MATH 451 FIRST MID-TERM

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Question	Marks
1	12
2	25
3	28
4	25
5	10

Question 1. Let H be a nonempty subset of the group G. Prove that H is a subgroup of G iff $a\,b^{-1} \in H$ for all a, $b \in H$.

First suppose that H is a subgroup of G. Then H is closed under multiplication and taking inverses. Hence if a, $b \in H$, then $b^{-1} \in H$ and so $ab^{-1} \in H$.

Next suppose that $\emptyset \neq H \subseteq G$ is such that $ab^{-1} \in H$ for all a, $b \in H$. Since $H \neq \emptyset$, there exists an element $a \in H$ and hence $1 = aa^{-1} \in H$. It follows that if $a \in H$, then $a^{-1} = 1\,a^{-1} \in H$. Finally suppose that a, $b \in H$. Then $b^{-1} \in H$ and so $ab = a(b^{-1})^{-1} \in H$. Thus H is a subgroup of G.

Question 2. (a) Prove that the alternating group A_5 does *not* have a subgroup which is isomorphic to the symmetric group S_4.

(b) Prove that the alternating group A_5 does *not* have a subgroup of order 15.

(a) Suppose that $H \leqslant A_5$ is a subgroup such that $H \cong S_4$. Then $|H| = |S_4| = 24$. Applying Lagrange's Theorem, we must have that $24 = |H|$ divides $|A_5| = 60$, which is a contradiction.

(b) Suppose that $H \leqslant A_5$ is a subgroup such that $|H| = 15$. Consider the transitive action of A_5 on the coset space $S = A_5/H$ and let

$$\varphi : A_5 \to \mathrm{Sym}(S)$$

be the associated homomorphism. Let $N = \ker \varphi$. Since A_5 acts transitively on S and $|S| > 1$, we have that $N \neq A_5$. Hence, since A_5 is simple, it follows that $N = 1$. But this means that φ is an injection of A_5 into $\mathrm{Sym}(S)$, which is impossible since $|A_5| = 60$ and $|\,\mathrm{Sym}(S)| = 24$.

Question 3. Suppose that G is a finite group and that S is a G-set. For each $s \in S$, let O_s denote the corresponding G-orbit.

(a) Prove that if $s \in S$, then $[\,G : G_s\,] = |O_s|$.

(b) Prove that if G is a finite p-group and p does *not* divide $|S|$, then there exists a fixed point for the action of G; i.e. an element $s \in S$ such that $gs = s$ for all $g \in G$.

(*Hint:* Let $s_1, \cdots s_t$ be representatives of the distinct G-orbits and consider the equation $|S| = |O_{s_1}| + \cdots + |O_{s_t}|$.)

(a) It is easily checked that if g, $h \in G$, then

$$g\,s = h\,s \quad \text{iff} \quad gG_s = hG_s.$$

Hence we can define an injective map $\varphi : G/G_s \to O_s$ by $\varphi(gG_s) = g\,s$. To see that φ is also surjective, let $r \in O_s$ be arbitrary. Then there exists $g \in G$ such that $g\,s = r$ and hence $\varphi(gG_s) = g\,s = r$.

(b) Let $s_1, \cdots s_t$ be representatives of the distinct G-orbits. Then

$$|S| = |O_{s_1}| + \cdots + |O_{s_t}|.$$

Since p does not divide $|S|$, there exists i such that p does not divide $|O_{s_i}|$. Since

$$|O_{s_i}| = [G : G_{s_i}] = |G|/|G_{s_i}|$$

and G is a p-group, it follows that $|O_{s_i}| = 1$ and so s_i is a fixed point for the action of G.

Question 4. If G is a group and $H \leqslant G$ is a subgroup, then the *centralizer* of H in G is defined to be

$$C_G(H) = \{\, g \in G \mid gh = hg \text{ for all } h \in H \,\}$$

and the *normalizer* of H in G is defined to be

$$N_G(H) = \{\, g \in G \mid gHg^{-1} = H \,\}$$

(a) Prove that $N_G(H)$ is a subgroup of G.

(b) Prove that $C_G(H)$ is a normal subgroup of $N_G(H)$.
(*Hint:* This can either be proved directly or else by considering a suitable homomorphism $\varphi : N_G(H) \to \mathrm{Aut}(H)$).

(a) First note that $1\,H\,1^{-1} = H$ and so $1 \in N_G(H)$. Next if $g \in N_G(H)$, then $gHg^{-1} = H$ and so $H = g^{-1}Hg$. Thus $g^{-1} \in N_G(H)$. Finally if g, $h \in N_G(H)$, then

$$ghH(gh)^{-1} = ghHh^{-1}g^{-1} = g(hHh^{-1})g^{-1} = gHg^{-1} = H$$

and so $gh \in N_G(H)$. Thus $N_G(H)$ is a subgroup of G.

(b) First note that if $g \in C_G(H)$, then $gHg^{-1} = H$ and so $C_G(H) \subseteq N_G(H)$. It follows that

$$C_G(H) = \{g \in N_G(H) \mid ghg^{-1} = h \text{ for all } h \in H\}.$$

Next note that if $g \in N_G(H)$, then $gHg^{-1} = H$ and so we can define an associated automorphism $c_g \in \mathrm{Aut}(H)$ by $c_g(h) = ghg^{-1}$. Consider the map

$$\varphi : N_G(H) \to \mathrm{Aut}(H)$$

defined by $\varphi(g) = c_g$. Then it is easily checked that φ is a homomorphism; and that

$$\ker\varphi = \{g \in N_G(H) \mid ghg^{-1} = h \text{ for all } h \in H\} = C_G(H).$$

Hence $C_G(H)$ is a normal subgroup of $N_G(H)$.

Question 5. Let G be a finite group. Prove that if $G \smallsetminus \{1\}$ is a single conjugacy class, then $|G| = 2$.

Suppose that $G \smallsetminus \{1\}$ is a single conjugacy class. If $a \in G \smallsetminus \{1\}$, then

$$|G| - 1 = |G \smallsetminus \{1\}| = |a^G| = [G : C_G(a)] = |G|/|C_G(a)|.$$

Thus $|G| - 1$ divides $|G|$ and this implies that $|G| = 2$.