Eleven Euclidean Distances are Enough

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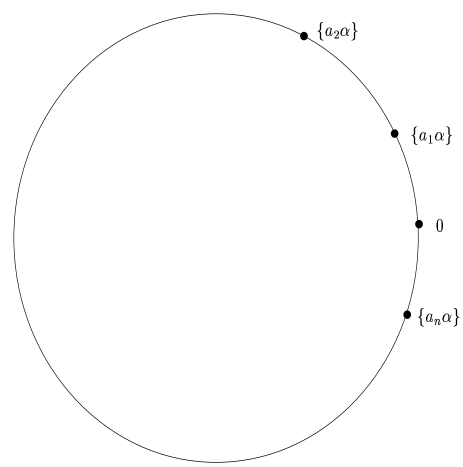
Introduction

Rational numbers are characterised by the fact that the sequence of fractional parts of their integer multiples is periodic, and therefore consists of only finitely many distinct elements. Kronecker showed that for any irrational number, the corresponding sequence is dense in the unit interval. Bohl, Sierpinski and Weyl, independently of each other, proved in the early 1900s that the sequence is, in fact, uniformly distributed in the unit interval. It is customary to think of such sequences as arising out of rotations on a circle of unit circumference, with the rotation labelled rational or irrational depending on the number being considered. Irrational rotations are of interest in ergodic theory and the study of dynamical systems.



A classic result in the study of real rotations is the three distance theorem, proved independently by several authors (see [5] and [6]) in the 1950s in response to a conjecture of Steinhaus. The theorem states that there are at most three distinct gaps between consecutive elements in the set of fractional parts of the first n multiples of any irrational number α. Formally, we have the following:

Theorem 1A Let α be any real number, and n a positive integer. Let $(a_1, a_2, \ldots, a_n)$ be a permutation of $\{1, 2, \ldots, n\}$ such that

$$0 < \{a_1\alpha\} < \{a_2\alpha\} < \cdots \{a_n\alpha\} < 1$$

Define $g_\alpha(0) = \{a_1\alpha\}$ and $g_\alpha(n) = 1 - \{a_n\alpha\}$. For $1 \leq k \leq n-1$, let $g_\alpha(k) = \{a_{k+1}\alpha\} - \{a_k\alpha\}$. Define

$$S_\alpha(n) = \{g_\alpha(k) : 0 \leq k \leq n\}$$

Then $|S_\alpha(n)| \leq 3$.

Generalisations

Chung and Graham [2] generalised the three distance theorem as follows:

Theorem 2 Let $\alpha, \lambda_1, \lambda_2, \ldots, \lambda_d$ be real numbers, and let $n_1, n_2, \ldots, n_d$ be positive integers. For $1 \leq i \leq d, 1 \leq k \leq n_i$, let $a_{i,k_i} = \{k_i\alpha + \lambda_i\}$, where $\{x\}$ denotes the fractional part of x. Then there are at most $3d$ distinct gaps between consecutive a_{i,k_i}.

Liang [4] gave a very short proof of the above $3d$ distance theorem. Another noteworthy result is due to Geelen and Simpson [3], who established the following:

Theorem 3 Let α and β be real numbers, and let n_1 and n_2 be positive integers. For $0 \leq k_1 < n_1, 0 \leq k_2 < n_2$, let $a_{k_1,k_2} = \{k_1\alpha + k_2\beta\}$. Then there are at most $n_1 + 3$ distinct gaps between consecutive a_{k_1,k_2}.

Chevallier [1] obtained the following higher-dimensional analogue of the three-distance theorem for the subsequence of best simultaneous approximation denominators.

Theorem 4 Let N be a best simultaneous approximation denominator with respect to the Euclidean norm of the d-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_d)$. Then there is a norm on R^d such that the Voronoi diagrams of the first N points of the sequence $(\{k\alpha_1\}, \{k\alpha_2\}, \ldots, \{k\alpha_d\})$ with respect to this norm are of at most C_d different forms, where C_d is a constant that depends only on the dimension d.

A New Formulation

The purpose of this article is to show that the central tenet of the three-distance theorem, namely the finiteness of the set of minimal distances, can be generalised to higher dimensions under a suitable interpretation. We begin by rephrasing the theorem in a form that lends itself to the generalisation we seek.

We think of the three distance theorem as a statement about champions in a tournament. The players in the tournament are edges connecting $\{j\alpha\}$ and $\{k\alpha\}, 1 \leq j < k \leq n$, two edges play each other if and only if they overlap, and an edge loses only against edges of shorter length that it plays against. Defeated edges are allowed to play (and defeat) other overlapping edges. According to the three distance theorem, there are at most three distinct values for the lengths of undefeated edges. Thus the theorem can be restated as follows:

Theorem 1B Let α be any real number, and n a positive integer. Define $d_\alpha(j,k) = ||\{k\alpha\} - \{j\alpha\}||$ where $||x||$ denotes the distance between x and the integer nearest to x. Let $I_{j,k}$ be the "geodesic" joining $\{j\alpha\}$ with $\{k\alpha\}$, i.e., if $m_{j,k} = \min(\{j\alpha\}, \{k\alpha\})$ and $M_{j,k} = \max(\{j\alpha\}, \{k\alpha\})$, we define

$$I_{j,k} = \begin{cases} [m_{j,k}, M_{j,k}) & \text{if } M_{j,k} - m_{j,k} \leq 1/2 \\ [0, m_{j,k}) \cup [M_{j,k}, 1) & \text{otherwise} \end{cases}$$

Let $S_\alpha(n) = \{d_\alpha(j,k) : d_\alpha(p,q) < d_\alpha(j,k) \Rightarrow I_{p,q} \cap I_{j,k} = \emptyset, 0 \leq j,k,p,q, \leq n\}$. Then $|S_\alpha(n)| \leq 3$.

We first prove a two-dimensional version of this theorem. We show that if the players are edges connecting $(\{j\alpha\}, \{j\beta\})$ and $(\{k\alpha\}, \{k\beta\})$ and two edges play each other if and only if their *projections along either co-ordinate axis* overlap, there are at most 11 distinct values for the lengths of undefeated edges. Numerical evidence suggests that the true value could be as small as 3.

Theorem P Let α and β be real numbers, and let n be a positive integer. Define the "circular" metric $d_{\alpha,\beta}(j,k) = \sqrt{||(k-j)\alpha||^2 + ||(k-j)\beta||^2}$. Let $I^1_{j,k}$ and $I^2_{j,k}$ be the geodesics joining $\{j\alpha\}$ with $\{k\alpha\}$ and $\{j\beta\}$ with $\{k\beta\}$ respectively. Define

$$S_{\alpha,\beta}(n) = \{d_{\alpha,\beta}(j,k) : d_{\alpha,\beta}(p,q) < d_{\alpha,\beta}(j,k) \Rightarrow I^1_{p,q} \cap I^1_{j,k} = I^2_{p,q} \cap I^2_{j,k} = \emptyset, 0 \le j,k,p,q \le n\}$$

Then $|S_{\alpha,\beta}(n)| \le 11$.

Proof We begin by classifying the denominators of simultaneous rational approximations to (α,β). Let $[[x]] = \{x\} - 1/2$. We say that q is a denominator of type $(+,-)$ if $[[q\alpha]] \ge 0$ and $[[q\beta]] < 0$. Denominators of type $(-,+), (+,+)$ and $(-,-)$ are defined analogously. Types $(+,+)$ and $(-,-)$ are said to be opposites to each other, as are types $(+,-)$ and $(-,+)$. We write $q_1 \,||\, q_2$ if q_1 and q_2 are of the same type and $q_1 \not\parallel q_2$ if q_1 and q_2 are not of the same type. We also write $q_1 \perp q_2$ if they are of opposite type, and $q_1 \sim q_2$ if they are *not* of opposite type.

We define the *length* of an integer q with respect to α and β as $\ell(q) = d_{\alpha,\beta(0,q)}$. Consider the edge $L_{a,b}$ joining $(\{a\alpha\},\{a\beta\})$ and $(\{b\alpha\},\{b\beta\})$, with $1 \le a < b \le n$. Let $d = b - a$.

Let Q_1 denote the least integer with the property that $\ell(Q_1) \le \ell(q)$ for all q, $1 \le q \le n/2$. For $n/2 < q \le n$, we say that q is *primary* if $\ell(q) < \ell(Q_1)$.

Lemma P1 If d is primary, $\ell(d)$ can take at most five distinct values.

We will prove the lemma later. Proceeding with the proof of the theorem, define $Q_1^\perp = \{q : 1 \le q \le n - Q_1, q \not\parallel Q_1\}$. Note that $Q_1^\perp$ is non-empty if $Q_1 \ge 2$. Let Q_2 be the least integer in $Q_1^\perp$ such that $\ell(q) \le \ell(Q_2)$ for all $q \in Q_1^\perp$.

Suppose q^* is not primary. We consider three cases.

CASE 1: $\mathbf{q^* \sim Q_1}$

Note that one of $L_{a,a+Q_1}$ or $L_{b-Q_1,b}$ will be admissible (i.e., have both subscripts lying within $[0,n]$), and will defeat the edge $L_{a,b}$ (i.e., is shorter than $L_{a,b}$ and overlaps it along one of the co-ordinate axes).

CASE 2: $\mathbf{q^* \perp Q_1}$ and $\mathbf{1 \leq q^* \leq n - Q_1}$

Clearly, $\ell(q^*) \geq \ell(Q_2)$. Moreover, if equality does not hold, one of $L_{a,a+Q_2}$ or $L_{a-Q_1,a}$ will defeat $L_{a,b}$.

CASE 3: $\mathbf{q^* \perp Q_1}$ and $\mathbf{n - Q_1 < q^* \leq n}$

Note that if $Q_1 = 1, q^* = n$ is the only possibility, and can be dealt with separately. Otherwise, Q_2 exists, and we say that q^* is *secondary* if $\ell(q^*) < \ell(Q_2)$. Observe that if q^* is not secondary, one of $L_{a,a+Q_2}$ or $L_{a-Q_1,a}$ will defeat $L_{a,b}$.

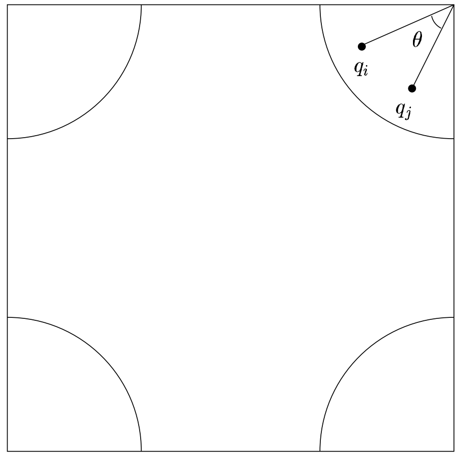
Lemma P2 If q^* is secondary, $\ell(q^*)$ can take at most four distinct values.

Assuming Lemmas P1 and P2, it follows that at most eleven distinct gaps survive. It remains to prove the two lemmas.

Proof of Lemma P1 Consider four quarter-circles of radius $R = \ell(Q_1)$ centred at the four corners of the unit square. Suppose there exist $q_i, 1 \leq i \leq 7$ with $n/2 < q_1 < q_2 < \cdots < q_7 \leq n$ and $\ell(q_i) < R$. Then there must be a pair $(i,j), 1 \leq i < j \leq 7$ such that the angle θ subtended between the edges L_{0,q_i} and L_{0,q_j} is less than $\pi/3$. But then we have $\ell(q_j - q_i) < R$, yielding a contradiction, since $1 \leq q_j - q_i < n/2$.

Furthermore, the only way to have six primary q_i avoiding $\ell(q_j - q_i) < R$ is to arrange them along the vertices of a regular hexagon, leading to identical values of $\ell(q_i)$. It follows that $\ell(q^*)$ can take at most five distinct values if q^* is primary. ∎

Proof of Lemma P2 We claim that there does not exist $q < Q_1$ satisfying $||q\alpha|| < (\ell(Q_2)/2)$ and $||q\beta|| < (\ell(Q_2)/2)$. Suppose there is such a q. Observe that $\ell(Q_1 - q) < \ell(Q_2)$ and $Q_1 - q \not\parallel Q_1$. This contradicts the definition of Q_2.



Now suppose that $n - Q_1 < q_1 < q_2 < \cdots < q_5 < n$, with $\ell(q_i) < \ell(Q_2)$ and $q_i \perp Q_1$. Since $q_1 \,||\, q_2 \,||\, \ldots \,||\, q_5$, it is easy to see that there exist i and j satisfying $||(q_i - q_j)\alpha|| < \ell(Q_2)/2$, $||(q_i - q_j)\beta|| < \ell(Q_2)/2$ and $q_i - q_j < Q_1$, contradicting our claim above, and proving the lemma. ∎

Higher Dimensions

For higher dimensions, the above argument can be adapted to obtain similar results. We prove the following theorem which implies, in particular, that there are at most 74 distances in three dimensions.

Theorem H Let $\alpha \doteq (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m$, and let n be a positive integer. Define

$$d_\alpha(j,k) = \sqrt{\sum_{i=1}^{n} ||(k-j)\alpha_i||^2}$$

For $1 \le r \le m$, let $I^r_{j,k}$ denote the geodesic joining $\{j\alpha_r\}$ with $\{k\alpha_r\}$. Define

$$S_\alpha(n) = \{d_\alpha(j,k) : d_\alpha(p,q) < d_\alpha(j,k) \Rightarrow I^r_{p,q} \cap I^r_{j,k} = \emptyset \text{ for all } r.\}$$

Then

$$|S_\alpha(n)| \le (2^m + 1)(\lceil \sqrt{m}\, \rceil^m) + 2$$

Proof Let $[[x]] = \{x\} - 1/2$. As in the proof of Theorem 1, we assign, to each denominator q an m-tuple of signs. The i^{th} sign is positive if $[[q\alpha_i]] \ge 0$

and negative otherwise.

The *length* of an integer q with respect to α is defined as $\ell(q) = d_\alpha(0, q)$. Let Q_1 denote the least integer with the property that $\ell(Q_1) \leq \ell(q)$ for all q, $1 \leq q \leq n/2$. For $n/2 < q \leq n$, we say that q is *primary* if $\ell(q) < \ell(Q_1)$.

Lemma H1 There are at most $(2\lceil\sqrt{m}\,\rceil)^m$ primary denominators in $I\!R^m$ for any given α.

As in the planar case, define $Q_1^\perp = \{q : 1 \leq q \leq n - Q_1, q \not| Q_1\}$, and let Q_2 be the least integer in $Q_1^\perp$ such that $\ell(q) \leq \ell(Q_2)$ for all $q \in Q_1^\perp$. Note that $Q_1^\perp$ is non-empty if $Q_1 \geq 2$.

Consider the line $L_{a,b}$ joining $(\{a\alpha\}, \{a\beta\})$ and $(\{b\alpha\}, \{b\beta\})$, with $1 \leq a < b \leq n$. Let $q^* = b - a$. Suppose q^* is not primary. We consider two cases.

CASE 1: $\mathbf{q^* \sim Q_1}$

Note that one of $L_{a,a+Q_1}$ or $L_{b-Q_1,b}$ will be admissible, and will defeat $L_{a,b}$.

CASE 2: $\mathbf{q^* \perp Q_1}$ and $\mathbf{1 \leq q^* \leq n - Q_1}$

As before, $\ell(q^*) \geq \ell(Q_2)$ and if equality does not hold, one of $L_{a,a+Q_2}$ or $L_{a-Q_1,a}$ will defeat $L_{a,b}$.

CASE 3: $\mathbf{q^* \perp Q_1}$ and $\mathbf{n - Q_1 < q^* \leq n}$

Note that if $Q_1 = 1, q^* = n$. Otherwise, Q_2 exists, and we say that q^* is *secondary* if $\ell(q^*) < \ell(Q_2)$. If q^* is not secondary, one of $L_{a,a+Q_2}$ or $L_{a-Q_1,a}$ will be admissible, and will defeat $L_{a,b}$.

Lemma H2 If q^* is secondary, $\ell(q^*)$ can take at most $(\lceil\sqrt{m}\,\rceil^m + 1)$ distinct values.

Note that the statement of the theorem is a direct consequence of Lemmas H1 and H2. We now prove these lemmas.

Proof of Lemma H1 If the number of distinct values q satisfying $\ell(q) < \ell(Q_1)$ exceeds $(2\lceil\sqrt{m}\,\rceil)^m$, at least $1 + \lceil\sqrt{m}\,\rceil^m$ of these values must be of the same type. By the pigeonhole principle, there exists q_1 and q_2 with $||(q_2 - q_1)\alpha_i|| < \ell(Q_1)/\sqrt{m}$ for all $i, 1 \le i \le m$. It follows that $\ell(q_2 - q_1) < \ell(Q_1)$. But $q_2 - q_1 < n/2$, contradicting the definition of Q_1. Thus at most $(2\lceil\sqrt{m}\,\rceil)^m$ denominators can be primary. ∎

Proof of Lemma H2 We claim that there are does not exist $q < Q_1$ satisfying $||Q\alpha_i|| < \ell(Q_2)/\sqrt{m}$. Suppose there exists such a q. Note that $\ell(Q_1 - q) < \ell(Q_2)$ and $Q_1 - q \not| Q_1$, contradicting the definition of Q_2.

Now suppose that for $k = (\lceil\sqrt{m}\,\rceil^m + 1$, we have

$$n - Q_1 < q_1 < q_2 < \cdots < q_k < n$$

with $\ell(q_i) < \ell(Q_2)$. Since $q_1||q_2||\cdots||q_k$, there exist i and j with

$$||(q_i - q_j)\alpha_r|| < \ell(Q_2)/\sqrt{m}, 1 \le r \le m$$

and $q_i - q_j < Q_1$, contradicting our claim above, and proving the lemma. ∎

Acknowledgements

I am grateful to Professors József Beck and Endre Szemerédi for useful discussions. I also thank Professors Ronald Graham and Vera Sós for providing pointers to literature.

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