# MATH 451 SECOND MID-TERM

NAME: John Q. Public

| Question | Marks |
|---|---|
| 1 | 20 |
| 2 | 20 |
| 3 | 20 |
| 4 | 20 |
| 5 | 20 |

**Question 1.** Throughout this question, let $p$ be a prime.

(a) Suppose that $G$ is a finite $p$-group and that $X$ is a nonempty $G$-set such that $|X| \not\equiv 0 \mod p$. Prove that there exists a point $x \in X$ such that $g\,x = x$ for all $g \in G$.

(b) Suppose that $G$ is a $p$-group and that $H \trianglelefteq G$. Prove that if $H \neq \{\,1\,\}$, then $H \cap Z(G) \neq \{\,1\,\}$.
(*Hint:* Consider the action of $G$ on $H \smallsetminus \{\,1\,\}$ by conjugation.)

(a) Let $|G| = p^n$. Let $X = \Omega_1 \sqcup \cdots \sqcup \Omega_t$ be the decomposition of $X$ into $G$-orbits. If $\alpha \in \Omega_i$, then $|\Omega_i| = [G : G_\alpha] = p^{m_i}$ for some $0 \leq m_i \leq n$. Since $|X| = \sum_{i=1}^t |\Omega_i|$ and $p$ does not divide $|X|$, there exists $1 \leq i_0 \leq t$ such that $m_{i_0} = 0$. Hence, letting $\Omega_{i_0} = \{x\}$, it follows that $g\,x = x$ for all $g \in G$.

(b) Let $|G| = p^n$. If $g \in G$, then $gHg^{-1} = H$ and so $g(\,H \smallsetminus \{1\}\,)g^{-1} = H \smallsetminus \{1\}$. Thus $G$ acts by conjugation on $H \smallsetminus \{1\}$. Since $|H| = p^m$ for some $1 \leq m \leq n$, it follows that $p$ does not divide $|H \smallsetminus \{1\}|$. Hence there exists $h \in H \smallsetminus \{1\}$ such that $ghg^{-1} = h$ for all $g \in G$. Clearly $h \in H \cap Z(G)$.

**Question 2.** (a) State the Third Sylow Theorem.

(b) Prove that there does not exist a simple group of order 5500.

(c) Give an example of a nonabelian group of order 5500.

(a) Suppose that $G$ is a finite group of order $n = p^e m$, where $p$ is a prime, $e \geq 1$ and $p$ does not divide $m$. If $s$ is the number $s$ of Sylow $p$-subgroups of $G$, then $s$ divides $m$ and $s \equiv 1 \mod p$.

(b) Suppose $G$ is a simple group of order $5^3 \times 11 \times 2^2$. If $s$ is the number of Sylow 5-subgroups of $G$, then $s$ divides 44 and $s \equiv 1 \mod 5$. Since $G$ is simple, $s \neq 1$ and so $s = 11$. By considering the transitive action of $G$ by conjugation on the set of its Sylow 5-subgroups, we see that there is an embedding of $G$ into $S_{11}$. But this is impossible, since $5^3$ does not divide $|S_{11}|$.

(c) Since $|\operatorname{Aut}(C_{11})| = 10$, there exist embeddings

$$C_2 \hookrightarrow \operatorname{Aut}(C_{11}) \quad \text{and} \quad C_5 \hookrightarrow \operatorname{Aut}(C_{11}),$$

which give rise to corresponding nonabelian semidirect products. Thus the non-abelian groups of order 5500 include:

- $(\, C_{11} \rtimes C_2 \,) \times C_{250}$
- $(\, C_{11} \rtimes C_5 \,) \times C_{100}$
- etc.

**Question 3.** Suppose that $G$ be a simple group of order 168. Let $P$ be a Sylow 7-subgroup of $G$ and let $H = N_G(P)$.

(a) Prove that $|H| = 21$.

(b) Prove that $N_G(H) = H$. (*Hint:* Notice that $H \leqslant N_G(H) \leqslant G$.)

(c) Prove that there exists an element $g \in G$ such that $gHg^{-1} \neq H$ and $gHg^{-1} \cap H \neq \{ 1 \}$.

(a) If $s$ is the number of Sylow 7-subgroups of $G$, then $s$ divides 24 and $s \equiv 1 \mod 7$. Since $G$ is simple, $s \neq 1$ and so $s = 8$. By considering the transitive action of $G$ by conjugation on the set of its Sylow 7-subgroups, we see that $[G : N_G(P)] = 8$ and hence $|H| = |N_G(P)| = 21$.

(b) Since $H \leqslant N_G(H) \leqslant G$, it follows that $d = [G : N_G(H)]$ divides $[G : H] = 8$. Also by considering the transitive action of $G$ on the coset space $G/N_G(H)$, we see that there is an embedding of $G$ into $S_d$. Thus 7 divides $|S_d|$ and so $d = 8$. It follows that $N_G(H) = H$.

(c) Suppose that $gHg^{-1} \cap H = 1$ whenever $gHg^{-1} \neq H$. Then the 8 distinct conjugates of $H$ intersect pairwise in 1. Hence

$$|( \bigcup_{g \in G} H^g ) \smallsetminus \{1\}| = 8 \times 20 = 160.$$

But this means that $G$ has a unique Sylow 2-subgroup, which is a contradiction.

**Question 4.** Prove that $\langle\, x, y \mid x^2 = 1,\, y^2 = 1,\, (xy)^3 = 1 \,\rangle$ is a presentation of $S_3$.

Let $X = \{x, y\}$ and let $N$ be the normal closure of $\{x^2, y^2, (xy)^3\}$ in $F(X)$. For each $w \in F(X)$, let $\bar{w} = wN \in F(X)/N$. By von Dyck's Theorem, there exists a surjective homomorphism $\varphi : F(X)/N \to S_3$ such that $\varphi(\bar{x}) = (1\,2)$ and $\varphi(\bar{y}) = (2\,3)$. In particular, $|F(X)/N| \geq 6$. On the other hand, let

$$\bar{w} = \bar{x}^{n_1}\bar{y}^{m_1} \cdots \bar{x}^{n_t}\bar{y}^{m_t} \in F(X)/N,$$

where each $n_i$, $m_i \in \mathbb{Z}$. Since $\bar{x}^2 = 1$ and $\bar{y}^2 = 1$, we can suppose that each $0 \leq n_i, m_i \leq 1$. Using the relations $\bar{x}\bar{y}\bar{x}\bar{y}\bar{x}\bar{y} = 1$ and $\bar{x} = \bar{x}^{-1}$ and $\bar{y} = \bar{y}^{-1}$, we can now reduce $\bar{w}$ to one of the following words:

$$1, \bar{x}, \bar{y}, \bar{x}\bar{y}, \bar{y}\bar{x}, \bar{x}\bar{y}\bar{x}.$$

Thus $|F(X)/N| \leq 6$ and it follows that $\varphi : F(X)/N \to S_3$ is an isomorphism.

**Question 5.** Recall that if $\pi \in \mathrm{Sym}(X)$, then $\mathrm{supp}(\pi) = \{x \in X \mid \pi(x) \neq x\}$. Let $S_\infty$ and $A_\infty$ be the subgroups of $\mathrm{Sym}(\mathbb{N}^+)$ defined by

- $S_\infty = \{\, \pi \in \mathrm{Sym}(\mathbb{N}^+) : |\,\mathrm{supp}(\pi)| < \infty \,\}$
- $A_\infty = \{\, \pi \in \mathrm{Sym}(\mathbb{N}^+) : |\,\mathrm{supp}(\pi)| < \infty$ and $\pi$ is an even permutation $\}$.

Prove that $A_\infty$ is the *unique* nontrivial proper normal subgroup of $S_\infty$.

For each $n \geq 1$, define

$$G_n = \{\, \pi \in S_\infty \mid \mathrm{supp}(\pi) \subseteq \{1, \cdots, n\} \,\}$$

and

$$H_n = \{\, \pi \in A_\infty \mid \mathrm{supp}(\pi) \subseteq \{1, \cdots, n\} \,\}.$$

Then we have that

- $G_n \cong S_n$ and $H_n \cong A_n$.
- $S_\infty = \bigcup_{n \geq 1} G_n$ and $A_\infty = \bigcup_{n \geq 1} H_n$.

Suppose that $N$ is a nontrivial proper normal subgroup of $S_\infty$ and let $1 \neq \pi \in N$. Then there exists $n_0 \geq 5$ such that $\pi \in G_{n_0}$. It follows that $N \cap G_n$ is a nontrivial normal subgroup of $G_n$ for each $n \geq n_0$; and this implies that either $N \cap G_n = H_n$ or $N \cap G_n = G_n$. In particular, $H_n \leqslant N \cap G_n$ and so

$$A_\infty = \bigcup_{n \geq n_0} H_n \leqslant N.$$

It is easily checked that $[S_\infty : A_\infty] = 2$. Hence, since $N$ is a *proper* subgroup of $S_\infty$, it follows that $N = A_\infty$.