bar{\beta}$ then $\bar{\beta} \leq \bar{\alpha}$ and for all $\bar{\gamma} \leq \bar{\beta}$, $\hat{K}(\bar{\gamma}) \geq \hat{J}(\bar{\gamma})$.

Proof. First we prove the lemma for $\delta = 1$. This will, in essence, prove the lemma for all δ successor (assuming the limit case is true as well).

Suppose the lemma fails, and let $[\bar{\alpha}, n]_J$ be $\leq_J$-least such that there exists $K \in \mathcal{E}_{\alpha+2}$ a limit root of J with $\hat{K}_n^{(n-1)}(\bar{\alpha}) < \hat{J}_n^{(n-1)}(\bar{\alpha})$. Assume for ease of notation that $n = 0$ and that $K \in \mathcal{E}_{\alpha+2}$ is a 0-close limit root of J. Then we have that $\bar{\alpha}$ is definable over $J_{\alpha+2}(V_{\lambda+1})$ from J and $\hat{K}(\bar{\alpha})$ as the least ordinal sent by $\hat{J}$ above $\hat{K}(\bar{\alpha})$. Hence for all n we have

$$(k_0 \circ \cdots \circ k_n)(\bar{\alpha}) = (j_0 \circ \cdots \circ j_n)(\bar{\alpha}).$$

So for all n we have

$$\hat{K}_n^{(n-1)}(\bar{\alpha}_n) < \hat{J}_n^{(n-1)}(\bar{\alpha}_n)$$

where $\bar{\alpha}_n = (j_0 \circ \cdots \circ j_{n-1})(\bar{\alpha})$.

Let β be least such that for some $K \in \mathcal{E}_{\alpha+2}$ a limit root of J, $\hat{K}(\bar{\alpha}) = \beta$. Then we have that $\beta < \hat{J}(\bar{\alpha})$ and $\beta \geq \sup_{\bar{\beta}<\bar{\alpha}} \hat{J}(\bar{\beta})$.

We claim that $\beta \in \operatorname{rng} \hat{J}$, which is a contradiction. To see this, we claim that β is definable from $[J]_\sim$ and $\hat{J}(\bar{\alpha})$ over $J_{\alpha+3}(V_{\lambda+1})$. And this follows since for any $S \in [J]_\sim$, for all large enough n, if

$$\bar{\alpha}' = (\hat{S}_n^{(n-1)})^{-1}(\hat{J}(\bar{\alpha}))$$

then β is least such that for some $K \in \mathcal{E}_{\alpha+2}$ a limit root of $S_n^{(n-1)}$, $\hat{K}(\bar{\alpha}') = \beta$. Hence since $[J]_\sim \in \operatorname{rng} \hat{J}$, we have $\beta \in \operatorname{rng} \hat{J}$.

Now we prove the lemma for δ a limit, assuming the lemma is true for all $\delta' < \delta$.

Suppose the lemma fails for δ and $\langle K^{\delta'} | \, \delta' < \delta \rangle$ is a limit root sequence with K the common part witnessing this failure. Let $\delta' < \delta$ be least such that for some $(\bar{\beta}, m)$,

$$(\hat{K}^{\delta'})_m^{(m-1)}(\bar{\beta}) > \hat{K}_m^{(m-1)}(\bar{\beta}).$$

Without loss of generality, by renaming, we can assume that $\delta' = 0$. Let $[\bar{\beta}, m]_{K^0}$ be $\leq_{K^0}$-least such that for some $(\bar{\beta}_0, m_0) \in [\bar{\beta}, m]_{K^0}$,

$$(\hat{K}^0)_{m_0}^{(m_0-1)}(\bar{\beta}_0) > \hat{K}_{m_0}^{(m_0-1)}(\bar{\beta}_0).$$

Then in fact by the previous lemma, there is an n_0 such that for all $m' \geq n_0$ and

$$(\bar{\beta}', m'), (\bar{\beta}_{n_0}, n_0) \in [\bar{\beta}, m]_{K^0},$$

we have

$$(\hat{K}^0)_{m'}^{(m'-1)}(\bar{\beta}') = (\hat{K}^0)_{m_0}^{(m_0-1)}(\bar{\beta}_0) > \hat{K}_{m_0}^{(m_0-1)}(\bar{\beta}_0) \geq \hat{K}_{m'}^{(m'-1)}(\bar{\beta}') = \hat{K}_{n_0}^{(n_0-1)}(\bar{\beta}_{n_0}).$$

The last inequality follows from the fact that K is a δ-limit root of K^0.

Again by renaming, we can assume without loss of generality that $n_0 = 0$. Let $(\bar{\beta}_0, 0) \in [\bar{\beta}, m]_{K^0}$. Then for all n we have

$$(k_0 \circ \cdots \circ k_n)(\bar{\beta}_0) = (k_0^0 \circ \cdots \circ k_n^0)(\bar{\beta}_0).$$

So for all n we have

$$\hat{K}_n^{(n-1)}(\bar{\beta}_n) < \hat{K}_n^{0,(n-1)}(\bar{\beta}_n)$$

where $\bar{\beta}_n = (k_0^0 \circ \cdots \circ k_{n-1}^0)(\bar{\beta}_0)$.

Let β be least such that for some $K \in \mathcal{E}_{\alpha+2}$ a δ-limit root of J, $\hat{K}(\bar{\alpha}) = \beta$. Then we have that $\beta < \hat{J}(\bar{\alpha})$ and $\beta \geq \sup_{\bar{\beta}<\bar{\alpha}} \hat{J}(\bar{\beta})$.

We claim that $\beta \in \operatorname{rng} \hat{J}$, which is a contradiction. To see this, we claim that β is definable from $[J]_\sim$ and $\hat{J}(\bar{\alpha})$ over $J_{\alpha+3}(V_{\lambda+1})$. And this follows since for any $S \in [J]_\sim$, for all large enough n, if

$$\bar{\alpha}' = (\hat{S}_n^{(n-1)})^{-1}(\hat{J}(\bar{\alpha}))$$

then β is least such that for some $K \in \mathcal{E}_{\alpha+2}$ a δ-limit root of $S_n^{(n-1)}$, $\hat{K}(\bar{\alpha}') = \beta$. Hence since $[J]_\sim \in \operatorname{rng} \hat{J}$, we have $\beta \in \operatorname{rng} \hat{J}$. □

Lemma 2.6.12. *Let $\alpha < \Theta$ be good and $J \in \mathcal{E}_{\alpha+\omega}$. Then for some $\gamma < \omega \cdot \omega$ there is $K \in \mathcal{E}_\alpha$ which is a γ-limit root of J such that there is a saturated set E and $\bar{\alpha}$ such that $K \in E$ and for all $K' \in CL(E)$, K' extends to an embedding*

$$\hat{K}' : J_{\bar{\alpha}}(V_{\bar{\lambda}_K+1}) \to J_\alpha(V_{\lambda+1}).$$

Proof. Let $c : \omega \cdot \omega \to \omega$ be an injection. We attempt to construct an $\omega \cdot \omega$-limit root sequence $\langle K^n | \, n < \omega \cdot \omega \rangle$ following c at $\alpha + 2$ such that for all $n < \omega \cdot \omega$ we have K^n extends to

$$\hat{K}^n : J_{\bar{\alpha}_n}(V_{\bar{\lambda}_K+1}) \to J_\alpha(V_{\lambda+1}),$$

and for all $i < \omega$, $\bar{\alpha}_{\omega \cdot i} > \bar{\alpha}_{\omega \cdot (i+1)}$. Clearly we can't actually construct such a sequence. Hence our attempt must fail at some point, at which point the lemma will hold.

We construct the sequence as follows by induction for $i < \omega$. Let $K^0 = J$. Having constructed $\langle K^n | \, n \leq \omega \cdot i \rangle$, if there exists an extension $\langle K^n | \, n \leq \omega \cdot (i+1) \rangle$ a limit root sequence following $c \restriction \omega \cdot (i+1)$ at $\alpha + 2$ such that

$$\bar{\alpha}_{\omega \cdot (i+1)} < \lim_{n \to \omega \cdot (i+1)} \bar{\alpha}_n,$$

then use any such extension. Otherwise there is some $m < \omega$ and an extension

$$\langle K^n |\, n \leq \omega \cdot i + m \rangle$$

such that for all further extensions following c at $\alpha + 2$, $\langle K^n |\, n \leq \omega \cdot (i+1) \rangle$ we have

$$\bar{\alpha}_{\omega \cdot (i+1)} = \bar{\alpha}_{\omega \cdot i + m},$$

in which case such extensions form a saturated set as desired by the lemma. $\square$

Corollary 2.6.13. *Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then for all $\alpha < \Theta$ good, strong inverse limit reflection at α holds.*

Chapter 3

Structural Properties of $L(V_{\lambda+1})$

3.1 Club filter on λ^+

3.1.1 Comparison with $L(\mathbb{R})$

We first give a brief summary of ultrafilters on ω_1 and larger regular cardinals in $L(V_{\lambda+1})$. The starting point is the following theorem of Solovay.

Theorem 3.1.1 (Solovay (see [KW10])). *Assume $AD^{L(\mathbb{R})}$. Then in $L(\mathbb{R})$ the club filter on ω_1 is a countably complete ultrafilter.*

The proof proceeds by considering for $A \subseteq \omega_1$ the following sup game on ω_1.

$$\begin{array}{cccccccc} I & & \alpha_0 & & \alpha_1 & & \cdots \\ II & & & \beta_0 & & \beta_1 & \cdots \end{array}$$

Where the rules are that

$$\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots < \omega_1,$$

and I wins if $\sup_{i<\omega} \alpha_i \in A$. In fact, as we are assuming only $AD^{L(\mathbb{R})}$, an integer version of this game must be played, and a boundedness property used to show that if I has a winning strategy then A contains a club.

For larger regular cardinals we have the following.

Theorem 3.1.2 (Steel (see [Ste95])). *Assume $AD^{L(\mathbb{R})}$. Then in $L(\mathbb{R})$ for all $\kappa < \Theta$, κ is measurable, and this is witnessed by the ω-club filter.*

3.1.2 Partition measures on λ^+

We prove in this section results due to Woodin which are similar to the above results for $L(\mathbb{R})$. In particular we consider the club filter partitioned into stationary sets.

Lemma 3.1.3 (Woodin)**.** *Assume there is $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Suppose that $\gamma < \Theta$ and cof(γ) $> \lambda$. Let $S = \{\alpha < \gamma \mid cof(\alpha) = \omega\}$ and let $\mathcal{F}$ be the ω-club filter on γ. Then there is a partition $\langle S_\alpha \mid \alpha < \eta\rangle \in L(V_{\lambda+1})$ of S such that $\eta < \lambda$ and for all $\alpha < \eta$, in $L(V_{\lambda+1})$ $\mathcal{F} \restriction S_\alpha$ is an ultrafilter.*

Proof. Assume towards a contradiction that there is a partition $\langle S_\alpha \mid \alpha < \lambda\rangle \in L(V_{\lambda+1})$ of S into stationary subsets in $L(V_{\lambda+1})$. We can assume that γ is least such that

1. $\lambda^+ \leq \gamma$ and $\mathrm{cof}(\gamma) > \lambda$,
2. and there exists such a partition of $\{\alpha < \gamma \mid \mathrm{cof}(\alpha) = \omega\}$.

So we have that $j(\gamma) = \gamma$. Let $\langle T_\alpha \mid \alpha < \lambda\rangle = j(\langle S_\alpha \mid \alpha < \lambda\rangle)$. By elementarity $\langle T_\alpha \mid \alpha < \lambda\rangle$ is a partition of S into stationary subsets in $L(V_{\lambda+1})$.

Let $C = \{\alpha < \gamma \mid j(\alpha) = \alpha\}$. Let $\xi \in C \cap T_{\mathrm{crit}\,(j)}$. Let α be such that $\xi \in S_\alpha$. Then $j(\xi) = \xi \in T_{(j(\alpha))}$. But this is a contradiction.

Finally, since $\mathcal{F}$ is λ^+-complete we are done. □

In fact we have the following stronger theorem of Woodin.

Theorem 3.1.4 (Woodin [Woo11])**.** *Suppose there is an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Then in $L(V_{\lambda+1})$, Θ is a limit of measurable cardinals, and this is witnessed by the club filter on a stationary set.

3.1.3 Weak-club filter

In this section we use inverse limit reflection to obtain results related to the club filter on λ^+ in $L(V_{\lambda+1})$. We cannot quite show that the ω-club filter restricted to the cofinality ω ordinals is an ultrafilter in $L(V_{\lambda+1})$, but we obtain a couple approximations to this result. Namely, we show that the weak ω-club filter is an ultrafilter in $L(V_{\lambda+1})$, and that any two disjoint stationary (in V) subsets of the cofinality ω ordinals must not be in $L(V_{\lambda+1})$.

These results extend to higher ordinals of cofinality greater than λ, though for simplicity of notation we prove them for λ^+. We will simply state these extensions below, as the proofs are nearly identical.

Fix λ, $\bar\lambda < \lambda$ and a surjection $\bar\rho : V_{\bar\lambda+1} \to L_{\bar\lambda^+}(V_{\bar\lambda+1})$ definable over $L_{\bar\lambda^+}(V_{\bar\lambda+1})$. Also let E be a saturated set of inverse limits such that for all $(J, \vec{j}) \in E$, J extends to

$$\hat{J} : L_{\bar\lambda^+ +1}(V_{\bar\lambda+1}) \to L_{\lambda^+ +1}(V_{\lambda+1}).$$

Assume ρ is a surjection $\rho : V_{\lambda+1} \to L_{\lambda^+}(V_{\bar\lambda+1})$ and for all $(J, \vec{j}) \in E$, $\hat{J}(\bar\rho) = \rho$ (see the remark before Theorem 2.5.7). We will say that A tags b (over $V_{\lambda+1}$) if $\rho(A) = b$, and similarly for $\bar\rho$.

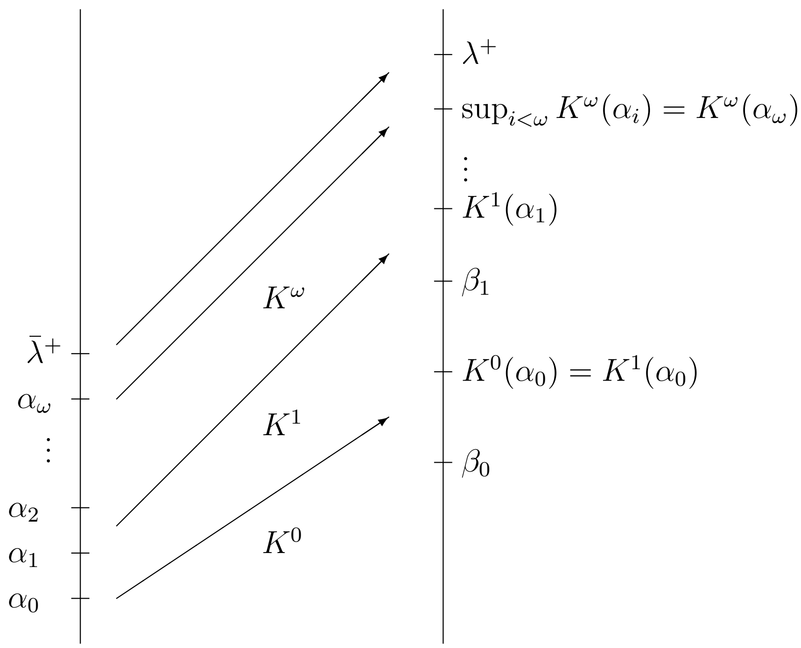

Figure 3.1: The game $G(\langle \alpha_i | \, i < \omega \rangle, E)$, where K^ω is the common part of $\langle K^i | \, i < \omega \rangle$.

Consider the following game $G(\langle \alpha_i | \, i < \omega \rangle, E)$ (see Figure 3.1.3), where $\langle \alpha_i | \, i < \omega \rangle$ is an increasing sequence of ordinals $< \bar{\lambda}^+$.

$$\begin{array}{lcccc c} I & \beta_0 & & \beta_1 & & \cdots \\ II & & (K^0, \vec{k}^0), A_0 & & (K^1, \vec{k}^1), A_1 & \cdots \end{array}$$

With the following rules:

1. $\beta_0 < \beta_1 < \cdots < \lambda^+$ are limit ordinals.

2. For all i, $(K^i, \vec{k}^i) \in E$, and K^{i+1} is a limit root of K^i.

3. Let $\hat{K}^i$ be the extension of K^i to $L_{\bar{\lambda}^+}(V_{\bar{\lambda}+1})$. Then we have
$$\beta_0 < \hat{K}^0(\alpha_0) < \beta_1 < \hat{K}^1(\alpha_1) < \beta_2 < \cdots < \lambda^+.$$

4. For all i, $\bar{\rho}(A_i) = \alpha_i$.

5. For all i and $n \leq i$, $K^{i+1}(A_n) = K^i(A_n)$.

II wins if the game goes on ω-many steps. This is a closed game for I, and hence determined.

We first show that II can win the analogous one step game.

Lemma 3.1.5. *Let E be saturated such that for all $(J, \vec{j}) \in E$, J extends to*
$$\hat{J} : L_{\bar{\lambda}^+ +1}(V_{\bar{\lambda}+1}) \to L_{\lambda^+ +1}(V_{\lambda+1}).$$
Then for all $(J, \vec{j}) \in E$, there exists an $\alpha < \bar{\lambda}^+$ such that for all $\beta < \lambda^+$ there is a $(K, \vec{k}) \in E$, a limit root of J, such that $\hat{K}(\alpha) \geq \beta$.

Proof. Let $(J, \vec{j}) \in E$. Then for $i = i(E, J)$ (see Definition 2.1.9), we have that for all $\gamma < \lambda^+$ there exists a $(K, \vec{k}) \in E$ such that $\hat{K}_i(\bar{\gamma}) = \gamma$ and hence $\hat{K}(\bar{\gamma}) \geq \gamma$ for some $\bar{\gamma} < \bar{\lambda}^+$. So by regularity of λ^+ there is an $\alpha < \bar{\lambda}^+$ such that for cofinally many $\beta < \lambda^+$ there is $(K, \vec{k}) \in E$, a limit root of J, such that $\hat{K}(\alpha) \geq \beta$, which is what we wanted. □

Lemma 3.1.6. *Let E be saturated such that for all $(J, \vec{j}) \in E$, J extends to*
$$\hat{J} : L_{\bar{\lambda}^+ +1}(V_{\bar{\lambda}+1}) \to L_{\lambda^+ +1}(V_{\lambda+1}).$$
Then there exists an increasing sequence $\langle \alpha_i | \, i < \omega \rangle$ such that II has a quasi-winning strategy in $G(\langle \alpha_i | \, i < \omega \rangle, E)$.

Proof. Suppose towards a contradiction that for all $\vec{\alpha} \in [\bar{\lambda}^+]^\omega$ that I has a winning strategy $\sigma^{\vec{\alpha}}$ in $G(\vec{\alpha}, E)$. We use the regularity of λ^+ to play against all of these winning strategies simultaneously.

Choose a sequence $\vec{\alpha}^*$ as follows. Let

$$\beta_0^* = \sup_{\vec{\alpha} \in [\bar{\lambda}^+]^\omega} \sigma^{\vec{\alpha}}(\emptyset) < \lambda^+.$$

Let $K^0 \in E, A_0$ and α_0^* be such that $\hat{K}^0(\alpha_0^*) > \beta_0^*$ and A_0 tags α_0^*. After having chosen $K^0, \dots, K^n \in E$ and $\alpha_0^*, \dots, \alpha_n^*$, let

$$\beta_{n+1}^* = \sup\{\sigma^{\vec{\alpha}}(\langle K^0, A_0, \dots, K^n, A_n \rangle) |\, \vec{\alpha} \in [\bar{\lambda}^+]^\omega, \forall i \leq n\, (\alpha_i = \alpha_i^*)\}.$$

Let $K^{n+1} \in E, A_{n+1}$ and α_{n+1}^* be such that K^{n+1} is a limit root of K^n, for all $i \leq n$, $K^{n+1}(A_i) = K^n(A_i)$, $\hat{K}^{n+1}(\alpha_{n+1}^*) > \beta_{n+1}^*$ and A_{n+1} tags α_{n+1}^*.

We then play $\langle K^0, A^0, K^1, A^1, \dots, \rangle$ in the game $G(\vec{\alpha}^*, E)$, against the winning strategy $\sigma^{\vec{\alpha}^*}$. But by the way we chose K^i, A_i and α_i^*, this must be a winning play by II. Hence $\sigma^{\vec{\alpha}^*}$ is not a winning strategy for I, a contradiction. $\square$

Lemma 3.1.7. *Let E be saturated such that for all $(J, \vec{j}) \in \bar{E}$, J extends to*

$$\hat{J} : L_{\bar{\lambda}^+ + 1}(V_{\bar{\lambda}+1}) \to L_{\lambda^+ + 1}(V_{\lambda+1}).$$

Suppose that $\langle \alpha_i |\, i < \omega \rangle$ is an increasing sequence of ordinals $< \bar{\lambda}^+$ and

$$(\beta_0, K^0, A_0, \beta_1, K^1, A_1, \dots)$$

is a winning play for II in $G(\langle \alpha_i |\, i < \omega \rangle, E)$. Let K be the common part of $\langle K^i |\, i < \omega \rangle$. Then

$$\hat{K}(\sup_{i<\omega} \alpha_i) = \sup_{i<\omega} \beta_i.$$

Proof. Note that we have for all i and $n \leq i$ that $C_n := K^{i+1}(A_n) = K^i(A_n)$. Hence we have that $K(A_n) = C_n$ and therefore $\hat{K}(\alpha_n) = \gamma_n$, where γ_n is tagged by C_n. And by the rules of the game, we have

$$\beta_0 < \gamma_0 < \beta_1 < \gamma_1 < \cdots.$$

Hence $\hat{K}(\sup_{i<\omega} \alpha_i) = \sup_{i<\omega} \beta_i$ follows by continuity. $\square$

Theorem 3.1.8. *Assume strong inverse limit reflection at α for $\alpha > \lambda^+$ good. Let*

$$S_\omega = \{\beta < \lambda^+ |\, cof(\beta) = \omega\}.$$

Then if $S \in L_\alpha(V_{\lambda+1})$ and $S \subseteq S_\omega$ is stationary (in V), then $S_\omega \setminus S$ is not stationary.

Proof. Suppose E, $\bar{\alpha}$, γ and $\bar{S}$ are such that $E \in L_\gamma(V_{\lambda+1})$ and for all $(K, \vec{k}) \in CL(E)$, K extends to

$$\hat{K} : L_{\bar{\alpha}}(V_{\bar{\lambda}+1}) \to L_\alpha(V_{\lambda+1})$$

and $\hat{K}(\bar{S}) = S$. We show that for any $\langle \alpha_i | \, i < \omega \rangle$ such that II has a quasi-winning strategy in $G(\langle \alpha_i | \, i < \omega \rangle, E)$, that if $\alpha = \sup \alpha_i \in \bar{S}$ then S contains a ω-club.

Suppose this is not the case, so $S_\omega \setminus S$ is stationary. We have since α is good that in $L_\gamma(V_{\lambda+1})$ II has a quasi-winning strategy in $G(\langle \alpha_i | \, i < \omega \rangle, E)$. Let $M \prec L_\gamma(V_{\lambda+1})$ be such that $|M| = \lambda$, $S, J, E \in M$, $V_\lambda \subseteq M$, and $M \cap \lambda^+ \in S_\omega \setminus S$. Let $\langle \beta_i | \, i < \omega \rangle$ be increasing and cofinal in $M \cap \lambda^+$ such that for all i, $\beta_i \in M$. Then if I plays a legal subsequence of $\langle \beta_i | \, i < \omega \rangle$, at each stage there is a winning response by player II in M. Suppose without loss of generality (by passing to a subsequence) that the game is played as $(\beta_0, K^0, A_0, \beta_1, K^1, A_1 \ldots)$ with $(K^i, \vec{k}^i) \in M$ for all i. Let K be the common part of $\langle K^i | \, i < \omega \rangle$ as computed in $L(V_{\lambda+1})$. Then by the previous lemma we have that $\hat{K}(\sup \alpha_i) = \sup M \cap \lambda^+ \in S$ by elementarity. But this is a contradiction. So $S_\omega \setminus S$ is not stationary. $\square$

Applying Corollary 2.6.13 we have the following.

Corollary 3.1.9. *Assume there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Then there are no disjoint stationary subsets S_1 and S_2 of $\{\beta < \lambda^+ | \, cof(\beta) = \omega\}$ such that $S_1, S_2 \in L(V_{\lambda+1})$.

It is unclear whether or not the conclusion follows from just an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$, as it requires strong inverse limit reflection.

Definition 3.1.10. Suppose that $C \subseteq \gamma$ for γ a limit with uncountable cofinality. Then we say that C is *weakly club* if there exists a structure $(M, \ldots)$ in a countable language such that

$$C = \{\alpha < \gamma | \, \exists (X, \ldots) \prec (M, \ldots), \sup(X \cap \gamma) = \alpha\}.$$

We say that $S \subseteq \gamma$ is *weakly stationary* if for all $C \subseteq \gamma$ weakly club, $S \cap C \neq \emptyset$. The *weak club filter on* γ is the filter generated by the set of weakly club subsets of γ. We define weakly ω-club and the weak ω-club filter analogously, restricting to countable elementary substructures.

Corollary 3.1.11. *Suppose there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Let $S_\omega = \{\beta < \lambda^+ | \, cof(\beta) = \omega\}$. Then in $L(V_{\lambda+1})$ the weak club filter restricted to S_ω is an ultrafilter.*

Proof. Assume that there exists an α such that α is good and there exists $S \in L_\alpha(V_{\lambda+1})$, $S \subseteq \lambda^+$ such that both S and $S_\omega \setminus S$ are weakly stationary in $L(V_{\lambda+1})$. But by Theorem 2.5.9 inverse limit reflection holds at α. So by the proof of Theorem 3.1.8, there is a weakly club $C \in L(V_{\lambda+1})$ such that either $C \subseteq S$ or $C \subseteq S_\omega \setminus S$, a contradiction. $\square$

We can prove similar results in exactly the same way for limit ordinals $\gamma > \lambda^+$ such that $\mathrm{cof}(\gamma) > \lambda$. For instance we have the following.

Theorem 3.1.12. *Suppose there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

and that $\gamma < \Theta$ is such that $\mathit{cof}(\gamma) > \lambda$. Let

$$S_\omega = \{\beta < \gamma \mid \mathit{cof}(\beta) = \omega\}.$$

Then if $S \in L_\alpha(V_{\lambda+1})$ and $S \subseteq S_\omega$ is stationary (in V), then $S_\omega \setminus S$ is not stationary.

3.2 Perfect set property

In this section we prove an approximation to the Perfect Set Property in $L(V_{\lambda+1})$. We regard $V_{\lambda+1}$ as a topological space with basic open sets $O_{(a,\alpha)}$, where $\alpha < \lambda$, $a \subseteq V_\alpha$ and

$$O_{(a,\alpha)} = \{b \in V_{\lambda+1} \mid b \cap V_\alpha = a\}.$$

Since $\mathrm{cof}(\lambda) = \omega$, this is a metric topology, and it is complete. Xianghui Shi and Woodin showed a similar result using Theorem 4.5.2.

Lemma 3.2.1. *Assume $X \subseteq V_{\lambda+1}$, $X \in L(V_{\lambda+1})$, and $|X| > \lambda$. Let $\alpha < \Theta$ be good such that $X \in L_\alpha(V_{\lambda+1})$. Suppose that $E \subseteq \mathcal{E}$ is saturated and $\bar{\alpha}$ are such that for all $(J, \vec{j}) \in E$, J extends to*

$$\hat{J} : L_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

and $X \in \mathit{rng}\, \hat{J}$. Then for any $(J, \vec{j}) \in \mathcal{E}$ there is an $\bar{A} \in V_{\bar{\lambda}+1}$ and $E' \subseteq E$ such that for

$$Y = \{A \in V_{\lambda+1} \mid \exists (K, \vec{k}) \in E' \text{ a limit root of } J \text{ such that } K(\bar{A}) = A\}$$

we have $Y \subseteq X$ and $|Y| > \lambda$.

Proof. This follows immediately by letting $\bar{X}$ be such that $\hat{J}(\bar{X}) = X$, and setting $E' = \{(K, \vec{k}) \in E \mid \hat{K}(\bar{X}) = X\}$. Then using the fact that $|X| > \lambda$, the lemma follows. ☐

Lemma 3.2.2. *Suppose E is a saturated set of inverse limits and $(J, \vec{j}) \in E$. Let Z be the set of $A \in V_{\bar{\lambda}+1}$ such that*

$$|\{K(A) \mid (K, \vec{k}) \in E \text{ is a limit root of } J\}| < \lambda.$$

Then $|Z| \leq \bar{\lambda}$.

Proof. Let $\kappa < \lambda$ and let Z_κ be the set of $A \in V_{\bar{\lambda}+1}$ such that

$$|\{K(A)|\, (K, \vec{k}) \in E \text{ is a limit root of } J\}| < \kappa.$$

Suppose $|Z_\kappa| > \bar{\lambda}$. Then if $\bar{T}$ is the tree of initial segments of elements of Z_κ. We have $|[\bar{T}]| > \bar{\lambda}$. Let $J(\bar{T}) = T$. Then by elementarity, $|[T]| > \lambda$. But by definition of Z_κ we have that

$$|\bigcup\{K''\bar{T}|\, (K, \vec{k}) \in E \text{ is a limit root of } J\}| \leq \bar{\lambda} \cdot \kappa < \lambda.$$

We claim this is a contradiction. To see this, let i be such that for all $b \in V_{\lambda+1}$ there exists $(K, \vec{k}) \in E$ a limit root of J such that $b \in \operatorname{rng} K_i$ and $K(\bar{T}) = T$. Let $T_i = (j_0 \circ \cdots j_{i-1})^{-1}(T)$. Then $|T_i| = \lambda$ and for all $b \in T_i$, there exists $(K, \vec{k}) \in E$ a limit root of J such that $b \in \operatorname{rng} K_i$. But then $(j_0 \circ \cdots \circ j_{i-1})(b) \in T$. And hence, since $j_0 \circ \cdots \circ j_{i-1}$ is injective,

$$|\bigcup\{K''\bar{T}|\, (K, \vec{k}) \in E \text{ is a limit root of } J\}| = \lambda,$$

a contradiction.

The lemma follows by noting that $\operatorname{cof}(\lambda) = \omega$, so $|Z| \leq \bar{\lambda}$. □

Theorem 3.2.3. *Suppose there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Assume $X \subseteq V_{\lambda+1}$, $X \in L(V_{\lambda+1})$, and $|X| > \lambda$. Then there is a perfect set $Y \subseteq X$ such that $|Y| > \lambda$ and $Y \in L(V_{\lambda+1})$. In fact, for all $a, \alpha \in V_\lambda$ such that $a \subseteq V_\alpha$ and there exists $b \in Y$ such that $a = b \cap V_\alpha$, we have

$$|Y \cap O_{(a,\alpha)}| > \lambda.$$

Proof. By Σ_1-reflection, if there a counterexample to the Theorem, then there is one below the least stable δ of $L(V_{\lambda+1})$. So we prove the Theorem for subsets of $V_{\lambda+1}$ in $L_\delta(V_{\lambda+1})$.

Let $\alpha < \delta$ be good and let $X \in L_\alpha(V_{\lambda+1})$ be such that $X \subseteq V_{\lambda+1}$. By strong inverse limit reflection, there is $E \subseteq \mathcal{E}$ saturated, $\bar{\alpha}$, and $\bar{X}$ such that for all $(J, \vec{j}) \in CL(E)$, J extends to

$$\hat{J} : L_{\bar{\alpha}+1}(V_{\bar{\lambda}+1}) \to L_{\alpha+1}(V_{\lambda+1})$$

and $\hat{J}(\bar{X}) = X$. Let $\langle \lambda_i |\, i < \omega \rangle$ be increasing and cofinal in λ, and let $\langle \kappa_i |\, i < \omega \rangle$ be increasing and cofinal in $\bar{\lambda}$.

Let $T \subseteq V_{\bar{\lambda}}$ be a tree defined as follows. For $i < \omega$ let

$$T_i = \{B \in V_{\kappa_i+1} : |\{A \in V_{\bar{\lambda}+1}|\, A \in \bar{X}, B = A \cap V_{\kappa_i}\}| > \bar{\lambda}\}$$

and

$$T = \{(A_{i_0}, \ldots, A_{i_n})|\, \forall m \leq n(A_{i_m} \in T_{i_m} \text{ and } \forall s < m(A_{i_s} = A_{i_m} \cap V_{\kappa_{i_s}}))\}.$$

Let I be the set

$$I = \{\vec{s} \in [\lambda]^{<\omega} | \, \forall i < \mathrm{len}(\vec{s})(s_i < \lambda_i)\}.$$

Now let

$$F : \{(\vec{A}, s) | \, \exists n, \vec{i}(\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T, s \in I, |s| = n)\} \to E$$

have the following properties:

1. For all $\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T$, $s \in I$, $|s| = n-1$, if
$$F(\vec{A}, s^\frown \langle \alpha \rangle) = (K, \vec{k}) \text{ and } F((\vec{A}, s^\frown \langle \beta \rangle)) = (K', \vec{k}')$$
for $\alpha < \beta < \lambda_n$, then $K(A_{i_n}) \neq K'(A_{i_n})$.
2. For all $\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T$, $s \in I$, $|s| = n$, and $m < n$ if
$$F(\vec{A}, s) = (K, \vec{k}) \text{ and } F(\vec{A} \restriction m+1, s \restriction m) = (K', \vec{k}'),$$
then $K(A_{i_m}) = K'(A_{i_m})$.
3. For all $\vec{A} = (A_{i_0}, \ldots, A_{i_n}) \in T$, $s \in I$, $|s| = n-1$ then $F(\vec{A}, s^\frown \langle \alpha \rangle)$ is a limit root of $F(\vec{A} \restriction n, s)$ for $\alpha < \lambda_{n-1}$.

Also assume that F is maximal with these properties, in the sense that F cannot be extended to some F' also satisfying these properties.

Let Z be the set of $A \in \bar{X}$ such that there exists a sequence $\langle i_n | \, n < \omega \rangle$ such that for $A_{i_n} = A \cap V_{\kappa_{i_n}}$, for all $n < \omega$, and $s \in I$, if $|s| = n$ then $((A_{i_0}, \ldots, A_{i_n}), s) \in \mathrm{dom}(F)$. We claim that $|\bar{X} \setminus Z| \leq \bar{\lambda}$. To see this, suppose that $A \in \bar{X} \setminus Z$. Then there exists $\vec{A}$ and s such that $F(\vec{A}, s) = (K, \vec{k})$, and

$$|\{K'(A) | \, (K', \vec{k}') \in E \text{ is a limit root of } K \text{ and } K'(\vec{A}) = K(\vec{A})\}| < \lambda.$$

But for every K, there are $\leq \bar{\lambda}$ many such A with this property. Hence $|\bar{X} \setminus Z| \leq \bar{\lambda}$.

So finally, let $A \in Z$, and let $\langle i_n | n < \omega \rangle$ be such that for all $n < \omega$, $A_{i_n} = A \cap V_{\kappa_{i_n}}$, and for $s \in I$, if $|s| = n$ then $((A_{i_0}, \ldots, A_{i_n}), s) \in \mathrm{dom}(F)$. Set

$$K^{s,n} = F((A_{i_0}, \ldots, A_{i_n}), s).$$

Also for $x \in \lambda^\omega$, let K^x be the common part of $\left\langle K^{x \restriction n, n} | \, n < \omega \right\rangle$, and set

$$P = \{K^x(A) | \, x \in \lambda^\omega, \forall i < \omega(x_i < \lambda_i)\}.$$

Clearly P is a perfect subset of X by definition of E and the fact that $A \in Z \subseteq \bar{X}$. Furthermore by definition of F we have $|P| > \lambda$. Note that for any $s \in I$, if we set

$$P^s = \{K^x(A) | \, x \in \lambda^\omega, \forall i < \omega(x_i < \lambda_i), \forall i < |s|(s_i = x_i)\}$$

then P^s is a perfect subset of P, $|P^s| > \lambda$ and

$$P^s = P \cap O_{(A_{i_n}, \kappa_{i_n})}$$

where $n = |s|$. And hence we have the final part of the conclusion. $\square$

Chapter 4

$U(j)$-representations

In this chapter we introduce the notion of a $U(j)$-representation, which was first defined by Woodin. For a more thorough introduction see [Woo11]. In the first section we prove a number of closure properties, which will form the basis of proving that sets in $L(V_{\lambda+1})$ have such representations. In Sections 4.2 and 4.3 we will prove certain properties of fixed point measures which we will then use in Section 4.4 to show that these representations extend considerably far in $L(V_{\lambda+1})$.

4.1 Definition and Closure Properties

For this chapter we fix $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ elementary. We will use the notation $j_{(i)}$ to denote the i-th iterate of j to distinguish it from our inverse limit notation.

Definition 4.1.1 (Woodin)**.** Let $U(j)$ be the set of $U \in L(V_{\lambda+1})$ such that in $L(V_{\lambda+1})$ the following hold:

1. U is a λ^+-complete ultrafilter.

2. For some $\gamma < \Theta$, U is generated by $U \cap L_\gamma(V_{\lambda+1})$.

3. For all sufficiently large $n < \omega$, $j_{(n)}(U) = U$ and for some $A \in U$,

$$\{a \in A |\, j_{(n)}(a) = a\} \in U.$$

For each ordinal κ, let $\Theta^{L_\kappa(V_{\lambda+1})}$ denote the supremum of the ordinals α such that there is a surjection $\rho : V_{\lambda+1} \to \alpha$ such that $\{(a,b)|\, \rho(a) < \rho(b)\} \in L_\kappa(V_{\lambda+1})$. Suppose that $\kappa < \Theta$ and $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$. Then $\mathcal{E}(j,\kappa)$ is the set of all elementary embeddings $k : L_\kappa(V_{\lambda+1}) \to L_\kappa(V_{\lambda+1})$ such that there exists $n, m < \omega$ such that $k_{(n)} = j_{(m)} \upharpoonright L_\kappa(V_{\lambda+1})$.

Suppose that $\kappa < \Theta$ and that $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$. For each $\delta \leq \lambda$ let $\mathcal{F}^\delta(\mathcal{E}(j,\kappa))$ be the filter on $P(\kappa) \cap L(V_{\lambda+1})$ generated by the sets

$$\{D_\sigma |\, \sigma \in [\mathcal{E}(j,\kappa)]^\delta\},$$

where for each $\sigma \in [\mathcal{E}(j,\kappa)]^\delta$,

$$D_\sigma = \{b \in L_\kappa(V_{\lambda+1}) |\, k(b) = b \text{ for all } k \in \sigma\}.$$

Lemma 4.1.2 (Woodin)**.** *Suppose $\kappa < \Theta$, $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$ and that $j(\kappa) = \kappa$. Then there is $\delta < crit(j)$ and a partition $\{S_\alpha |\, \alpha < \delta\} \in L(V_{\lambda+1})$ of $L_\kappa(V_{\lambda+1})$ into $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$-positive sets such that for each $\alpha < \delta$,*

$$\mathcal{F}^\lambda(\mathcal{E}(j,\kappa)) \restriction S_\alpha \in U(j).$$

Proof. First, we have that since $j(\kappa) = \kappa$ that

$$j(\mathcal{E}(j,\kappa)) = \mathcal{E}(j,\kappa) \text{ and } j(\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))) = \mathcal{F}^\lambda(\mathcal{E}(j,\kappa)).$$

Now we show that there is no sequence $\langle S_\alpha |\, \alpha < \mathrm{crit}\,(j)\rangle \in L(V_{\lambda+1})$ of pairwise disjoint $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$-positive sets. This follows since

$$\{a \in L_\kappa(V_{\lambda+1}) |\, j(a) = a\} \in \mathcal{F}^\lambda(\mathcal{E}(j,\kappa)),$$

and hence if

$$j(\langle S_\alpha |\, \alpha < \mathrm{crit}\,(j)\rangle) = \langle T_\alpha |\, \alpha < j(\mathrm{crit}\,(j))\rangle\,,$$

then there exists a β such that $\beta \in T_{\mathrm{crit}\,(j)}$ and $j(\beta) = \beta$. But then by elementarity, there exists an $\alpha < \mathrm{crit}\,(j)$ such that $\beta \in S_\alpha$. But then $j(\beta) = \beta \in T_\alpha$, a contradiction.

Now, since $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$ is λ^+-complete, there must exists a $\delta < \mathrm{crit}\,(j)$ and a partition $\{S_\alpha |\, \alpha < \delta\} \in L(V_{\lambda+1})$ of $L_\kappa(V_{\lambda+1})$ into $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$-positive sets such that for each $\alpha < \delta$, $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa)) \restriction S_\alpha$ is an ultrafilter.

For $\alpha < \delta$, let U_α be the ultrafilter given by $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa)) \restriction S_\alpha$. We have that U_α is λ^+-complete since $\mathcal{F}^\lambda(\mathcal{E}(j,\kappa))$ is λ^+-complete. Furthermore we have that

$$B_\alpha := \{a \in S_\alpha |\, j(a) = a)\} \in U_\alpha.$$

And hence we have that $j(U_\alpha) = U_\alpha$, since for all $\beta \in B_\alpha$, $\beta \in S_\alpha \iff \beta \in j(S_\alpha)$. So we have that for all $\alpha < \delta$, $U_\alpha \in U(j)$. □

Suppose that $\kappa < \Theta$ and $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$. Suppose that $\langle a_i |\, i < \omega\rangle$ is a sequence of elements of $L_\kappa(V_{\lambda+1})$ such that for all $i < \omega$, there exists an $n < \omega$ such that $j_{(n)}(a_i) = a_i$. Let $U(j,\kappa,\langle a_i |\, i < \omega\rangle)$ denote the set of $U \in U(j)$ such that there exists $n < \omega$ such that for all $k \in \mathcal{E}(j,\kappa)$, if $k(a_i) = a_i$ for all $i \leq n$, then

$$\{a \in L_\kappa(V_{\lambda+1}) |\, k(a) = a\} \in U.$$

Definition 4.1.3 (Woodin)**.** Suppose $\kappa < \Theta$, κ is weakly inaccessible in $L(V_{\lambda+1})$, and $\langle a_i | \, i < \omega \rangle$ is an ω-sequence of elements of $L_\kappa(V_{\lambda+1})$ such that for all $i < \omega$ there is an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.

Suppose that $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$. Then Z is *$U(j, \kappa, \langle a_i | \, i < \omega \rangle)$-representable* if there exists an increasing sequence $\langle \lambda_i | \, i < \omega \rangle$, cofinal in λ and a function

$$\pi : \bigcup \{ V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} | \, i < \omega \} \to U(j, \kappa, \langle a_i | \, i < \omega \rangle)$$

such that the following hold:

1. For all $i < \omega$ and $(a, b, i) \in \mathrm{dom}(\pi)$ there exists $A \subseteq (L(V_{\lambda+1}))^i$ such that $A \in \pi(a, b, i)$.
2. For all $i < \omega$ and $(a, b, i) \in \mathrm{dom}(\pi)$, if $m < i$ then
$$(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m) \in \mathrm{dom}(\pi)$$
and $\pi(a, b, i)$ projects to $\pi(a \cap V_{\lambda_m}, b \cap V_{\lambda_m}, m)$.
3. For all $x \subseteq V_\lambda$, $x \in Z$ if and only if there exists $y \subseteq V_\lambda$ such that
   (a) for all $m < \omega$, $(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) \in \mathrm{dom}(\pi)$,
   (b) the tower
   $$\langle \pi(x \cap V_{\lambda_m}, y \cap V_{\lambda_m}, m) | \, m < \omega \rangle$$
   is well founded.

For $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$ we say that Z is *$U(j)$-representable* if there exists $(\kappa, \langle a_i | \, i < \omega \rangle)$ such that Z is $U(j, \kappa, \langle a_i | \, i < \omega \rangle)$.

We first show some basic facts about this definition.

Lemma 4.1.4 (Woodin)**.** *Suppose that Z is $U(j, \kappa, \langle a_i | \, i < \omega \rangle)$-representable and $\langle b_i | \, i < \omega \rangle$ is such that for all i there exists an n such that*

$$j_{(n)}(b_i) = b_i.$$

Then Z is $U(j, \kappa, \langle (a_i, b_i) | \, i < \omega \rangle)$-representable.

Proof. It is enough to show that

$$U(j, \kappa, \langle a_i | \, i < \omega \rangle) \subseteq U(j, \kappa, \langle (a_i, b_i) | \, i < \omega \rangle).$$

To see this, suppose that $U \in U(j, \kappa, \langle a_i | \, i < \omega \rangle)$. So there exists an n such that for all $k \in \mathcal{E}(j, \kappa)$, if $k(a_i) = a_i$ for all $i \leq n$, then

$$\{ a \in L_\kappa(V_{\lambda+1})) | \, k(a) = a \} \in U.$$

But requiring that $k((a_i, b_i)) = (a_i, b_i)$ for all $i \leq n$ is an even stronger condition. So $U \in U(j, \kappa, \langle (a_i, b_i) | \, i < \omega \rangle$. $\square$

Lemma 4.1.5 (Woodin). *Suppose that Z is $U(j, \kappa_0, \langle a_i | \, i < \omega\rangle)$-representable, $\kappa_0 < \kappa_1 < \Theta$ and $\kappa_1 \leq \Theta^{L_{\kappa_1}(V_{\lambda+1})}$. Then Z is $U(j, \kappa_1, \langle b_i | \, i < \omega\rangle)$-representable where $b_0 = (a_0, \kappa_0)$ and $b_i = a_i$ for $i > 0$.*

We now want to show certain closure properties for $U(j)$-representations. While closure under λ-unions and existential quantification is pretty much immediate, we will see that closure under complementation is much more involved and requires a property called the Tower Condition.

Lemma 4.1.6 (Woodin). *Suppose that $\lambda < \kappa < \Theta$, $\kappa \leq \Theta^{L_\kappa(V_{\lambda+1})}$ and cof(κ) $> \lambda$. Let N be the collection of sets which are $U(j, \kappa, \langle a_i | \, i < \omega\rangle)$ in $L(V_{\lambda+1})$ for some $\langle a_i | \, i < \omega\rangle$. If $N_0 \subseteq N$, $|N_0| \leq \lambda$, then $\bigcup N_0 \in N$.*

Proof. For each $Z \in N_0$ there is $\big\langle (\lambda_i^Z, a_i^Z) | \, i < \omega \big\rangle$ and

$$\pi_Z : \bigcup\{V_{\lambda_i^Z+1} \times V_{\lambda_i^Z+1} \times \{i\} | \, i < \omega\} \to U(j, \kappa, \big\langle a_i^Z | \, i < \omega\big\rangle)$$

witnessing that in $L(V_{\lambda+1})$, Z is $U(j)$-representable. We can assume without loss of generality that there is a $\langle \lambda_i | \, i < \omega\rangle$ such that for all $Z \in N_0$, $\big\langle \lambda_i^Z | \, i < \omega \big\rangle = \langle \lambda_i | \, i < \omega\rangle$.

Let $\langle \kappa_i | \, i < \omega\rangle$ be the critical sequence of j, and let $\langle Z_\alpha | \, \alpha < \lambda\rangle$ be an enumeration of N_0. For each $n < \omega$, let

$$a_n = \{a_i^{Z_\alpha} | \, \alpha < \kappa_n \text{ and } j_{(n)}(a_i^{Z_\alpha}) = a_i^{Z_\alpha}\}.$$

Since cof(κ) $> \lambda$ we have that

1. For all $n < \omega$, $a_n \in L_\kappa(V_{\lambda+1})$ and $|a_n| < \lambda$.

2. For all $Z \in N_0$, for all $i < \omega$, $a_i^Z \in \bigcup\{a_n | \, n < \omega\}$.

3. For all $n < \omega$, there exists $m < \omega$ such that $j_{(m)}(a_n) = a_n$.

Hence by the above lemma we are done. □

Lemma 4.1.7 (Woodin). *Suppose that $Z \subseteq \{x \times y | \, x, y \in V_{\lambda+1}\}$ and Z is $U(j, \kappa, \langle a_i | \, i < \omega\rangle)$ -representable. Let*

$$Y = \{x \in V_{\lambda+1} | \, \exists y \in V_{\lambda+1}\, (x \times y \in Z)\}.$$

Then Y is $U(j, \kappa, \langle a_i |, i < \omega\rangle)$-representable.

Proof. Let

$$\pi : \bigcup\{V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} | \, i < \omega\} \to U(j, \kappa, \vec{a})$$

witness that Z is $U(j, \kappa, \vec{a})$-representable where $\langle \lambda_i | \, i < \omega\rangle$ is the critical sequence of j. We construct a function

$$\pi_Y : \bigcup\{V_{\lambda_i+1} \times V_{\lambda_i+1} \times \{i\} | \, i < \omega\} \to U(j, \kappa, \vec{a})$$

which witnesses that Y is $U(j, \kappa, \vec{a})$-representable.

The main point is the following continuity property. We have that if $x, y \in V_{\lambda+1}$ then

$$x \times y = \{(a, b) \in x_i \times y_i |\, i < \omega, x_i = x \cap V_{\lambda_i}, y_i = y \cap V_{\lambda_i}\},$$

and on the other hand if $\langle x_i |\, i < \omega\rangle$ and $\langle y_i |\, i < \omega\rangle$ are sequences such that for all i, $x_i = x_{i+1} \cap V_{\lambda_i}$ and $y_i = y_{i+1} \cap V_{\lambda_i}$ then for $x = \bigcup_i x_i$ and $y = \bigcup_i y_i$ we have

$$x \times y = \bigcup_i x_i \times y_i.$$

We define π_Y such that the following hold:

1. Suppose $(a, b) \in V_{\lambda+1} \times V_{\lambda+1}$ and $a = x \times y$ for some $x, y \in V_{\lambda+1}$. Then for all $i < \omega$,

$$\pi(a \cap V_{\lambda_i}, b \cap V_{\lambda_i}, i) = \pi_Y(x \cap V_{\lambda_i}, c \cap V_{\lambda_i}, i)$$

   where $c = y \times b$.

2. If $x_0, c_0 \subseteq V_{\lambda_i}$ are such that there is no $(a, b), (x, y) \in V_{\lambda+1} \times V_{\lambda+1}$ with $a = x \times y$, $x_0 = x \cap V_{\lambda_i}$ and $c_0 = c \cap V_{\lambda_i}$ then

$$(x_0, c_0, i) \notin \mathrm{dom}(\pi_Y).$$

Then clearly π_Y witnesses that Y is $U(j, \kappa, \langle a_i |\, i < \omega\rangle$-representable. □

We now introduce the Tower Condition, a continuous ill-foundedness condition, which is the key property needed to show closure of $U(j)$-representations under complementation.

Definition 4.1.8 (Woodin)**.** Suppose $A \subseteq U(j)$, $A \in L(V_{\lambda+1})$, and $|A| \leq \lambda$. The *Tower Condition for* A is the following statement: There is a function $F : A \to L(V_{\lambda+1})$ such that the following hold:

1. For all $U \in A$, $F(U) \in U$.

2. Suppose $\langle U_i |\, i < \omega\rangle \in L(V_{\lambda+1})$ and for all $i < \omega$, there exists $Z \in U_i$ such that

$$Z \subseteq L(V_{\lambda+1})^i, \qquad U_i \in A, \qquad \text{and} \qquad U_{i+1} \text{ projects to } U_i.$$

   Then the tower $\langle U_i |\, i < \omega\rangle$ is wellfounded in $L(V_{\lambda+1})$ if and only if there exists a function $f : \omega \to L(V_{\lambda+1})$ such that for all $i < \omega$,

$$f \restriction i \in F(U_i).$$

The *Tower Condition for* $U(j)$ is the statement that for all $A \subseteq U(j)$ if $A \in L(V_{\lambda+1})$ and $|A| \leq \lambda$ then the Tower Condition holds for A.

Lemma 4.1.9 (Woodin)**.** *Assume that the tower condition holds for $U(j)$. Suppose that $\gamma_0 < \gamma_1 < \gamma_2$ are weakly inaccessible cardinals in $L(V_{\lambda+1})$ such that $j(\gamma_0, \gamma_1, \gamma_2) = (\gamma_0, \gamma_1, \gamma_2)$ and such that*

$$L_{\gamma_0}(V_{\lambda+1}) \prec L_{\gamma_1}(V_{\lambda+1}) \prec L_{\gamma_2}(V_{\lambda+1}) \prec L_{\Theta}(V_{\lambda+1}).$$

Suppose that Z is $U(j, \gamma_0, \langle a_i |\, i < \omega\rangle)$-representable. Then there exists a sequence $\langle b_i |\, i < \omega\rangle$ such that $V_{\lambda+1} \setminus Z$ is $U(j, \gamma_2, \langle b_i |\, i < \omega\rangle)$-representable.

Proof. Fix π witnessing that Z is $U(j, \gamma_0, \langle a_i |\, i < \omega\rangle)$-representable in $L(V_{\lambda+1})$. Let

$$\langle \kappa_i |\, i < \omega\rangle$$

be the critical sequence of j and assume that for all $i < \omega$, $\lambda_i = \kappa_i$, $|a_i| \leq \kappa_i$ and

$$\forall n > i \left(j_{(n)}(a_i) = a_i \text{ and } j_{(n)}(\pi \restriction V_{\kappa_i+\omega}) = \pi \restriction V_{\kappa_i+\omega} \right).$$

For each $n < \omega$ let $\mathcal{E}_n$ be the set of all elementary embeddings

$$k : L_{\gamma_1}(V_{\lambda+1}) \to L_{\gamma_1}(V_{\lambda+1})$$

such that $k(\gamma_0) = \gamma_0$, $k \in \mathcal{E}(j, \gamma_1)$ and for all $i \leq n$,

$$k(a_i) = a_i \text{ and } k(\pi \restriction V_{\kappa_i+\omega}) = \pi \restriction V_{\kappa_i+\omega}.$$

Note that $\langle \mathcal{E}_n |\, n < \omega\rangle \in L_{\gamma_2}(V_{\lambda+1})$.

For each $\sigma \in [\mathcal{E}_n]^\lambda$, let

$$D_\sigma = \{a \in L_{\gamma_1}(V_{\lambda+1}) |\, k(a) = a \text{ for all } k \in \sigma\},$$

and let $\mathcal{F}_n$ be the filter generated by $\{D_\sigma |\, \sigma \in [\mathcal{E}_n]^\lambda\}$.

We have that for all $n < \omega$ and $m > n$ that $j_{(m)}(\mathcal{E}_n) = \mathcal{E}_n$. So we have that for all $n < \omega$ there is partition $\langle S_{n,\alpha} |\, \alpha < \delta_n\rangle \in L_{\gamma_2}(V_{\lambda+1})$ of $(L_{\gamma_1}(V_{\lambda+1}))^n$ into $\mathcal{F}_n$-positive sets such that $\delta_n < \kappa_{n+1}$ and for each $\alpha < \delta_n$, $\mathcal{F}_n \restriction S_{n,\alpha}$ is an ultrafilter (see Lemma 4.1.2). For each $n < \omega$ and $\alpha < \delta_n$ let $U_{n,\alpha}$ be the ultrafilter on $S_{n,\alpha}$ given by $\mathcal{F}_n$.

Let $A = \{U_{n,\alpha} |\, n < \omega, \alpha < \delta_n\}$. We have that $A \in L(V_{\lambda+1})$, $|A| \leq \lambda$, $A \subseteq U(j)$ and $\operatorname{rng}(\pi) \subseteq A$.

Let $F : A \to L(V_{\lambda+1})$ be a function witnessing the tower condition such that $F \in L(V_{\lambda+1})$. We have that

$$\langle F(U_{n,\alpha}) |\, n < \omega, \alpha < \delta_n\rangle \in L_{\gamma_2}(V_{\lambda+1}).$$

There exists a set $T^0_A \subseteq (L_{\gamma_1}(V_{\lambda+1}))^{<\omega}$ such that the following hold:

1. $T^0_A \in L_{\gamma_2}(V_{\lambda+1})$.

2. For all $U \in A$, there exists $B \subseteq T^0_A$ such that $B \in U$.

3. For all $U_1 \in A$ and $U_2 \in A$, if $U_1 \neq U_2$ then $F(U_1) \cap F(U_2) \cap T^0_A = \emptyset$.

4. For all $s \in T^0_A$ there exists $U \in A$ such that $s \in F(U)$.

Such a T^0_A exists as $|A| \leq \lambda$ and each $U \in A$ is λ^+-complete. We also have that for each $s \in T^0_A$, there is exactly one $U \in A$ such that $s \in F(U)$, which we denote U_s. Furthermore, for each $n < \omega$,

$$|\{U_s | \, s \in T^0_A \cap (L_{\gamma_1}(V_{\lambda+1}))^n\}| \leq \delta_n < \kappa_{n+1}.$$

Let T_A be the set of all $s \in T^0_A$ such that

$$(L(V_{\lambda+1}), s) \equiv_{\Sigma_2} (L(V_{\lambda+1}), t)$$

with parameters from $V_\lambda \cup \{T^0_A, F, A, \pi, Z, \gamma_0, \gamma_1, \gamma_2\}$, for U_s-almost all t. We have that T_A is closed under initial segments and it satisfies 1-4 above. T_A also satisfies that if $s \in T_A$ and t is an initial segment of s, then U_s projects to U_t. Hence we have that for each $f \in [T_A]$, the tower $\langle U_{f \restriction i} | \, i < \omega \rangle$ is wellfounded by definition of the tower function.

For each $U \in A$, let $\mathrm{Ult}(L_{\gamma_2}(V_{\lambda+1}), U)$ be the ultrapower computed using only functions $f : L_{\gamma_1}(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $f \in L_{\gamma_2}(V_{\lambda+1})$. We let $[f]_U$ denote the element of $\mathrm{Ult}(L_{\gamma_2}(V_{\lambda+1}), U)$ given by f. If $\mathrm{rng}\,(f) \subseteq \gamma_2$, we let ξ^f_U be the ordinal in the transitive collapse of $\mathrm{Ult}(L_{\gamma_2}(V_{\lambda+1}), U)$ given by $[f]_U$. We have that $\xi^f_U < \gamma_2$ since $U \in U(j, \gamma_1)$ and $L_{\gamma_2}(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

Let $\sigma \in [\mathcal{E}(j, \gamma_1)]^\lambda$ be such that for all $n < \omega$,

$$D_{\sigma_n} \cap (L_{\gamma_1}(V_{\lambda+1}))^n \subseteq T_A$$

where $\sigma_n = \mathcal{E}_n \cap \sigma$. And let $C = \{\xi < \gamma_1 | \, \forall k \in \sigma(k(\xi) = \xi)\}$. Note that $C^{<\omega} \subseteq T_A$.

We have the following: Suppose $(a, b) \in V_{\lambda+1} \times V_{\lambda+1}$ and that the tower

$$\langle U_i | \, i < \omega \rangle = \langle \pi(a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i) | \, i < \omega \rangle$$

is not wellfounded. There is a sequence of functions $\langle f_i | \, i < \omega \rangle$ such that for all i,

$$f_i : (L_{\gamma_0}(V_{\lambda+1}))^i \to C$$

and such that for all $i_1 < i_2 < \omega$, $j_{i_1,i_2}(\xi^{f_{i_1}}_{U_{i_1}}) > \xi^{f_{i_2}}_{U_{i_2}}$. Here

$$j_{i_1,i_2} : \mathrm{Ult}(L_{\gamma_2}(V_{\lambda+1}), U_{i_1}) \to \mathrm{Ult}(L_{\gamma_2}(V_{\lambda+1}), U_{i_2})$$

is given by the fact that U_{i_2} projects to U_{i_1}. This follows from the fact that $\gamma_0 < \gamma_1$ are both weakly inaccessible and $L_{\gamma_0}(V_{\lambda+1}) \prec L_{\gamma_1}(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

Now suppose that $a \in V_{\lambda+1} \setminus Z$. Then for each $b \in V_{\lambda+1}$, the tower

$$\langle \pi(a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i) | \, i < \omega \rangle$$

is not wellfounded. Since F witnesses that the tower condition holds for $A \supseteq \operatorname{rng}(\pi)$, for each $b \in V_{\lambda+1}$, there is no function $h : \omega \to L(V_{\lambda+1})$ such that for all $i < \omega$,

$$h \restriction i \in F(\pi(a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i)).$$

Thus there exists

$$e : \{(a \cap V_{\kappa_i}, x, i) | \, i < \omega, x \subseteq V_{\kappa_i}\} \times (L_{\gamma_0}(V_{\lambda+1}))^{<\omega} \to C$$

such that for all $b \in V_{\lambda+1}$ and $i_1 < i_2 < \omega$, $j_{i_1,i_2}(\xi^{f_{i_1}}_{U_{i_1}}) > \xi^{f_{i_2}}_{U_{i_2}}$, where for each $i < \omega$, $U_i = \pi(a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i)$, and for all $s \in L_{\gamma_0}(V_{\lambda+1})$,

$$f_i(s) = e((a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i), s).$$

To define such an e, let T be the tree given by F restricted to $\pi(a \cap V_{\kappa_i}, b_i, i)$ such that $i < \omega$, $b_i \subseteq V_{\kappa_i}$ and $(a \cap V_{\kappa_i}, b_i, i) \in \operatorname{dom}(\pi)$. Then for each $s \in F(\pi(a \cap V_{\kappa_i}, b_i, i))$ let $e((a \cap V_{\kappa_i}, b_i, i), s)$ be the rank of s in T.

Define a function

$$e_a : \{(a \cap V_{\kappa_i}, x, i) | \, i < \omega, x \subseteq V_{\kappa_i}\} \to \gamma_1$$

as follows. For each $i < \omega$ and $x \in V_{\kappa_i+1}$,

$$e_a(a \cap V_{\kappa_i}, x, i) = \xi^f_U$$

where $U = \pi(a \cap V_{\kappa_i}, x \cap V_{\kappa_i}, i)$ and for all $s \in L_{\gamma_0}(V_{\lambda+1})$,

$$f(s) = e((a \cap V_{\kappa_i}, x, i), s).$$

Finally for each $n < \omega$, let $s_n = \langle e_a \restriction V_{\kappa_i+\omega} | \, i \leq n \rangle$. We have that for all $n < \omega$, $s_n \in T_A$, since for all $k \in \sigma_n$, $k(s_n) = s_n$. So the tower $\langle U_{s_n} | \, n < \omega \rangle$ is wellfounded.

Let T^* be the set of all $(a \cap V_{\kappa_n}, s) \in V_\lambda \times T_A$ such that $a \in V_{\lambda+1} \times Z$ and there is a function e as above such that $s = \langle e_a | \, V_{\kappa_i+\omega} | \, i \leq n \rangle$.

Suppose $(a, e) \in [T^*]$. Then by definition of T_A,

$$e : \{(a \cap V_{\kappa_i}, x, i) | \, i < \omega, x \subseteq V_{\kappa_i}\} \to \mathrm{Ord}$$

and for all $b \subseteq V_\lambda$, if

$$\langle U_i | \, i < \omega \rangle = \langle \pi(a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i) | \, i < \omega \rangle$$

and

$$\langle \xi_i | \, i < \omega \rangle = \langle e(a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i) | \, i < \omega \rangle$$

then for all $i_1 < i_2$, $j_{i_1,i_2}(\xi_{i_1}) > \xi_{i_2}$. So for all $b \in V_{\lambda+1}$, the tower $\langle \pi(a \cap V_{\kappa_i}, b \cap V_{\kappa_i}, i) | \, i < \omega \rangle$ is not wellfounded, and so $a \notin Z$.

Thus

1. $\{a \in V_{\lambda+1} |\, (a, e) \in [T^*] \text{ for some } e\} = V_{\lambda+1} \setminus Z$.

2. For each $(a, e) \in [T^*]$, the tower $\langle U_i |\, i < \omega\rangle$ is wellfounded where for all $i < \omega$, $U_i = U_{e\restriction i}$.

3. For each $(a_0, e_0) \in [T^*]$ and $i < \omega$,

$$|\{U_{e_1\restriction i} |\, (a_1, e_1) \in [T] \text{ and } a_1 \cap V_{\kappa_i} = a_0 \cap V_{\kappa_i}\}| < \kappa_{i+1}.$$

Let $\langle b_i |\, i < \omega\rangle = \langle \pi \restriction V_{\kappa_i+\omega} |\, i < \omega\rangle$. Then $V_{\lambda+1} \setminus Z$ is $U(j, \gamma_2, \langle b_i |\, i < \omega\rangle)$-representable.

□

4.2 The Tower Condition

For the proof of the Tower Condition we do not actually use inverse limit reflection. Instead, we use the structure of the inverse limits together with their 'naive extensions' above λ. Because of this difference we define for $\alpha < \Theta$,

$$\mathcal{E}^e_\alpha = \{(J, \vec{j}) |\, (J, \vec{j} \restriction V_{\lambda+1}) \in \mathcal{E}, \forall i (j_i : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1}))\}.$$

Suppose that $(J, \vec{j}) \in \mathcal{E}^e_\alpha$. Then we say that $a \in L_\alpha(V_{\lambda+1})$ is in the *extended range of* J if for all $i < \omega$, $a \in \operatorname{rng}(j_0 \circ \cdots \circ j_i)$. We set $J^{\text{ext}}(b) = a$ if for some $n < \omega$, for all $i \geq n$,

$$(j_0 \circ \cdots \circ j_i)^{-1}(a) = b.$$

Again, we omit the sequence of embeddings from our notation.

Lemma 4.2.1. *Suppose α is good and $(J, \langle j_i\rangle) \in \mathcal{E}^e_\alpha$ is an inverse limit such that for all i, $j_i(j_i) = j \restriction L_\alpha(V_{\lambda+1})$. Let $U \in L_\alpha(V_{\lambda+1})$ be in the extended range of J and such that for some i, $j_{(i)}(U) = U$. Let*

$$j_0(U_0) = U, j_1(U_1) = U_0, \ldots.$$

Then there exists an n such that for all $m \geq n$, $U^n = U^m$. Furthermore, for this n we have that for all $m \geq n$, $j_m^{(m-1)}(U) = U$ (see Section 2.1).

Proof. Note that $j_{(n)}$ denotes the nth iterate of j, and j_n denotes the nth element of the inverse limit sequence. Let m be such that $j_{(m-1)}(U) = U$. We prove by induction that for $n \geq m$ we have $j_n(U^n) = U^n$. First suppose that $m = 1$. Then $j(U) = U$. We have that

$$j(U) = U \Rightarrow j_0(j_0)(U) = U \Rightarrow j_0(U^0) = U^0.$$

And hence $U^0 = U$. The fact that $j_n(U^n) = U^n$ follows by induction.

Now suppose that $m > 1$. Assume by induction that we have proved the result for all $m' < m$. Then we have for $n = m - 1$

$$j_{(n)}(U) = U \Rightarrow (j_0(j_0))_{(n)}(U) = U \Rightarrow (j_0)_{(n)}(U^0) = U^0 \Rightarrow j_{(n-1)}(U^0) = U^0.$$

And then using the induction hypothesis on U^0 and $\langle j_i | \, i \geq 1 \rangle$ we have the first result.

To see the second result, simply note that $U^{m-1} = j_m(U^m) = U^m$, and hence

$$\begin{aligned} &j_m(U^m) = U^m \Rightarrow j_m(U^{m-1}) = U^{m-1} \Rightarrow \\ &\quad j_m^{(m-1)}((j_0 \circ \cdots \circ j_{m-1})(U^{m-1})) = (j_0 \circ \cdots \circ j_{m-1})(U^{m-1}) \Rightarrow j_m^{(m-1)}(U) = U, \end{aligned}$$

for any $m \geq n$, for n satisfying the first part of the conclusion (where $U^{-1} = U$). $\square$

Lemma 4.2.2. *Suppose that $A \in L_\Theta(V_{\lambda+1})$, $|A| \leq \lambda$ and for all $a \in A$, there exists an i such that $j_{(i)}(a) = a$. Then there exists a sequence $\langle B_i | \, i < \omega \rangle$ and $(K, \langle k_i | \, i < \omega \rangle) \in \mathcal{E}^e_\eta$ for some $\eta < \Theta$ good such that,*

1. *for all $i < \omega$, $B_i = (k_0 \circ \cdots \circ k_{i-1})(B_0)$,*
2. *$A \subseteq \lim_{i \to \omega} B_i := \{a | \, \exists n \, \forall i \geq n (a \in B_i)\}$,*
3. *for all $i < \omega$, $k_i(k_i) = j \restriction L_\alpha(V_{\lambda+1})$,*
4. *for all $a \in \lim_{i \to \omega} B_i$, there is an $i < \omega$ such that $k_i^{(i-1)}(a) = a$,*
5. *for all $a \in \lim_{i \to \omega} B_i$, there is an $i < \omega$ such that $a \in rng\,(K_i^{(i-1)})^{ext}$.*

Proof. Let $C = \langle U_\alpha | \, \alpha < \lambda \rangle$ be an enumeration of A, and let $\eta < \Theta$ be good and large enough so that $C, A \in L_\eta(V_{\lambda+1})$. Let $(K, \langle k_i | \, i < \omega \rangle) \in \mathcal{E}^e_\eta$ be such that for all $i < \omega$, $k_i(k_i) = j \restriction L_\alpha(V_{\lambda+1})$,

$$k_0(C_0) = C, \; k_0(A_0) = A$$

and for $i > 0$,

$$k_i(C_i) = C_{i-1}, \; k_i(A_i) = A_{i-1}.$$

Let $\bar{\lambda} = \bar{\lambda}_K$. Set

$$B_0 = \lim_{i \to \omega} C_i \restriction \bar{\lambda}.$$

Let $B_i = (k_0 \circ \cdots \circ k_{i-1})(B_0)$.

We want to show that for $\alpha < \mathrm{crit}\,(k_0)$, that $U_\alpha \in \lim_{i \to \omega} B_i$. But this follows by Lemma 4.2.1. To see this, by induction define U^i_α for $i < \omega$ as follows: $k_0(U^0_\alpha) = U_\alpha$ and for $i \geq 0$, $k_{i+1}(U^{i+1}_\alpha) = U^i_\alpha$. Then by the lemma we have that for some n, $U^i_\alpha = U^n_\alpha$ for all $i \geq n$. Hence $U^n_\alpha \in B_0$. We want that for all $i \geq n$, $U_\alpha \in B_{i+1}$. But this follows since

$$U^n_\alpha = U^i_\alpha \in B_0 \Rightarrow (k_0 \circ \cdots \circ k_i)(U^i_\alpha) = U_\alpha \in B_{i+1}.$$

Similarly, we have that for $\alpha < (k_0 \circ \cdots \circ k_{i-1})(\operatorname{crit} k_i)$, $U_\alpha \in \lim_{i\to\omega} B_i$. For ease of notation we prove this for $i = 1$. The proof for $i > 1$ is very similar. So we want for $\alpha < k_0(\operatorname{crit} k_1)$, that $U_\alpha \in \lim_{i\to\omega} B_i$. To see this, by induction define U^i_α for $i < \omega$ as follows: $k_1^{(0)}(U^1_\alpha) = U_\alpha$ and for $i \geq 1$, $k_{i+1}^{(0)}(U^{i+1}_\alpha) = U^i_\alpha$. Then by Lemma 4.2.1 we have that for some n, $U^i_\alpha = U^n_\alpha$ for all $i \geq n$. We want to see that $U^n_\alpha \in B_1$. We have

$$B_1 = k_0(\lim_{i\to\omega} C_i \restriction \bar{\lambda}) = \lim_{i\to\omega} k_0(C_i) \restriction k_0(\bar{\lambda}),$$

and furthermore

$$k_0(k_1 \circ \cdots \circ k_i)(k_0(C_i)) = C.$$

Hence using that $k_0(\operatorname{crit} k_1) = \operatorname{crit}(k_0(k_1 \circ \cdots \circ k_i))$ and $\alpha < k_0(\operatorname{crit} k_1)$ we have that $U^n_\alpha \in B_1$. We show that for all $i \geq n$, $U_\alpha \in B_{i+1}$. But this follows since

$$U^n_\alpha = U^i_\alpha \in B_1 \Rightarrow k_0(k_1 \circ \cdots \circ k_i)(U^i_\alpha) = U_\alpha \in B_{i+1}.$$

Note that

$$B_i = k_0(k_1 \circ \cdots \circ k_{i-1})(k_0(B_0)) = k_0(k_1 \circ \cdots \circ k_{i-1})(B_1).$$

But

$$\sup_{i<\omega}(k_0 \circ \cdots \circ k_{i-1})(\operatorname{crit} k_i) = \lambda.$$

So $A \subseteq \lim_i B_i$.

Note that we have for all $U \in \lim_{i\to\omega} B_i$, that there is an i such that $j_i^{(i-1)}(U) = U$, using the proof of Lemma 4.2.1 together with above argument. $\square$

Theorem 4.2.3. *Suppose $A \subseteq U(j)$, $A \in L(V_{\lambda+1})$, $|A| \leq \lambda$. Then the tower condition for A holds.*

Proof. Let $A \subset U(j)$, $|A| = \lambda$, and $A \in L(V_{\lambda+1})$. Let $\langle B_i | \, i < \omega\rangle$, $\eta < \Theta$ be good, $L_\eta(V_{\lambda+1}) \prec_{\Sigma_1} L_\Theta(V_{\lambda+1})$, and $(J, \langle j_i | \, i < \omega\rangle) \in \mathcal{E}^e_{\eta+2}$ be such that, $A \subseteq \lim_{i\to\omega} B_i$, for all $i < \omega$, $j_i(j_i) = j \restriction L_\eta(V_{\lambda+1})$, and for $i < \omega$,

$$B_i = (j_0 \circ \cdots \circ j_{i-1})(B_0).$$

Since $|B_0| < \lambda$, λ-DC holds in $L(V_{\lambda+1})$, and each measure in A is λ^+-complete, there is a tower function $F_0 \in L_\eta(V_{\lambda+1})$ for B_0. Define for $i > 0$,

$$(j_0 \circ \cdots \circ j_{i-1})(F_0) = F_i.$$

Let $B := \lim_{i\to\omega} B_i$, and for $U \in B$ define

$$F(U) = \bigcap\{F_i(U) \cap \{a \in L(V_{\lambda+1}) | \, j_{(i)}(a) = a\} | \, i < \omega, U \in B_i \text{ and } j_{(i)}(U) = U\}.$$

We want to show that F is a tower function for $B := \lim_{i\to\omega} B_i$. To see this suppose $\langle U_i | \, i < \omega\rangle$ is an illfounded tower with $U_i \in B$ for all $i < \omega$, and $f \in L_\eta(V_{\lambda+1})$ is such that

$$\forall i (f \restriction i \in F(U_i)).$$

Let $\langle \alpha_i | \, i < \omega\rangle \in L_\eta(V_{\lambda+1})$ be such that

$$j_{U_i,U_{i+1}}(\alpha_i) > \alpha_{i+1}.$$

For $i < \omega$, let m_i be least such that $U_i \in B_n$ for all $n \geq m_i$. Let $(K, \langle k_i | i < \omega\rangle) \in \mathcal{E}^e_{\eta+1}$ be a 0-close limit root of J such that the following hold:

1. For all i, $(k_0 \circ \cdots \circ k_i)(F_0) = F_i$ and $(k_0 \circ \cdots \circ k_i)(B_0) = B_i$.

2. For all $i < \omega$, $\alpha_i, f(i) \in \operatorname{rng}(K^{\text{ext}})$. Let α_i^n and $f^n(i)$ be such that

   $$k_0(\alpha_i^0) = \alpha_i,\ k_1(\alpha_i^1) = \alpha_i^0,\ k_2(\alpha_i^2) = \alpha_i^1, \ldots$$

   and

   $$k_0(f^0(i)) = f(i),\ k_1(f^1(i)) = f^0(i),\ k_2(f^2(i)) = f^1(i), \ldots$$

3. For all $i < n$, $k_i(j_n \restriction L_\eta(V_{\lambda+1})) = j_i(j_n \restriction L_\eta(V_{\lambda+1}))$.

4. For all n, let i_n be least such that $U_n \in \operatorname{rng}((j_0 \circ \cdots j_{i_n-1})(J^{\text{ext}}_{i_n}))$. Let $U_{n,i}$ be defined as follows:

   $$(j_0 \circ \cdots j_{i_n-1})(j_{i_n})(U_{n,0}) = U_n,\ (j_0 \circ \cdots j_{i_n-1})(j_{i_n+1})(U_{n,1}) = U_{n,0}, \ldots,$$
   $$(j_0 \circ \cdots j_{i_n-1})(j_{i_n+i+1})(U_{n,i+1}) = U_{n,i}, \ldots$$

   Then for $i, n < \omega$ and $m < i_n$, there are $U^m_{n,i}$ such that

   $$k_0(U^0_{n,i}) = U_{n,i},\ k_1(U^1_{n,i}) = U^0_{n,i}, \ldots, k_{i_n-1}(U^{i_n-1}_{n,i}) = U^{i_n-2}_{n,i}.$$

   Furthermore, for $i \geq i_n$

   $$k_i(U^{i_n-1}_{n,i-i_n}) = j_i(U^{i_n-1}_{n,i-i_n}).$$

It is easy to find such a $(K, \vec{k})$ using the proof of Lemma 2.1.7.

We have for all i that

$$\alpha_i \geq \alpha_i^0 \geq \alpha_i^1 \geq \cdots .$$

Let α_i^ω be the stable value. For $n < \omega$, by Lemma 4.2.1 $\langle U_{n,i} | \, i < \omega\rangle$ and $\langle f^i | \, i < \omega\rangle$ must stabilize for some i. Let $U_{n,\omega}$ and f^ω be the stable values, defining $U^m_{n,\omega}$ in the obvious way as above. Note that we have for all $n, i < \omega$,

$$\begin{aligned}(k_0 \circ \cdots \circ k_{i_n-1})(k_{i_n+i})(U_{n,i}) &= (k_0 \circ \cdots \circ k_{i_n-1})(k_{i_n+i})((k_0 \circ \cdots \circ k_{i_n-1})(U_{n,i}^{i_n-1})) \\ &= (k_0 \circ \cdots \circ k_{i_n-1})(k_{i_n+i}(U_{n,i}^{i_n-1})) \\ &= (k_0 \circ \cdots \circ k_{i_n-1})(j_{i_n+i}(U_{n,i}^{i_n-1})) \\ &= (k_0 \circ \cdots \circ k_{i_n-1})(j_{i_n+i})((k_0 \circ \cdots \circ k_{i_n-1})(U_{n,i}^{i_n-1})) \\ &= (j_0 \circ \cdots \circ j_{i_n-1})(j_{i_n+i})(U_{n,i}) = U_{n,i-1}\end{aligned}$$

where $U_{n,-1} = U_n$.

We want to show that $\langle U_{n,\omega}^{i_n-1} | \, n < \omega \rangle = \langle U'_n | \, n < \omega \rangle$ is an illfounded tower as witnessed by $\langle \alpha_i^\omega \rangle$, and for all $n < \omega$, $U'_n \in B_0$. But also that for all n, $f^\omega \restriction n \in F_0(U'_n)$, contradicting the fact that F_0 is a tower function for B_0.

The fact that for all n, $U'_n \in B_0$ follows since for all large enough i, $U_n \in B_i$ and $(k_0 \circ \cdots \circ k_{i-1})(U'_n) = U_n$.

To see that $\langle U'_n | \, n < \omega \rangle$ is illfounded, fix n and let n_0 be such that

$$f \restriction n \in F_{n_0}(U_n)$$

and for all $i \geq n_0$,

$$(k_0 \circ \cdots \circ k_i)(\alpha_n^\omega, \alpha_{n+1}^\omega, U'_n, U'_{n+1}, f^\omega \restriction n) = (\alpha_n, \alpha_{n+1}, U_n, U_{n+1}, f \restriction n).$$

Then we have that

$$j_{U_n, U_{n+1}}(\alpha_n) > \alpha_{n+1} \Rightarrow j_{U'_n, U'_{n+1}}(\alpha_n^\omega) > \alpha_{n+1}^\omega.$$

And since $f \restriction n \in F_{n_0}(U_n)$ we have that $f^\omega \restriction n \in F_0(U'_n)$.

Hence we have a contradiction to the fact that F_0 is a tower function for B_0. So F is a tower function for B, and the theorem follows as $A \subseteq B$. □

In fact, since we did not actually use inverse limit reflection, exactly the same proof gives the tower condition for $L(X, V_{\lambda+1})$. In this situation we start with assuming an elementary embedding $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$, and we make the same definition for a $U(j)$-representation and the Tower Condition, replacing each $L(V_{\lambda+1})$ with $L(X, V_{\lambda+1})$. We then have the following:

Theorem 4.2.4. *Suppose there exists an elementary embedding*

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1}).$$

Then the Tower Condition for $U(j)$ holds in $L(X, V_{\lambda+1})$.

Finally by a Theorem of Woodin (see [Woo11]) we have the following:

Corollary 4.2.5. *Suppose there exists an elementary embedding*

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1}).$$

Let Y be $U(j)$-representable in $L(X, V_{\lambda+1})$. Let $\kappa = \lambda^+$ and set

$$\eta = \sup\{(\kappa^+)^{L[A]} | \, A \subseteq \lambda\}.$$

Then every set

$$Z \in L_\eta(Y, V_{\lambda+1}) \cap V_{\lambda+2}$$

is $U(j)$-representable in $L(X, V_{\lambda+1})$.

4.3 Complexity of fixed point measures

In order to extend the $U(j)$-representations past the point given by Corollary 4.2.5 we need to analyze the fixed point measures which compose these representations. We consider a certain game on fixed points which was first considered in [Woo11].

Definition 4.3.1. Suppose $\gamma < \Theta^{L(V_{\lambda+1})}$ and

$$\langle a_i | \, i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$$

and we have:

1. $\gamma \leq \Theta^{L_\gamma(V_{\lambda+1})}$,
2. for all $i < \omega$, $a_i \subseteq a_{i+1} \subseteq \gamma$ and $|a_i| < \lambda$,
3. for all $i < \omega$, there exists an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.

Then let $G(j, \gamma, \langle a_i | \, i < \omega \rangle)$ denote the following game. Player I plays a sequence

$$\big\langle (\gamma_i, \big\langle b^i_m : \, m < \omega \big\rangle) : \, i < \omega \big\rangle$$

and player II plays a sequence $\langle \mathcal{E}_i : i < \omega \rangle$ such that the following hold:

1. $\mathcal{E}_i \subseteq \mathrm{Emb}(j, \gamma_i)$, $|\mathcal{E}_i| \leq \lambda$, and for each $k \in \mathcal{E}_i$ there exists $m < \omega$ such that $k(b^i_m) = b^i_m$.
2. $\gamma_0 = \gamma$, $\gamma_{i+1} < \gamma_i$ and there exists $m < \omega$ such that

   $$k(b^i_m) = b^i_m \Rightarrow k(\gamma_{i+1}) = \gamma_{i+1}$$

   for all $k \in \mathcal{E}_i$.
3. for all $i < \omega$, $\gamma_i \leq \Theta^{L_{\gamma_i}(V_{\lambda+1})}$,

4. $\langle b^0_m : m < \omega \rangle = \langle a_m : m < \omega \rangle$.
5. for all $m < \omega$, $b^i_m \subseteq b^i_{m+1} \subseteq \gamma_i$ and $|b^i_m| < \lambda$
6. for all $m < \omega$ there exists $m^* < \omega$ such that

$$k(b^i_{m^*}) = b^i_{m^*} \Rightarrow k(b^{i+1}_m) = b^{i+1}_m$$

for all $k \in \mathcal{E}_i$.

Definition 4.3.2. Suppose $\gamma < \Theta^{L(V_{\lambda+1})}$, $S \subseteq L_\gamma(V_{\lambda+1})$, and $\langle a_i | \, i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$ and we have:

1. $\gamma \leq \Theta^{L_\gamma(V_{\lambda+1})}$,
2. for all $i < \omega$, $a_i \subseteq a_{i+1} \subseteq \gamma$ and $|a_i| < \lambda$,
3. for all $i < \omega$, there exists an $n < \omega$ such that $j_{(n)}(a_i) = a_i$.
4. $S = \bigcup_{i<\omega} a_i$.

Then we say that $\langle a_i | \, i < \omega \rangle$ is a *j-partition of S*.

Suppose that $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Note that for all $S \subseteq L_\Theta(V_{\lambda+1}) \cap \mathrm{Ord}$ such that $|S| \leq \lambda$, there is a $\gamma < \Theta$ and a $\langle a_i | \, i < \omega \rangle \in (L_\gamma(V_{\lambda+1}))^\omega$ such that $\langle a_i | \, i < \omega \rangle$ is a j-partition of S.

Suppose that $\vec{\mathcal{E}}$ and S are such that for all $\alpha \in S$ there exists an i such that for all $k \in \mathcal{E}^i$, $k(\alpha) = \alpha$. Let $\vec{a}$ be defined by

$$a_i = \{\alpha \in S | \, \forall k \in \mathcal{E}_i \, (k(\alpha) = \alpha)\}.$$

We say that $\vec{a}$ is the partition of S with respect to $\vec{\mathcal{E}}$.

Suppose $\vec{a}$ is such that $a_0 \subseteq a_1 \subseteq \cdots$,

$$a_i \subseteq \{\alpha \in S | \, \forall k \in \mathcal{E}_i \, (k(\alpha) = \alpha)\}$$

and for all i, $|a_i| < \mathrm{crit}\, j_{(i)}$. Then we say that $\vec{a}$ is a *division of S with respect to $\vec{\mathcal{E}}$*.

Lemma 4.3.3. *Suppose that $\langle a_i | \, i < \omega \rangle$ is a j-partition of S with $S \subseteq \gamma$ for some γ. Suppose that $(J, \vec{j}) \in \mathcal{E}^e_{\eta+1}$ is such that $\eta \geq \gamma$ is good and for all $i < \omega$,*

$$J^{ext,(i-1)}_i(a_i) = a_i.$$

Then there exists a $(K, \vec{k}) \in \mathcal{E}^e_\eta$ a limit root of J such that for all $i < \omega$,

$$K^{ext,(i-1)}_i(a_i) = a_i.$$

Proof. This follows easily in the usual way by noticing that for α an ordinal, if $j_i(\alpha) = \alpha$, k_i is a square root of j_i and $\alpha \in \operatorname{rng} k_i$ then $k_i(\alpha) = \alpha$. Hence if $\langle a_i | \, i < \omega \rangle \in \operatorname{rng} k_n$ for all $n < \omega$, then we have that for all $n < \omega$ that

$$(j_0 \circ \cdots \circ j_{n-1})^{-1}(a_n) \in \operatorname{rng} j_n$$

and so $a_n \in \operatorname{rng} j_n^{(n)}$. □

Definition 4.3.4. Fix $\kappa < \Theta$ good with $\operatorname{cof}(\kappa) > \lambda$. Let $S \subseteq \kappa$ such that $|S| \leq \lambda$. Then we say that S is λ-*threaded* if the following hold:

1. Suppose $\alpha < \sup S$ is such that there exists $\vec{\beta} \in S^{<\omega}$ and $a \in V_\lambda$ such that α is definable over $L_\kappa(V_{\lambda+1})$ from $\vec{\beta}$ and a. Then $\alpha \in S$.
2. Suppose $\alpha \in S$ is a limit and $\operatorname{cof}(\alpha) < \lambda$. Then $S \cap \alpha$ is cofinal in α.

We say that S is *definably closed* if S satisfies (1).

Since λ-DC holds in $L(V_{\lambda+1})$, we have that for every $S \subseteq \kappa$ with $|S| \leq \lambda$, there is $S' \subseteq \kappa$ with $S' \supseteq S$ and $|S'| \leq \lambda$ such that S' is λ-threaded.

We put for E a set such that for all $k \in E$, $k : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ for some $\alpha < \Theta$,

$$\mathcal{F}(E) = \{\beta | \, \forall k \in E \, (k(\beta) = \beta)\}.$$

Lemma 4.3.5. *Suppose α is a limit ordinal. Then there exists a $\gamma < \alpha$ such that for all $\beta \in [\gamma, \alpha)$ if β_0 is such that $\beta = \gamma + \beta_0$, then for all $\delta < \alpha$, $\delta + \beta_0 < \alpha$.*

Proof. We prove this by induction. Suppose that α is such that there exists a $\beta < \alpha$ such that for some $\delta < \alpha$, $\beta + \delta \geq \alpha$. Let $\alpha^* \leq \alpha$ be the sup of ordinals $\gamma < \alpha$ such that for all $\beta, \delta < \gamma$, $\delta + \beta < \gamma$. Call the set of such ordinals A. Then clearly $\alpha^* \in A$. So $\alpha^* < \alpha$. Let α_0 be such that $\alpha^* + \alpha_0 = \alpha$.

We claim that $\alpha_0 < \alpha$. If not, then $\alpha^* + \alpha = \alpha$. But then $\alpha^* \cdot \omega \leq \alpha$, and $\alpha^* \cdot \omega \in A$, a contradiction.

But then by applying the lemma to α_0, we have that there exists a $\gamma < \alpha_0$ such that for all $\beta \in [\gamma, \alpha_0)$ if β_0 is such that $\beta = \gamma + \beta_0$, then for all $\delta < \alpha_0$, $\delta + \beta_0 < \alpha_0$. But if $\beta \in [\gamma, \alpha_0)$ then $\alpha^* + \beta < \alpha$ and for some β_0,

$$\alpha^* + \beta = \alpha^* + \gamma + \beta_0.$$

Hence $\gamma_0 = \alpha^* + \gamma$ witnesses the lemma for α. To see this let $\beta \in [\gamma_0, \alpha)$ and let β_0 be such that $\alpha^* + \gamma + \beta_0 = \beta$. Suppose $\delta \in [\alpha^*, \alpha)$. Let δ^* be such that $\alpha^* + \delta^* = \delta$. Then we have that $\delta^* < \alpha_0$, and hence $\delta^* + \beta_0 < \alpha_0$. But then

$$\alpha^* + \delta^* + \beta_0 = \delta + \beta_0 < \alpha^* + \alpha_0 = \alpha,$$

which proves the lemma. □

From the previous lemma, given a limit ordinal α there is a unique decomposition,

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_n$$

for some n with $\alpha_0 > \alpha_1 > \cdots > \alpha_n$, and such that for all for all $i < n$, for all

$$\beta \in [\alpha_0 + \cdots + \alpha_i, \alpha_0 + \cdots + \alpha_{i+1})$$

if β_0 is such that

$$\beta = \alpha_0 + \cdots + \alpha_i + \beta_0$$

then for all $\delta < \alpha_0 + \cdots + \alpha_{i+1}$, $\delta + \beta_0 < \alpha_0 + \cdots + \alpha_{i+1}$. We call $\langle \alpha_0, \ldots, \alpha_n \rangle$ the *addition decomposition of* α.

We define the function $c(\alpha, \beta)$ for $\beta < \alpha$ as follows. Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be the addition decomposition of α. Let i be largest such that $\alpha_0 + \cdots + \alpha_i \leq \beta$, and let β_0 be such that

$$\alpha_0 + \cdots + \alpha_i + \beta_0 = \beta.$$

Set $c(\alpha, \beta) = \beta_0$.

Also define the following functions:

$$\mathrm{ld}(\alpha) = \alpha_0 + \cdots + \alpha_{n-1}$$

if $n > 0$ and $\mathrm{ld}(\alpha) = 0$ otherwise, where $\langle \alpha_0, \ldots, \alpha_n \rangle$ is the addition decomposition of α, and

$$\mathrm{rd}(\alpha) = \alpha_n.$$

Lemma 4.3.6. *Suppose that $\beta + \gamma = \alpha$. Let $\langle \alpha_0, \ldots, \alpha_n \rangle$ be the addition decomposition of α. Then for some i, $\gamma = \alpha_i + \cdots + \alpha_n$.*

Proof. Note that we have for some i that $\beta = \alpha_0 + \cdots + \alpha_{i-1} + \beta'$ and $\gamma = \gamma' + \alpha_{i+1} + \cdots + \alpha_n$ for some β' and γ' such that $\beta' + \gamma' = \alpha_i$. Furthermore, if β and γ were a contradiction to the lemma, we would have that $\beta', \gamma' < \alpha_i$ and $\beta', \gamma' \neq 0$. But then $\beta' + \gamma' < \alpha_i$ by definition of the addition decomposition, a contradiction. □

Lemma 4.3.7. *Suppose that $\beta + \gamma = \alpha$ and $\gamma \neq \emptyset$. Then $rd(\gamma) = rd(\alpha)$.*

Proof. By the previous lemma, if $\langle \alpha_0, \ldots, \alpha_n \rangle$ is the addition decomposition of α, we have that $\gamma = \alpha_i + \cdots + \alpha_n$ for some i. So it is enough to see that $\langle \alpha_i, \ldots, \alpha_n \rangle$ is the addition decomposition of γ. But this is basically immediate. □

We fix a good limit ordinal $\kappa < \Theta$ for the rest of the section such that κ is regular in $L(V_{\lambda+1})$. Furthermore whenever we refer to $\hat{J}$ for some inverse limit J, we mean the extension of J to an embedding $\hat{J} : L_{\bar{\kappa}}(V_{\bar{\lambda}+1}) \to L_\kappa(V_{\lambda+1})$ for some $\bar{\kappa}$ and $\bar{\lambda}$.

We first consider a more restrictive version of the above game. This game, in some sense, captures a version of $G(j, \gamma \langle a_i | \, i < \omega \rangle)$ where only the 'local largeness' of the γ_i matter. Later on when we play $G(j, \gamma \langle a_i | \, i < \omega \rangle)$, we will do so by playing many versions of this more restrictive game.

Lemma 4.3.8. *Suppose that $\alpha_0 < \kappa$ is an ordinal with cof$(\alpha_0) > \lambda$ and $(J^0, \vec{j}^0) \in \mathcal{E}_{\kappa+\alpha_0+2}$ and $\alpha_0 \in$ rng $\hat{J}^0$. Then II has a quasi-winning strategy in the following game $G(\alpha_0, J^0)$*

$$\begin{array}{lllllll} I & \beta_0, \gamma_0 & & \beta_1, \gamma_1 & & & \cdots \\ II & & \alpha_1, (J^1, \vec{j}^1) & & \alpha_2, (J^2, \vec{j}^2) & & \cdots \end{array}$$

which has the following rules.

1. *For all i, $\alpha_i, \beta_i, \gamma_i < \kappa$.*
2. *$\beta_0 > \beta_1 > \beta_2 > \cdots$ and $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$.*
3. *$\alpha_0 > \beta_0$.*
4. *For all i, if cof$(\beta_i) > \lambda$ then cof$(\alpha_{i+1}) > \lambda$.*
5. *For all i, $(J^i, \vec{j}^i) \in \mathcal{E}_{\kappa+\alpha_i+2}$ and $\alpha_i \in$ rng $\hat{J}^i$.*
6. *For all i, $\gamma_i \in (ld(\alpha_i), \alpha_i)$, $(J^i)^{ext}(\gamma_i) = \gamma_i$ and $\alpha_{i+1} \geq \gamma_i$.*
7. *For all i, α_{i+1} is definable over $L_{\kappa+\alpha_i+2}(V_{\lambda+1})$ from parameters in*

$$\{\alpha_0, \ldots, \alpha_i\} \cup \{\gamma_i\} \cup \lambda,$$

   and $(J^{i+1})^{ext}(\alpha_{i+1}) = \alpha_{i+1}$.

8. *For all i, $\beta_{i+1} > ld(\beta_i)$ and $\beta_i > ld(\alpha_0)$.*

The first player to violate one of the rules loses.

Proof. We describe a quasi-winning strategy for II. First suppose that I plays β_0. Let

$$K^0 \in \mathcal{E}_{\kappa+\alpha_0+1}$$

be a 0-close limit root of J^0 such that $\beta_0 \in \operatorname{rng} \hat{K}^0$, $(K^0)^{\text{ext}}(\gamma_0) = \gamma_0$, and

$$\hat{J}^0(\bar{\alpha}_0) = \alpha_0 = \hat{K}^0(\bar{\alpha}_0)$$

for some $\bar{\alpha}_0$. Let $\bar{\beta}_0$ be such that $\hat{K}^0(\bar{\beta}_0) = \beta_0$. Now we have by elementarity that $\operatorname{ld}(\bar{\alpha}_0) < \bar{\beta}_0$. So let $\bar{\beta}_0^*$ be such that

$$\operatorname{ld}(\bar{\alpha}_0) + \bar{\beta}_0^* = \bar{\beta}_0.$$

Let β_0^- be the least β such that there exists $(K, \vec{k}) \in \mathcal{E}_{\kappa+\alpha_0+1}$ with

$$\bar{\lambda}_K < \bar{\beta}_0^*, \qquad \hat{K}(\bar{\beta}_0^*) = \beta, \qquad K^{\text{ext}}(\beta) = \beta, \text{ and } K^{\text{ext}}(\gamma_0) = \gamma_0.$$

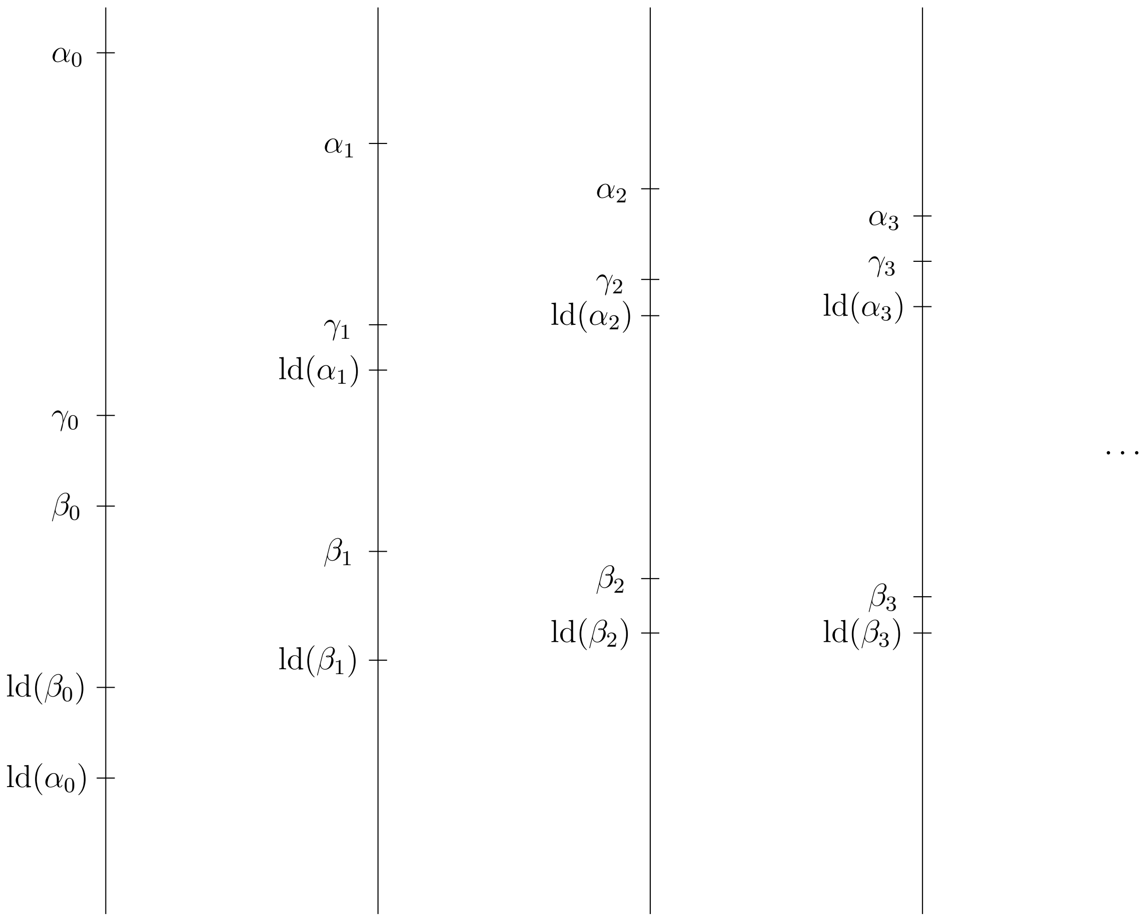

Figure 4.1: Typical play of $G(\alpha_0, J^0)$.

Let $\alpha_1 = \gamma_0 + \beta_0^-$. Clearly we have that α_1 is definable over $L_{\kappa+\alpha_0+2}(V_{\lambda+1})$ from γ_0 and $\bar{\beta}_0^*$. Furthermore we have that $\alpha_1 < \alpha_0$ since $\beta_0^- \le \hat{K}^0(\bar{\beta}_0^*)$, and for all $\delta < \alpha_0$,

$$\delta + \hat{K}^0(\bar{\beta}_0^*) < \alpha_0.$$

To see that $\beta_0^- \le \hat{K}^0(\bar{\beta}_0^*)$, note that

$$\beta_0^- \le (K^{\text{ext}})^{-1}(\hat{K}^0(\bar{\beta}_0^*))$$

since for large enough i, K_i^0 is a witness to this.

Let $(J^1, \vec{j}^1) \in \mathcal{E}_{\kappa+\alpha_0+1}$ be such that $\bar{\lambda}_{J^1} < \bar{\beta}_0^*$, $\hat{J}^1(\bar{\beta}_0^*) = \beta_0^-$, $(J^1)^{\text{ext}}(\beta_0^-) = \beta_0^-$ and $(J^1)^{\text{ext}}(\gamma_0) = \gamma_0$. Then clearly $(J^1)^{\text{ext}}(\alpha_1) = \alpha_1$. Also, if $\text{cof}(\beta_0) > \lambda$, then $\text{cof}(\bar{\beta}_0^*) > \bar{\lambda}_{J^1}$ and hence $\text{cof}(\alpha_1) > \lambda$.

Now suppose that I has played $\beta_0, \ldots, \beta_i$ and $\gamma_0, \ldots, \gamma_i$ satisfying the rules and II has responded with $\alpha_1, \ldots, \alpha_i$ and $(J^1, \vec{j}^1), \ldots, (J^i, \vec{j}^i)$ satisfying the rules. Also assume that II has chosen $\bar{\beta}_0, \ldots, \bar{\beta}_{i-1}$ and $K^0, \ldots, K^{i-1}$ satisfying that for all $n < i$, $\hat{K}^n(\bar{\beta}_n) = \beta_n$ and $\hat{J}^{n+1}(\bar{\beta}_n^*) = \beta_n^-$ where β_n^- is such that $\gamma_n + \beta_n^- = \alpha_{n+1}$.

Let $(K^i, \vec{k}^i) \in \mathcal{E}_{\kappa+\alpha_i+1}$ be such that for some $\bar{\beta}_i$

$$\hat{K}^i(\bar{\beta}_i) = \beta_i, \qquad \hat{K}^i(\bar{\beta}_{i-1}) = \beta_{i-1}, \text{ and } \hat{K}^i(\bar{\beta}_{i-1}^*) = \beta_{i-1}^-,$$

and

$$(K^i)^{\text{ext}}(\gamma_i, \gamma_{i-1}, \beta_{i-1}^-) = (\gamma_i, \gamma_{i-1}, \beta_{i-1}^-).$$

Now we have $\text{ld}(\bar{\beta}_{i-1}) < \bar{\beta}_i$. So let $\bar{\beta}_i^*$ be such that $\text{ld}(\bar{\beta}_{i-1}) + \bar{\beta}_i^* = \bar{\beta}_i$.

Let β_i^- be the least β such that there exists $(K, \vec{k}) \in \mathcal{E}_{\kappa+\alpha_i+1}$ with

$$\bar{\lambda}_K < \bar{\beta}_i^*, \qquad \hat{K}(\bar{\beta}_i^*) = \beta, \qquad K^{\text{ext}}(\beta) = \beta, \text{ and } K^{\text{ext}}(\gamma_i) = \gamma_i.$$

Let $\alpha_{i+1} = \gamma_i + \beta_i^-$. Clearly we have that α_{i+1} is definable over $L_{\kappa+\alpha_i+2}(V_{\lambda+1})$ from γ_i and $\bar{\beta}_i^*$. Furthermore we have that $\alpha_{i+1} < \alpha_i$ since

$$\beta_i^- \le \hat{K}^i(\bar{\beta}_i^*),$$

and $\bar{\beta}_{i-1} = \gamma + \bar{\beta}_{i-1}^*$ for some γ implies by Lemmas 4.3.6 and 4.3.7 that

$$\text{rd}(\bar{\beta}_{i-1}^*) = \text{rd}(\bar{\beta}_{i-1}) > \bar{\beta}_i^*$$

by definition of $\bar{\beta}_i^*$ and the fact that $\bar{\beta}_i > \bar{\beta}_{i-1}$. But applying $\hat{K}^i$ we get that

$$\text{rd}(\beta_{i-1}^-) > \hat{K}^i(\bar{\beta}_i^*) \ge \beta_i^-$$

which is enough to show that

$$\alpha_{i+1} = \gamma_i + \beta_i^- < \alpha_i = \gamma_{i-1} + \beta_{i-1}^-.$$

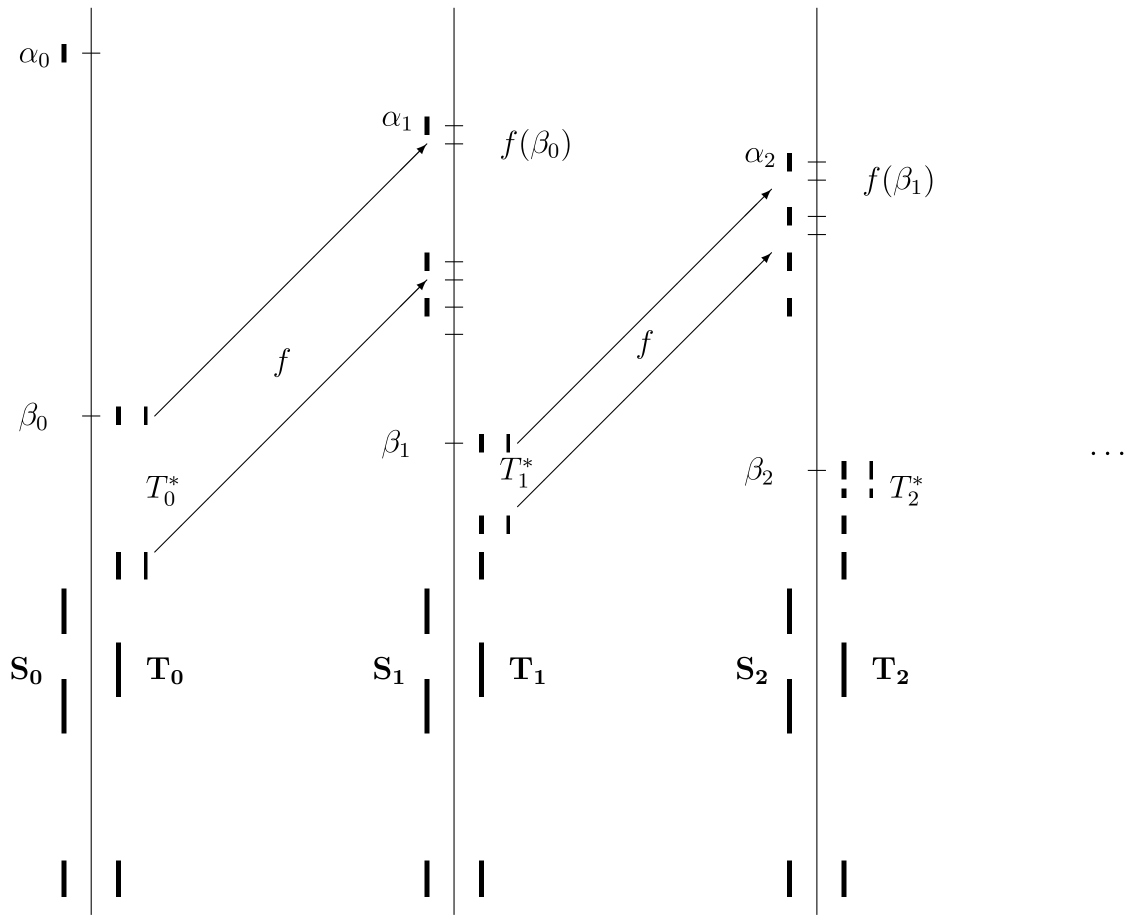

Figure 4.2: Strategy for $G(j, \kappa + \alpha_0, \vec{a})$.

To see that $\beta_i^- \leq \hat{K}^i(\bar{\beta}_i^*)$, note that

$$\beta_i^- \leq ((K^i)^{\text{ext}})^{-1}(\hat{K}^i(\bar{\beta}_i^*))$$

since for large enough m, K_m^i is a witness to this.

Let $(J^{i+1}, \vec{j}^{i+1}) \in \mathcal{E}_{\kappa+\alpha_i+1}$ be such that $\bar{\lambda}_{J^{i+1}} < \bar{\beta}_i^*$, $\hat{J}^{i+1}(\bar{\beta}_i^*) = \beta_i^-$, $(J^{i+1})^{\text{ext}}(\beta_i^-) = \beta_i^-$ and $(J^{i+1})^{\text{ext}}(\gamma_i) = \gamma_i$. Then clearly $(J^{i+1})^{\text{ext}}(\alpha_{i+1}) = \alpha_{i+1}$. Also, if $\text{cof}(\beta_i) > \lambda$, then $\text{cof}(\bar{\beta}_i^*) > \bar{\lambda}_{J^{i+1}}$ and hence $\text{cof}(\alpha_{i+1}) > \lambda$.

We have described a quasi-winning for II, which proves the lemma. $\square$

Theorem 4.3.9. *Let $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be elementary. Fix $\kappa < \Theta$ good and regular in $L(V_{\lambda+1})$. Suppose that S has a largest element α_0, S is λ-threaded, and $\langle a_i | \, i < \omega \rangle$ is a j-partition of S. Then rank$(j, \kappa + \alpha_0, \vec{a}) \geq \alpha_0$.*

Proof. We prove this by induction on α_0. Clearly, if $\alpha_0 = \alpha_0' + 1$ then $S \cap \alpha_0$ is λ-threaded and has largest element α_0', hence the induction is immediate.

Now assume that α_0 is a limit. There are two cases. Either $\mathrm{cof}(\alpha) < \lambda$ or $\mathrm{cof}(\alpha) > \lambda$.

First assume that $\mathrm{cof}(\alpha_0) < \lambda$. Then there must be a sequence $\langle \beta_i | \, i < \mathrm{cof}(\alpha_0)\rangle$ cofinal in α_0 such that for all $i < \mathrm{cof}(\alpha_0)$, $\beta_i \in S$. Hence we have that $S \cap \beta_i + 1$ is λ-threaded and has largest element β_i. So by induction we have that

$$\mathrm{rank}(j, \kappa + \beta_i, \langle a_i \cap \beta_i + 1 | \, i < \omega\rangle) \geq \beta_i.$$

But then clearly we have that $\mathrm{rank}(j, \kappa + \alpha_0, \vec{a}) \geq \sup_i \beta_i = \alpha_0$ since for all $i < \omega$ we have that for some n, $\beta_i \in a_n$.

Now assume that $\mathrm{cof}(\alpha_0) > \lambda$. Based on an arbitrary sequence $\beta_0 > \beta_1 > \cdots$ with $\beta_0 < \alpha_0$ we will choose responses α_i and $\vec{a}^i$ which are legal plays against a play by II in $G(j, \kappa + \alpha_0, \vec{a})$.

Let $\beta_0 < \alpha_0$. Let $\vec{\mathcal{E}}^0$ be a first play by II in $G(j, \kappa + \alpha_0, \vec{a})$ and set $S_0 = S$.

Let $(J^0, \vec{j}^0) \in \mathcal{E}^e_{\kappa+\alpha_0+\omega}$ be such that $\alpha_0 \in \mathrm{rng}\, \hat{J}^0$ and $(J^0)^{\mathrm{ext}}(\alpha_0) = \alpha_0$. Let $T_0 \subseteq \beta_0 + 1$ be λ-threaded with $\beta_0 \in T_0$.

For each $\beta \in T_0 \setminus \sup S_0$, we play a version of $G(\alpha_0, J^0)$ and define $f(\beta)$ by induction on the order of $T_0 \setminus \sup S_0$. We call this game $G(\alpha_0, J^0)[\beta]$ and let $\alpha[\beta]$ be a winning response by II to the play $\beta, f(\beta)$ by I. Let i be least such that

$$\alpha_0 \in \bigcap_{n \geq i} \mathcal{F}(\mathcal{E}^0_n).$$

Assume we have defined $f(\beta')$ and $\alpha[\beta']$ for all $\beta' \in \beta \cap (T_0 \setminus \sup S_0)$. Let γ be least such that for all $\beta' \in \beta \cap (T_0 \setminus \sup S_0)$,

$$\gamma > \alpha[\beta'], \qquad \forall n \geq i \, (\gamma \in \mathcal{F}(\mathcal{E}^0_n)), \text{ and} \qquad (J^0)^{\mathrm{ext}}(\gamma) = \gamma.$$

Set $f(\beta) = \gamma$.

Let

$$S_1 = \{\alpha[\beta] | \, \beta \in T_0 \setminus \sup S_0\} \cup (S_0 \cap \alpha_0)$$

and let $\vec{a}^1$ be a division of S_1 with respect to $\mathcal{E}^0$. Note that for all $\alpha \in S_1 \setminus \sup(S_0 \cap \alpha_0)$, there exists an i such that for some $\gamma \in \bigcap_{n \geq i} \mathcal{F}(\mathcal{E}^0_n)$, as α is definable from parameters in $\{\alpha_0, \gamma\} \cup \lambda$ over $L_{\kappa+\alpha_0+2}(V_{\lambda+1})$. Hence there exists an i' such that for all $n \geq i'$, $\alpha \in \mathcal{F}(\mathcal{E}^0_n)$. I then plays $(\vec{a}^1, \alpha[\beta_0])$.

Now assume that I has played

$$(\vec{a}, \alpha_0), (\vec{a}^1, \alpha[\beta_0]), \ldots, (\vec{a}^n, \alpha[\beta_0, \ldots, \beta_{n-1}])$$

against $\vec{\mathcal{E}}^0, \vec{\mathcal{E}}^1, \ldots, \vec{\mathcal{E}}^{n-1}$ and $\beta_0 > \beta_1 > \cdots > \beta_{n-1}$. Assume we have defined the following as well.

1. $T_0, \ldots, T_{n-1}$ such that for $i < n$, $T_i \subseteq \beta_i + 1$ is λ-threaded and $\beta_i \in T_i$. Let
$$T_i^* = T_i \setminus (\sup(T_{i-1} \cap \beta_i)),$$
where $T_{-1} = S_0$.

2. Suppose that $\delta_0 > \cdots > \delta_{m-1}$ is such that $m \leq n$ and the following hold: $\delta_0 \in T_0^*$, and for all $i < m-1$, there is an i' such that $\beta_{i'} = \delta_i$, and $\delta_{i+1} \in T_{i'}^*$. Then
$$G(\alpha_0, J^0)[\delta_0, \ldots, \delta_{m-1}]$$
is an instance of $G(\alpha_0, J^0)$ with
$$f(\delta_0), f(\delta_0, \delta_1), \ldots, f(\delta_0, \ldots, \delta_{m-1})$$
defined and with $\alpha[\delta_0] > \cdots > \alpha[\delta_0, \ldots, \delta_{m-1}]$ a winning response by II against the play
$$(\delta_0, f(\delta_0)), (\delta_1, f(\delta_0, \delta_1)), \ldots, (\delta_{m-1}, f(\delta_0, \ldots, \delta_{m-1})).$$

3. For W_n the set of such tuples $(\delta_0, \ldots, \delta_{m-1})$ the function f is defined on W such that it is order preserving from lexicographically ordered tuples to ordinals. Furthermore for all $(\delta_0, \ldots, \delta_{m-1}) \in W_n$, if s is such that $\delta_{m-1} \in T_s^*$, then there is an i such that for all $n' \geq i$
$$f(\delta_0, \ldots, \delta_{m-1}) \in \mathcal{F}(\mathcal{E}_{n'}^s).$$

Now let $\beta_n < \beta_{n-1}$ and let $\vec{\mathcal{E}}^n$ be a play by II. We can assume without loss of generality that if
$$T_{n-1} \cap [\beta_n, \beta_{n-1}) \neq \emptyset$$
then $\beta_n \in T_{n-1}$.

Suppose first that $\beta_n \notin T_{n-1}$. Let $T_n \subseteq \beta_n + 1$ be λ-threaded such that $\beta_n \in T_n$ and $T_{n-1} \cap \beta_{n-1} \subseteq T_n$. For each $\delta \in T_n \setminus (\sup T_{n-1} \cap \beta_n)$ we define $f(\beta_{s(0)}, \ldots, \beta_{s(m-1)}, \delta)$ by induction, where s is longest such that for all $i < m-1$, there exists an i' such that $\beta_{s(i)} \in T_{i'}$ but $\beta_{s(i+1)} \notin T_{i'}$ and $s(m-1) = n-1$:

First we know that $\beta_{n-1} \in T_{n-1}$ and it is the least element of T_{n-1} greater than β_n. Hence $\mathrm{cof}(\beta_{n-1}) > \lambda$ since T_{n-1} is λ-threaded. Hence by definition of the game $G(\alpha_0, J^0)$, $\alpha^* = \alpha[\beta_{s(0)}, \ldots, \beta_{s(m-1)}]$ is such that $\mathrm{cof}(\alpha^*) > \lambda$. Let i be least such that for all $i' \geq i$, $\alpha^* \in \mathcal{F}(\mathcal{E}_{i'}^n)$. Let γ be least in $\bigcap_{i' \geq i} \mathcal{F}(\mathcal{E}_i^n) \cap \alpha^*$ such that for all
$$\delta' \in \delta \cap (T_n \setminus (\sup(T_{n-1} \cap \beta_n)))$$
we have
$$\alpha[\beta_{s(0)}, \ldots, \beta_{s(m-1)}, \delta'] > \gamma.$$

Set $f(s(0), \dots, s(m-1), \delta) = \gamma$.

Now let

$$S_n = (\{\alpha[\delta_0, \dots, \delta_{m'-1}] | (\delta_0, \dots, \delta_{m'-1}) \in \mathrm{dom}(f)\} \cap \alpha_{n-1}) \cup (S_{n-1} \cap \alpha_{n-1}).$$

Set

$$\alpha_n = \alpha[\beta_{s(0)}, \dots, \beta_{s(m-1)}, \beta_n],$$

and let $\vec{a}^n$ be a division of S_n with respect to $\vec{\mathcal{E}}^n$. I then plays $(\vec{a}^n, \alpha_n)$.

Now suppose that $\beta_n \in T_{n-1}$. Then we simply let $T_n = T_{n-1} \cap \beta_n + 1$ and we set

$$\alpha_n = \alpha[\delta_0, \dots, \delta_{m-1}]$$

where $(\delta_0, \dots, \delta_{m-1}) \in W_n$ is the unique sequence satisfying that $\delta_{m-1} = \beta_n$. We set $S_n = S_{n-1} \cap \alpha_n + 1$ and let $\vec{a}^n$ be a division of S_n with respect to $\vec{\mathcal{E}}^n$. I then plays $(\vec{a}^n, \alpha_n)$.

Clearly we have shown legal plays by I based on any finite sequence $\beta_0 > \beta_1 > \cdots$. Hence the induction is complete. $\square$

Corollary 4.3.10. *Suppose there exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$. Then the supremum of rank$(j, \kappa, \vec{a})$ for all possible κ and $\vec{a}$ is Θ.*

4.4 Representations in $L(V_{\lambda+1})$

Using the results of the previous few sections, we can show by a result of Woodin that $U(j)$-representations extend considerably far in $L(V_{\lambda+1})$. Uncovering the full extent of these representations in $L(V_{\lambda+1})$ has many interesting consequences which we will explore in the next section.

Theorem 4.4.1 (Cramer, Woodin)**.** *Suppose there is an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1}).$$

Let κ be least such that

$$J_\kappa(V_{\lambda+1}) \prec_{\Sigma_1(V_{\lambda+1} \cup \{V_{\lambda+1}\})} J_{\kappa+1}(V_{\lambda+1}).$$

Then every set $Z \in L_\kappa(V_{\lambda+1}) \cap V_{\lambda+2}$ is $U(j)$-representable in $L(V_{\lambda+1})$.

The following proof, which gives the representations from the Tower Condition and the results of Section 4.3, is due to Woodin.

Proof sketch. Let $G^*(j, \gamma, \vec{a})$ be the game defined the same as $G(j, \gamma, \vec{a})$ where in addition one requires that

1. γ is weakly inaccessible in $L(V_{\lambda+1})$ and $L_\gamma(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

2. For each $i < \omega$, γ_i is weakly inaccessible in $L(V_{\lambda+1})$ and $L_{\gamma_i}(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

Since the set of $\gamma < \Theta$ which are weakly inaccessible in $L(V_{\lambda+1})$ is cofinal in Θ, the theorems of the previous section carry over to $G^*(j, \gamma, \vec{a})$.

Let S be the set of all $(\gamma, \vec{a})$ such that $G^*(j, \gamma, \vec{a})$ is defined. For each $(\gamma, \vec{a}) \in S$, let $N(\gamma, \vec{a})$ be the set of all $Z \in L(V_{\lambda+1}) \cap V_{\lambda+2}$ such that Z is $U(j, \gamma, \vec{a})$-representable. Let $\rho(\gamma, \vec{a})$ be the least ordinal α such that

$$L_{\alpha+1}(V_{\lambda+1}) \cap V_{\lambda+2} \not\subseteq N(\gamma, \vec{a}).$$

We show that if $(\gamma, \vec{a}) \in S$ then it is not the case that

$$\rho(\gamma, \vec{a}) < \mathrm{rank}(G^*(\gamma, \vec{a})).$$

To see this, suppose towards a contradiction that this were the case. Then by replacing $(\gamma, \vec{a})$ by a move at some finite stage of $G^*(j, \gamma, \vec{a})$ by Player I, we can assume that for all initial moves $E_0 \subseteq \mathcal{E}(j, \gamma)$ there exists a response $(\gamma_1, \vec{b}^1) \in S$ such that $\rho(\gamma_1, \vec{b}^1) \geq \rho(\gamma, \vec{a})$.

Let $\kappa = \rho(\gamma, \vec{a})$. Suppose that $a \in L_\gamma(V_{\lambda+1})$, $i < \omega$ and $j_{(i)}(a, \gamma) = (a, \gamma)$. Let E_a be the set of all $k \in \mathcal{E}(j, \gamma)$ such that $k(a) = a$. For each $\sigma \in [E_a]^\lambda$ let

$$C_\sigma = \{b \in L_\gamma(V_{\lambda+1}) | \, k(b) = b \text{ for all } k \in \sigma\},$$

and let $\mathcal{F}_a$ be the filter generated by the sets C_σ for $\sigma \in [E_a]^\lambda$.

As before (see Lemma 4.1.2) we have that there exists a partition $\langle S_\alpha | \, \alpha < \eta_a \rangle$ of $L_\gamma(V_{\lambda+1})$ into $\mathcal{F}_a$-positive sets such that $\eta_a < \mathrm{crit}\,(j_{(i)})$ and for each $\alpha < \eta_a$, $\mathcal{F}_a \restriction S_\alpha$ is an ultrafilter in $L(V_{\lambda+1})$.

We claim that there exists a set $E_0 \in [\mathcal{E}(j, \gamma)]^\lambda$ and a function

$$e : L_\gamma(V_{\lambda+1}) \to U(j, \gamma, \vec{a})$$

such that the following hold:

1. For each $k \in E_0$ there exists $i < \omega$ such that $k(a_i) = a_i$.

2. For all $\vec{c} \in L_\gamma(V_{\lambda+1})$, if for all $i < \omega$, $k(c_i) = c_i$ for all $k \in E_0$ such that $k(a_i) = a_i$, then $\langle e(c_i) | \, i < \omega \rangle$ is a wellfounded tower and for all $i < \omega$, for $e(c_i)$-almost all $c \in L_\gamma(V_{\lambda+1})$,

$$(L_\gamma(V_{\lambda+1}), \langle c_n | \, n < i \rangle) \equiv (L_\gamma(V_{\lambda+1}), c)$$

with parameters in $V_\lambda \cup \{V_{\lambda+1}\}$.

This follows immediately by restricting to a measure one set of equivalent elements in the above sense, and then finding E_0 which generates that set.

Let $\langle \kappa_i | \, i < \omega \rangle$ be the critical sequence of j. We can assume without loss of generality that $\kappa_i \in a_i$ for all $i < \omega$. Let $N(E_0)$ be the set of all $Z_0 \subseteq V_{\lambda+1}$ such that there exists a function

$$\pi_0 : \bigcup \{ V_{\kappa_i+1} \times V_{\kappa_i+1} \times \{i\} | \, i < \omega \} \to U(j, \gamma_0, \langle a_i | \, i < \omega \rangle)$$

such that

1. $\gamma_0 < \gamma$ and $L_{\gamma_0}(V_{\lambda+1}) \prec_{\Sigma_2} L_\gamma(V_{\lambda+1})$.
2. γ_0 is weakly inaccessible in $L(V_{\lambda+1})$.
3. There exists $i < \omega$ such that for all $k \in E_0$, if $k(a_i) = a_i$ then $k(\gamma_0) = \gamma_0$.
4. π_0 witnesses that Z_0 is $U(j, \gamma_0, \vec{a})$-representable.
5. For all $i < \omega$ and $k \in E_0$, if $k(a_i, \gamma_0) = (a_i, \gamma_0)$ then $k(\pi_0 \restriction V_{\kappa_i+\omega}) = \pi_0 \restriction V_{\kappa_i+\omega}$.

Fix $c_0 \in V_{\lambda+1}$ and for each formula $\phi(x_0, x_1)$ let Z_ϕ be the set of all $a \in V_{\lambda+1}$ such that there exists $Z \in N(E_0)$ such that

$$(V_{\lambda+1}, Z) \models \phi[c_0, a].$$

It can be shown that for all ϕ, Z_ϕ is $U(j, \gamma, \vec{a})$-representable. We can then show however that every subset of $V_{\lambda+1}$ in $J_{\kappa+1}(V_{\lambda+1})$ is $U(j, \gamma, \vec{a})$-representable, which is a contradiction to the definition of κ (see [Woo11] for details). □

4.5 Consequences of $U(j)$-representations

In view of the results of the previous sections, we make the following conjecture, which we call the $U(j)$-conjecture.

Conjecture 4.5.1 ($U(j)$-conjecture)**.** *Suppose that there exists $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ elementary. Then in $L(V_{\lambda+1})$ every subset of $V_{\lambda+1}$ is $U(j)$-representable.*

There are many consequences of the existence of $U(j)$-representations. These include structural consequences very similar to the consequences of inverse limit reflection. For instance a perfect set property and the fact that the weak ω-club filter is an ultrafilter both follow from $U(j)$-representations (see reference). There are stronger consequences of $U(j)$-representations, however. An important example is the following theorem of Woodin which gives a version of generic absoluteness.

Theorem 4.5.2 (Woodin). *Suppose that $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is a proper elementary embedding (see [Woo11]), and that β is such that all elements of $L_\beta(V_{\lambda+1}) \cap V_{\lambda+2}$ are $U(j)$-representable. Let M_ω be the ω-th iterate of $L(V_{\lambda+1})$ by j, and let $j_{0,\omega} : L(V_{\lambda+1}) \to M_\omega$. Suppose that $g \in V$, g is M_ω-generic for a partial order $\mathbb{P} \in j_{0,\omega}(V_\lambda)$ and that $cof(\lambda) = \omega$ in $M_\omega[g]$. Then for all $\alpha < \beta$ there exists an elementary embedding*

$$\pi : L_\alpha(M_\omega[g] \cap V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$$

such that $\pi \restriction \lambda$ is the identity.

An application of this theorem, using the existence of $U(j)$-representations for $L(V_{\lambda+1})$ shown above, is the following.

Theorem 4.5.3 (Woodin). *Assume the $U(j)$-conjecture holds. Then*

$$Con(I_0) \to Con(I_0 \text{ at } \lambda \text{ and } 2^\lambda = \lambda^{++}).$$

The existence of $U(j)$-representations also has the important consequence of connecting $L(V_{\lambda+1})$ with models of determinacy. In particular we have the following.

Theorem 4.5.4 (Woodin). *Suppose that there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

where λ is a limit of supercompact cardinals and there is a proper class of Woodin cardinals. Assume the $U(j)$-conjecture holds. Then there is an inner model which satisfies AD^+ and $\exists\alpha\,(\Theta_\alpha$ is the largest Suslin cardinal$))$.

Interestingly, if the $U(j)$-conjecture holds, then there can be no direct connection between uniformization and these representations, as uniformization does not hold for every set in $L(V_{\lambda+1})$. To see this, recall the following fact, whose proof is the same as in the case of $L(\mathbb{R})$.

Fact 4.5.5 (Kechris and Solovay). *There is a subset of $V_{\lambda+1}$, $R \in L(V_{\lambda+1})$, $\Pi_1(V_{\lambda+1})$-definable over $L(V_{\lambda+1})$ such that R does not have a uniformization in $L(V_{\lambda+1})$.*

Proof. Let R be the set

$$R = \{(x, y) \in V_{\lambda+1} \times V_{\lambda+1} | \, y \text{ is not OD from } x \text{ in } L(V_{\lambda+1})\}.$$

Suppose that U is a uniformization of R such that $U \in L(V_{\lambda+1})$. Then U is OD from some $x \in V_{\lambda+1}$ in $L(V_{\lambda+1})$. But then if y is such that $(x, y) \in U$, then y is OD from x in $L(V_{\lambda+1})$, a contradiction to the definition of R. □

Chapter 5

Conclusion

5.1 Future directions

5.1.1 The E^0_α hierarchy

One important direction for future research is to what extent the above results can be extended beyond $L(V_{\lambda+1})$ to stronger large cardinals. A related question is, what are the extensions beyond $L(V_{\lambda+1})$? A variety of possible extensions have been proposed, and in some sense the extension of inverse limit reflection and $U(j)$-representations to these structures would give them something of a justification, especially in light of the failure of a general inverse limit X-reflection.

In this section we introduce an extension above $L(V_{\lambda+1})$ introduced by Woodin, which was motivated by construction of the minimal model of $AD_{\mathbb{R}}$.

Definition 5.1.1. Let $\left\langle E^0_\alpha(V_{\lambda+1}) |\, \alpha < \Upsilon_{V_{\lambda+1}} \right\rangle$ be the maximum sequence such that the following hold.

1. $E^0_0(V_{\lambda+1}) = L(V_{\lambda+1}) \cap V_{\lambda+2}$ and $E^0_1(V_{\lambda+1}) = L((V_{\lambda+1})^{\#}) \cap V_{\lambda+2}$.

2. Suppose $\alpha < \Upsilon_{V_{\lambda+1}}$ is a limit ordinal. Then
$$E^0_\alpha(V_{\lambda+1}) = L(\bigcup\{E^0_\beta(V_{\lambda+1}) |\, \beta < \alpha\}) \cap V_{\lambda+2}.$$

3. Suppose $\alpha + 1 < \Upsilon_{V_{\lambda+1}}$. Then for some $X \in E^0_{\alpha+1}(V_{\lambda+1})$, there exists a surjection $f : V_{\lambda+1} \to E^0_\alpha(V_{\lambda+1})$ such that $f \in L(X, V_{\lambda+1})$ and
$$E^0_{\alpha+1}(V_{\lambda+1}) = L(X, V_{\lambda+1}) \cap V_{\lambda+2},$$
and if $\alpha + 2 < \Upsilon_{V_{\lambda+1}}$ then
$$E^0_{\alpha+2}(V_{\lambda+1}) = L((X, V_{\lambda+1})^{\#}) \cap V_{\lambda+2}.$$

4. Suppose $\alpha < \Upsilon_{V_{\lambda+1}}$. Then there exists $X \subseteq V_{\lambda+1}$ such that $E^0_\alpha V_{\lambda+1} \subseteq L(X, V_{\lambda+1})$ and such that there is a proper elementary embedding

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1}).$$

5. Suppose that $\alpha < \Upsilon_{V_{\lambda+1}}$ is a limit ordinal and $N = E^0_\alpha(V_{\lambda+1})$. Then either
   (a) $(\mathrm{cof}(\Theta^N))^{L(N)} < \lambda$, or
   (b) $(\mathrm{cof}(\Theta^N))^{L(N)} > \lambda$ and for some $Z \in N$, $L(N) = (HOD_{V_{\lambda+1}\cup\{Z\}})^{L(N)}$.

6. Suppose that $\alpha + 1 < \Upsilon_{V_{\lambda+1}}$ is a limit ordinal and $N = E^0_\alpha(V_{\lambda+1})$. Then either
   (a) $(\mathrm{cof}(\Theta^N))^{L(N)} < \lambda$ and $E^0_{\alpha+1}(V_{\lambda+1}) = L(N^\lambda, N) \cap V_{\lambda+2}$, or
   (b) $(\mathrm{cof}(\Theta^N))^{L(N)} > \lambda$ and $E^0_{\alpha+1}(V_{\lambda+1}) = L(\mathcal{E}(N), N) \cap V_{\lambda+2}$, where

$$\mathcal{E}(N) = \{j \mid j : N \to N \text{ is elementary}\}.$$

Question 5.1.2. Does inverse limit reflection hold in the E^0_α hierarchy? Which sets in the E^0_α-hierarchy are $U(j)$-representable?

5.1.2 Reinhardt cardinals

A fundamental question in the theory of large cardinals is whether or not it is consistent for there to be an elementary embedding $V \to V$ without assuming the axiom of choice. We consider here a weaker question which could be more readily answered.

Question 5.1.3. Is it consistent for there to be an elementary embedding $j : V \to V$ such that for λ the limit of the critical sequence of j, λ-DC holds and λ is a limit of supercompacts?

We outline some motivation that the such cardinals are indeed inconsistent. To do this we introduce the following notion of Woodin.

Definition 5.1.4 (Woodin). We call the following statement the *Weak Uniqueness of Square Roots at* λ. Suppose λ is a limit of supercompact cardinals. For all $X \subseteq V_{\lambda+1}$, if

$$j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$$

is a proper elementary embedding such that $j(X) \in L_\omega(X, V_{\lambda+1})$, and if k_1 and k_2 are each square roots of j such that

1. $k_1 \restriction V_\lambda = k_2 \restriction V_\lambda$,

2. $k_1(L_\omega(X, V_{\lambda+1})) = k_2(L_\omega(X, V_{\lambda+1})) = L_\omega(X, V_{\lambda+1})$,

then $k_1 \restriction \Theta^{L(X,V_{\lambda+1})} = k_2 \restriction \Theta^{L(X,V_{\lambda+1})}$.

Theorem 5.1.5 (Woodin)**.** *Suppose that ZFC implies that the weak uniqueness of square roots holds. Then under ZF, assuming there is a proper class of supercompact cardinals, there is no elementary embedding $j : V \to V$ such that crit(j) is a limit of supercompact cardinals.*

Lemma 5.1.6 (Woodin)**.** *Suppose $X \subseteq V_{\lambda+1}$, $j : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ is a proper elementary embedding such that $j(X) \in L_\omega(X, V_{\lambda+1})$. Let $\Theta = \Theta^{L(X,V_{\lambda+1})}$ and suppose that for each $\eta < \Theta$, there exists a surjection $\rho : V_{\lambda+1} \to \eta$ such that ρ is Σ_1-definable in $(L_\Theta(X, V_{\lambda+1}), j \restriction L_\Theta(X, V_{\lambda+1}))$ with parameters from $V_{\lambda+1} \cup \Theta \cup \{L_\omega(X, V_{\lambda+1}))\}$. Then the weak uniqueness of square roots holds for (X, j) at λ.*

The connection here is the following question about $L(V_{\lambda+1})$.

Question 5.1.7. Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is elementary. Then is

$$V^{\#}_{\lambda+1} \in L(j \restriction L_\Theta(V_{\lambda+1}), V_{\lambda+1})?$$

In fact do we have

$$L(j \restriction L_\Theta(V_{\lambda+1}), V_{\lambda+1}) \cap V_{\lambda+2} \neq L(V_{\lambda+1}) \cap V_{\lambda+2}?$$

If we could answer this question in the affirmative, then a similar analysis throughout the E^0_α hierarchy could be a way of showing that certain $V \to V$ embeddings do not exist without using the Axiom of Choice.

Bibliography

[Dav64] Morton Davis. Infinite games of perfect information. In *Advances in game theory*, pages 85–101. Princeton Univ. Press, Princeton, N.J., 1964.

[Kan94] Akihiro Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*. Perspect. Math. Logic. Springer-Verlag, New York, 1994.

[Kec85] Alexander S. Kechris. Determinacy and the structure of $L(\mathbf{R})$. In *Recursion theory (Ithaca, N.Y., 1982)*, volume 42 of *Proc. Sympos. Pure Math.*, pages 271–283. Amer. Math. Soc., Providence, RI, 1985.

[Kun71] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *J. Symbolic Logic*, 36:407–413, 1971.

[KW10] Peter Koellner and W. Hugh Woodin. Large cardinals from determinacy. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1951–2119. Springer, Dordrecht, 2010.

[Lav97] Richard Laver. Implications between strong large cardinal axioms. *Ann. Pure Appl. Logic*, 90(1-3):79–90, 1997.

[Lav01] Richard Laver. Reflection of elementary embedding axioms on the $L[V_{\lambda+1}]$ hierarchy. *Ann. Pure Appl. Logic*, 107(1-3):227–238, 2001.

[Lav02] R. Laver. On very large cardinals. In *Paul Erdős and his mathematics, II (Budapest, 1999)*, volume 11 of *Bolyai Soc. Math. Stud.*, pages 453–469. János Bolyai Math. Soc., Budapest, 2002.

[Mos80] Yiannis N. Moschovakis. *Descriptive set theory*, volume 100 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980.

[Ste83] John R. Steel. Scales in $L(\mathbf{R})$. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 107–156. Springer, Berlin, 1983.

[Ste95] John R. Steel. $\mathrm{HOD}^{L(\mathbf{R})}$ is a core model below Θ. *Bull. Symbolic Logic*, 1(1):75–84, 1995.

[Woo11] W. Hugh Woodin. Suitable extender models II: beyond ω-huge. *J. Math. Log.*, 11(2):115–436, 2011.