MATH 451 FINAL EXAM

Question 1. Let G be a group and for each $g \in G$, let $i_g : G \to G$ be the map defined by $i_g(x) = gxg^{-1}$.

(a) Prove that $i_g \in \mathrm{Aut}(G)$ for each $g \in G$.

(b) Prove that $\mathrm{Inn}(G) = \{ i_g \mid g \in G \}$ is a subgroup of $\mathrm{Aut}(G)$.

(c) Prove that if $\pi \in \mathrm{Aut}(G)$ and $g \in G$, then $\pi\, i_g\, \pi^{-1} = i_{\pi(g)}$.

(d) Prove that if G is a centerless group, then $C_{\mathrm{Aut}(G)}(\,\mathrm{Inn}(G)\,) = 1$.

Question 2. Recall that if G is a group and $x \in G$, then the corresponding conjugacy class is $x^G = \{ gxg^{-1} \mid g \in G \}$.

(a) Suppose that G is a finite group and that $H \leqslant G$. Prove that H is a normal subgroup iff there exist elements $x_1, \cdots, x_k \in G$ such that

$$H = \bigcup_{i=1}^{k} x_i^G.$$

(b) Find a set $\pi_1, \cdots, \pi_\ell$ of representatives for the *distinct* conjugacy classes of the alternating group A_5 and compute the size of the conjugacy class of π_i for each $1 \le i \le \ell$.

(c) Using (a), (b) and Lagrange's Theorem, prove that A_5 is a simple group.

Question 3.

(a) Give the definition of a Sylow p-subgroup of a finite group G.

(b) State and prove the Third Sylow Theorem. (*In your proof, you may assume the result that the Sylow p-subgroups of a finite group G are conjugate.*)

(c) Prove that there does *not* exist a simple group of order 270.

Question 4. (a) Let p be a prime and let $C_p = \{ 1, a, \cdots, a^{p-1} \}$ be a cyclic group of order p. Prove that $\mathrm{Aut}(C_p)$ is abelian of order $p - 1$.

(b) Prove that if G is a group of order 15, then G is abelian.

(c) Prove that there exists a nonabelian group of order 30.

(d) Prove that if G is a group of order 30, then G contains a normal subgroup of order 15.

Question 5. Suppose that G is a group and that Ω is a G-set.

(a) Let E be the binary relation defined on Ω by

$$s\,E\,t \quad \text{iff} \quad (\,\exists g \in G\,)\ t = g\,s.$$

Prove that E is an equivalence relation.

(b) Suppose that $N \trianglelefteq G$. Let $\alpha \in \Omega$ and let $\Delta = \{\, h\,\alpha \mid h \in N\,\}$ be the corresponding N-orbit. Prove that if $g \in G$, then $g\,\Delta = \{\, g\,\beta \mid \beta \in \Delta\,\}$ is the N-orbit containing $g\,\alpha$.

(c) Suppose that p is a prime and that $G \leqslant S_p$. Prove that if G acts transitively on $X = \{\,1, \cdots, p\,\}$ and $1 \neq N \trianglelefteq G$, then N also acts transitively on X. (*Hint: Using (b)and the transitivity of G, show that all the orbits of N on X have the same cardinality.*)

Question 6. Let $C = \{\, z^n \mid n \in \mathbb{Z}\,\}$ be an infinite cyclic group and let $A = \{\,1, a\,\}$ be a cyclic group of order 2. Let $\varphi : A \to \mathrm{Aut}(C)$ be the homomorphism such that $\varphi_a(z^n) = z^{-n}$ for all $n \in \mathbb{Z}$ and let $D = C \rtimes A$ be the corresponding semi-direct product.

(a) Prove that if $b = za \in D$, then $b^2 = 1$.

(b) State von Dyck's Theorem.

(c) Prove that $\langle\, x, y \mid x^2 = 1,\, y^2 = 1\,\rangle$ is a presentation of D. (*Hint: It may be helpful to notice that these relations imply that $yx = (xy)^{-1}$.*)