MATHEMATICAL LOGIC: FALL 2009
TAKE HOME EXAMINATION

COLLABORATION WITH OTHER STUDENTS IS PERMITTED. HOWEVER, SOLUTIONS MUST BE WRITTEN INDIVIDUALLY AND ANY DISCUSSIONS MUST BE ACKNOWLEDGED. THE EXAM IS DUE BACK ON FRIDAY, DECEMBER 18.

1. Set Theory

Question 1.

(a) Let $\mathrm{SI}(\mathbb{N})$ be the set of *strictly increasing* functions $f : \mathbb{N} \to \mathbb{N}$; i.e. those functions f such that $f(n) < f(m)$ for all $0 \leq n < m$. Determine whether $\mathrm{SI}(\mathbb{N})$ is countable or uncountable.

(b) Let $\mathrm{D}(\mathbb{N})$ be the set of *decreasing* functions $f : \mathbb{N} \to \mathbb{N}$; i.e. those functions f such that $f(n) \geq f(m)$ for all $0 \leq n < m$. Determine whether $\mathrm{D}(\mathbb{N})$ is countable or uncountable.

(c) Prove that there exist exactly $2^{\aleph_0}$ open subsets of $\mathbb{R}$. (*Hint: each nonemepty open subset is a union of open intervals with rational endpoints.*)

Question 2. In this question, $\mathcal{F}$ denotes the set of all functions $\varphi : \omega_1 \to \omega_1$.

(a) Let $\approx$ be the binary relation defined on $\mathcal{F}$ by

$$\varphi \approx \psi \quad \Longleftrightarrow \quad \{\alpha \in \omega_1 \mid \varphi(\alpha) = \psi(\alpha)\} \text{ contains a club.}$$

Prove that $\approx$ is an equivalence relation.

(b) For each $\varphi \in \mathcal{F}$, let $[\varphi]$ denote the corresponding $\approx$-class and let

$$\mathcal{F}/E = \{[\varphi] \mid \varphi \in \mathcal{F}\}.$$

Prove that the cardinality of $\mathcal{F}/E$ is $2^{\aleph_1}$. (*Hint: recall that ω_1 can be expressed as the disjoint union of ω_1 stationary subsets.*)

(c) Let $\prec$ denote the binary relation defined on $\mathcal{F}/E$ by

$$[\varphi] \prec [\psi] \quad \Longleftrightarrow \quad \{\alpha \in \omega_1 \mid \varphi(\alpha) < \psi(\alpha)\} \text{ contains a club.}$$

Prove that $\prec$ is well-defined and that $\prec$ is a partial ordering of $\mathcal{F}/E$.

(d) Prove that there does not exist an infinite descending sequence

$$\varphi_0 \succ \varphi_1 \succ \varphi_2 \succ \cdots \succ \varphi_n \succ \cdots$$

of elements of $\mathcal{F}/E$.

(e) For each $\alpha < \omega_1$, let $\theta_\alpha \in \mathcal{F}$ be the map such that $\theta_\alpha(\zeta) = \alpha$ for all $\zeta \in \omega_1$ and let $Id \in \mathcal{F}$ be the map such that $Id(\zeta) = \zeta$ for all $\zeta \in \omega_1$. Prove that $[Id]$ is the least upper bound of $\{[\theta_\alpha] \mid \alpha < \omega_1\}$ in $\mathcal{F}/E$.

Question 3. Suppose that the map $f : \omega_1 \to \mathbb{R}$ satisfies the following condition:

(†) For each $\varepsilon > 0$ and each limit ordinal $\alpha < \omega_1$, there exists $\beta < \alpha$ such that

$$|f(\gamma) - f(\alpha)| < \varepsilon \quad \text{for all } \beta < \gamma < \alpha.$$

Prove that there exists $\alpha_0 < \omega_1$ such that $f(\delta) = f(\alpha_0)$ for all $\alpha_0 \leq \delta < \omega_1$. (*Hint: for each* $\varepsilon = 1/n$, *we can use* (†) *to define a corresponding regressive function* g_n *and then apply Fodor's Theorem.*)

2. Model Theory

Question 4. Let $\mathcal{A}$ be a countably infinite structure for the countable first order language $\mathcal{L}$. Prove that if $\mathrm{Aut}(\mathcal{A})$ has finitely many orbits on A^n for every $n \geq 1$, then $\mathrm{Th}(\mathcal{A})$ is ω-categorical.

Question 5. Let $\mathcal{L} = \{E\}$, where E is a binary relation symbol, and let T be the theory with axioms which state:

- E is the edge relation of a graph; i.e. E is irreflexive and symmetric.
- Every element is adjacent to exactly two other elements.
- For each $n \geq 3$, there do not exist any n-cycles.

(a) Prove that T is a complete theory. (*Hint: show that* T *is* $\aleph_1$*-categorical.*)

(b) Show that T has exactly $\aleph_0$ countable models up to isomorphism.

(c) Identify the countable atomic and countably saturated models of T.

(d) Prove that T does not admit elimination of quantifiers.

Question 6.

(a) For each $n \in \mathbb{N}$, let $\mathcal{L}_n = \{E_\ell \mid \ell \leq n+1\}$, where E_ℓ is a binary relation symbol, and let T_n be the theory with axioms which state:

- If $\ell \leq n+1$, then E_ℓ is an equivalence relation.
- If $\ell \leq n+1$, then there are exactly 2^ℓ E_ℓ-classes, each of which is infinite.
- If $\ell < n+1$, then every E_ℓ-class is the union of exactly two $E_{\ell+1}$-classes.

Prove that T_n is a complete theory in $\mathcal{L}_n$ and that T_n admits elimination of quantifiers.

(b) Let $\mathcal{L} = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n$ and let $T = \bigcup_{n \in \mathbb{N}} T_n$. Prove that T is a complete theory in $\mathcal{L}$ and that T admits elimination of quantifiers.

(c) Let $\mathcal{A}$ be the model for $\mathcal{L}$ defined by

- $A = \{x \in 2^{\mathbb{N}} \mid x(n) = 0 \text{ for all but finitely many } n\}$.
- $x \, E_\ell^{\mathcal{A}} \, y$ iff $x(n) = y(n)$ for all $n < \ell$.

Prove that $\mathcal{A}$ is a countable atomic model of T.

(d) Let $\mathcal{B}$ be the model for $\mathcal{L}$ defined by

- $B = A \times \mathbb{N}$, where A is as in part (c).
- $(x,i) \, E_\ell^{\mathcal{A}} \, (y,j)$ iff $x(n) = y(n)$ for all $n < \ell$.

Prove that $\mathcal{B}$ is a countably saturated model of T.

(e) Prove that T has $2^{\aleph_0}$ countable models up to isomorphism.