AUTOMORPHISM GROUPS OF ULTRAPRODUCTS OF FINITE SYMMETRIC GROUPS

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Abstract. It is consistent that there exists a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that every automorphism of the corresponding ultraproduct $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ is inner.

1. Introduction

Suppose that $\mathcal{U}$ is a nonprincipal ultrafilter over $\mathbb{N}$ and that $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ is the corresponding ultraproduct of the finite symmetric groups. If CH holds, then $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ is a saturated structure and hence

$$| \mathrm{Aut}(\prod_{\mathcal{U}} \mathrm{Sym}(n))| = 2^{\aleph_1} > \aleph_1 = | \prod_{\mathcal{U}} \mathrm{Sym}(n)|.$$

(For example, see Chang-Keisler [1].) In particular, if CH holds, then $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ has many outer automorphisms. Of course, it is well-known that if $n \neq 6$, then every automorphism of $\mathrm{Sym}(n)$ is inner; and consequently, it appears to be difficult to exhibit an *explicit* example of an outer automorphism of $\prod_{\mathcal{U}} \mathrm{Sym}(n)$. The main result of this paper confirms that this is indeed a genuine difficulty.

Theorem 1.1. *It is consistent that there exists a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that every automorphism of $\prod_{\mathcal{U}} \mathrm{Sym}(n)$ is inner.*

No knowledge of set theory is needed in order to understand the proof of Theorem 1.1, which is a purely algebraic consequence of the following remarkable result on ultraproducts of the fields $\mathbb{F}_p$ of prime order p. (Recall that a structure $\mathcal{M}$ is said to be *rigid* if the identity map is the only automorphism of $\mathcal{M}$.)

Theorem 1.2 (Shelah [6]). *It is consistent that there exists a nonprincipal ultrafilter $\mathcal{F}$ over the set $\mathbb{P}$ of primes such that the field $\prod_{\mathcal{F}} \mathbb{F}_p$ is rigid.*

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This paper is organized as follows. In Section 2, we will recall the notion of an internal automorphism of an ultraproduct; and we will prove that if the field $\prod_{\mathcal{F}} \mathbb{F}_p$ is rigid, then every automorphism of $\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)$ is internal. In Section 3, we will recall two basic results concerning regular permutation representations. Finally, in Section 4, we will present the proof of Theorem 1.1.

2. The group of internal automorphisms

In this section, we will recall the notion of an internal automorphism of an ultraproduct; and we will prove that if the field $\prod_{\mathcal{F}} \mathbb{F}_p$ is rigid, then every automorphism of $\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)$ is internal.

Suppose that G_i, $i \in I$, are groups and that $\mathcal{U}$ is a nonprincipal ultrafilter over the index set I. Then an automorphism $\varphi \in \mathrm{Aut}(\prod_{\mathcal{U}} G_i)$ is said to be *internal* if there exist automorphisms $\varphi_i \in \mathrm{Aut}(G_i)$ such that for all $(x_i)_{\mathcal{U}} \in \prod_{\mathcal{U}} G_i$, we have that

$$\varphi(\,(x_i)_{\mathcal{U}}\,) = (\,\varphi_i(x_i)\,)_{\mathcal{U}}.$$

The group of internal automorphisms of $\prod_{\mathcal{U}} G_i$ is denoted by $\mathrm{Int}(\prod_{\mathcal{U}} G_i)$. Clearly we have that

$$\mathrm{Inn}(\prod_{\mathcal{U}} G_i) \trianglelefteq \mathrm{Int}(\prod_{\mathcal{U}} G_i) \leqslant \mathrm{Aut}(\prod_{\mathcal{U}} G_i).$$

Example 2.1. Recall that if $n \neq 6$, then every automorphism of $\mathrm{Sym}(n)$ is inner. It follows that if $\mathcal{U}$ is any nonprincipal ultrafilter over $\mathbb{N}$, then

$$\mathrm{Int}(\prod_{\mathcal{U}} \mathrm{Sym}(n)) = \mathrm{Inn}(\prod_{\mathcal{U}} \mathrm{Sym}(n)).$$

Hence Theorem 1.1 is equivalent to the consistency of a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$ such that

$$\mathrm{Aut}(\prod_{\mathcal{U}} \mathrm{Sym}(n)) = \mathrm{Int}(\prod_{\mathcal{U}} \mathrm{Sym}(n)).$$

Lemma 2.2. *If $\mathcal{F}$ is a nonprincipal ultrafilter over the set $\mathbb{P}$ of primes such that the field $\prod_{\mathcal{F}} \mathbb{F}_p$ is rigid, then*

$$\mathrm{Aut}(\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)) = \mathrm{Int}(\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)).$$

Proof. From now on, if K is any field and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(K)$, then $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ denotes the corresponding element of $PSL_2(K)$. Following Kegel-Wehrfritz [3, 1.L.6], we can define an isomorphism $\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p) \to PSL_2(\prod_{\mathcal{F}} \mathbb{F}_p)$ by

$$\left(\begin{bmatrix} a_p & b_p \\ c_p & d_p \end{bmatrix} \right)_{\mathcal{F}} \mapsto \begin{bmatrix} (a_p)_{\mathcal{F}} & (b_p)_{\mathcal{F}} \\ (c_p)_{\mathcal{F}} & (d_p)_{\mathcal{F}} \end{bmatrix}$$

By a well-known theorem of Schreier and van der Waerden [4], if K is any field with $|K| > 3$, then every automorphism of $PSL_2(K)$ is induced via conjugation by an element of

$$P\Gamma L_2(K) = PGL_2(K) \rtimes \mathrm{Aut}(K).$$

In particular, since the field $\prod_{\mathcal{F}} \mathbb{F}_p$ is rigid, every automorphism of $PSL_2(\prod_{\mathcal{F}} \mathbb{F}_p)$ is induced via conjugation by an element of $PGL_2(\prod_{\mathcal{F}} \mathbb{F}_p)$. This implies that every automorphism of $\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)$ is induced via conjugation by an element of $\prod_{\mathcal{F}} PGL_2(\mathbb{F}_p)$ and hence

$$\mathrm{Aut}(\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)) = \mathrm{Int}(\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)).$$

□

3. Regular permutation representations

In this section, we will recall two basic results concerning regular permutation representations. Here the *left regular permutation representation* of the group H is the embedding $\lambda : H \to \mathrm{Sym}(H)$ defined by $\lambda(h)(x) = h\,x$; and the *right regular permutation representation* of H is the embedding $\rho : H \to \mathrm{Sym}(H)$ defined by $\rho(h) = x\,h^{-1}$. The proof of Theorem 1.1 makes use of the following well-known result. (For example, see Hall [2, Theorem 6.3.1].)

Lemma 3.1. *If H is any group, then:*

(a) $C_{\mathrm{Sym}(H)}(\lambda[H]) = \rho[H]$; *and*
(b) $C_{\mathrm{Sym}(H)}(\rho[H]) = \lambda[H]$.

If H is any group, then the *holomorph* of H is the subgroup $\mathrm{Hol}(H)$ of $\mathrm{Sym}(H)$ defined by

$$\mathrm{Hol}(H) = \lambda[H] \rtimes \mathrm{Aut}(H).$$

It is easily checked that we also have that $\mathrm{Hol}(H) = \rho[\,H\,] \rtimes \mathrm{Aut}(H)$. The proof of Theorem 1.1 also makes use of the following well-known result. (For example, see Hall [2, Theorem 6.3.2].)

Lemma 3.2. *If H is any group, then*

$$N_{\mathrm{Sym}(H)}(\,\lambda[\,H\,]\,) = N_{\mathrm{Sym}(H)}(\,\rho[\,H\,]\,) = \mathrm{Hol}(H).$$

Finally recall that if Ω is any set, then a subgroup L of $\mathrm{Sym}(\,\Omega\,)$ is said to be *regular* if for each pair of (not necessarily distinct) elements x, $y \in \Omega$, there exists a unique element $\pi \in L$ such that $\pi(x) = y$. In this case, the permutation group $(\,L, \Omega\,)$ is isomorphic to the left regular permutation representation of L in $\mathrm{Sym}(\,L\,)$; and, of course, is also isomorphic to the right regular permutation representation.

4. The proof of Theorem 1.1

In this section, we will present the proof of Theorem 1.1. Throughout, we fix a nonprincipal ultrafilter $\mathcal{F}$ over the set $\mathbb{P}$ of primes such that the field $\prod_{\mathcal{F}} \mathbb{F}_p$ is rigid. For each prime $p \in \mathbb{P}$, let $\Gamma_p = PSL_2(\mathbb{F}_p)$. Let $G_{\mathcal{F}} = \prod_{\mathcal{F}} \mathrm{Sym}(\,\Gamma_p\,)$ and $H_{\mathcal{F}} = \prod_{\mathcal{F}} \Gamma_p$. Then it is clearly enough to show that every automorphism of $G_{\mathcal{F}}$ is inner.

For each prime $p \in \mathbb{P}$, let $\lambda_p : \Gamma_p \to \mathrm{Sym}(\,\Gamma_p\,)$ and $\rho_p : \Gamma_p \to \mathrm{Sym}(\,\Gamma_p\,)$ be the left regular and right regular permutation representations. Consider the action of $G_{\mathcal{F}}$ on $H_{\mathcal{F}}$ defined by

$$(\pi_p)_{\mathcal{F}} \cdot (\gamma_p)_{\mathcal{F}} = (\,\pi_p(\gamma_p))_{\mathcal{F}}\,.$$

Then, identifying the group $G_{\mathcal{F}}$ with its image under the corresponding embedding $G_{\mathcal{F}} \to \mathrm{Sym}(\,H_{\mathcal{F}}\,)$, we have that $\prod_{\mathcal{F}} \lambda_p[\,\Gamma_p\,]$ corresponds to the image $\lambda[\,H_{\mathcal{F}}\,]$ of the left regular permutation representation of $H_{\mathcal{F}}$. Also notice that under this identification, we have that

$$\mathrm{Alt}(\,H_{\mathcal{F}}\,) \leqslant G_{\mathcal{F}} \leqslant \mathrm{Sym}(\,H_{\mathcal{F}}\,).$$

Hence, by Scott [5, 11.4.7], it follows that $\mathrm{Aut}(\,G_{\mathcal{F}}\,)$ is precisely the normalizer of $G_{\mathcal{F}}$ in $\mathrm{Sym}(\,H_{\mathcal{F}}\,)$.

Next let $\Gamma = PSL_2(\mathbb{Z})$. Then it is well-known that Γ is generated by the following two elements:

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad\qquad b = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

For each prime $p \in \mathbb{P}$, let $a_p,\ b_p \in PSL_2(\mathbb{F}_p)$ be the images of $a,\ b \in PSL_2(\mathbb{Z})$ under the canonical surjective homomorphism $PSL_2(\mathbb{Z}) \to PSL_2(\mathbb{F}_p)$; and define the elements $\alpha,\ \beta \in \prod_{\mathcal{F}} \mathrm{Sym}(\Gamma_p)$ by $\alpha = (\rho_p(a_p))_{\mathcal{F}}$ and $\beta = (\rho_p(b_p))_{\mathcal{F}}$. Applying Lemma 3.2, since $\Gamma_p = \langle a_p, b_p \rangle$, it follows that

$$C_{\mathrm{Sym}(\Gamma_p)}(\rho_p(a_p), \rho_p(b_p)) = \lambda_p[\Gamma_p];$$

and hence, by the Łoś Theorem [1], we have that

$$C_{G_{\mathcal{F}}}(\alpha, \beta) = \prod\nolimits_{\mathcal{F}} \lambda_p[\Gamma_p].$$

Now suppose that $\pi \in \mathrm{Aut}(G_{\mathcal{F}})$ is any automorphism. Let $\pi(\alpha) = (g_p)_{\mathcal{F}}$ and let $\pi(\beta) = (h_p)_{\mathcal{F}}$. Then we have that

$$C_{G_{\mathcal{F}}}((g_p)_{\mathcal{F}}, (h_p)_{\mathcal{F}}) = \prod\nolimits_{\mathcal{F}} \Delta_p,$$

where $\Delta_p = C_{\mathrm{Sym}(\Gamma_p)}(g_p, h_p)$. Recall that if we will identify $G_{\mathcal{F}}$ with its image under the embedding $G_{\mathcal{F}} \to \mathrm{Sym}(H_{\mathcal{F}})$, then

$$C_{G_{\mathcal{F}}}(\alpha, \beta) = \prod\nolimits_{\mathcal{F}} \lambda_p[\Gamma_p]$$

corresponds to the image $\lambda[H_{\mathcal{F}}]$ of the left regular permutation representation of $H_{\mathcal{F}}$; and since π corresponds to conjugation by a suitable element of the normalizer of $G_{\mathcal{F}}$ in $\mathrm{Sym}(H_{\mathcal{F}})$, it follows that $\prod_{\mathcal{F}} \Delta_p$ also corresponds to a regular subgroup of $\mathrm{Sym}(H_{\mathcal{F}})$. This easily implies that

$$A = \{\, p \in \mathbb{P} \mid \Delta_p \text{ is a regular subgroup of } \mathrm{Sym}(\Gamma_p)\,\} \in \mathcal{F}.$$

Furthermore, by Thomas [7], there exists a first-order sentence σ such that if L is any group, then

$$L \vDash \sigma \iff L \cong PSL_2(K) \text{ for some field } K.$$

Since $\prod_{\mathcal{F}} \Delta_p \cong \prod_{\mathcal{F}} \lambda_p[\Gamma_p] \cong PSL_2(\prod_{\mathcal{F}} \mathbb{F}_p)$, it follows that

$$B = \{\, p \in \mathbb{P} \mid \Delta_p \cong PSL_2(K) \text{ for some field } K\,\} \in \mathcal{F}.$$

If $p \in A \cap B$, then $|\Delta_p| = |\Gamma_p|$ and it follows that Δ_p and $\lambda_p[\Gamma_p]$ are isomorphic regular subgroups of $\mathrm{Sym}(\Gamma_p)$, which implies that Δ_p is conjugate to $\lambda_p[\Gamma_p]$ inside $\mathrm{Sym}(\Gamma_p)$. It follows that $\prod_{\mathcal{F}} \Delta_p$ is conjugate to $\prod_{\mathcal{F}} \lambda_p[\Gamma_p]$ inside $G_{\mathcal{F}}$. Hence, after adjusting π by an inner automorphism of $G_{\mathcal{F}}$ if necessary, we can suppose that

$$\pi[\prod_{\mathcal{F}} \lambda_p[\Gamma_p]] = \prod_{\mathcal{F}} \lambda_p[\Gamma_p].$$

Applying Lemma 3.2, it follows that π corresponds to conjugation by an element of $\mathrm{Hol}(H_{\mathcal{F}}) = \lambda[H_{\mathcal{F}}] \rtimes \mathrm{Aut}(H_{\mathcal{F}})$. By Lemma 2.2, we have that

$$\mathrm{Aut}(H_{\mathcal{F}}) = \mathrm{Aut}(\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p)) = \mathrm{Int}(\prod_{\mathcal{F}} PSL_2(\mathbb{F}_p));$$

and hence $\lambda[H_{\mathcal{F}}] \rtimes \mathrm{Aut}(H_{\mathcal{F}})$ corresponds to a subgroup of $G_{\mathcal{F}}$. Thus π is an inner automorphism of $G_{\mathcal{F}}$, as required.

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