1. Show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

(Hint: In how many ways can a team of n people be chosen from a group of n men and n women?) [10 points]

2. Find a polynomial $p(x)$ such that the coefficient of x^{187} in the expansion of $1/p(x)$ gives the number of ways of making change for \$1.87 using $1¢, 5¢, 10¢$ and $25¢$ coins. [10 points]

3. Explain the basic principles of error-detecting and error-correcting codes, with examples. [10 points]

4. a) State Hall's marriage theorem, clarifying notation, if any. [4 points]

b) Show that a bipartite graph with a Hamiltonian cycle has at least two different perfect matchings. [6 points]

5. Let G be a graph on six vertices, with the vertices labelled $2, 3, 4, 5, 6, 7$. There is an edge between u and v in G if the labels are relatively prime. The cost of an edge is the product of the labels of the vertices it connects. Find a minimum-cost spanning tree in G. [10 points]

6. Given a graph G, the complement of G, denoted by $\overline{G}$ is defined on the same vertex set as G, with an edge present in $\overline{G}$ if and only if it is absent in G. Show that if G is not connected, $\overline{G}$ must be connected. [10 points]

7. The outdegree of a vertex v in a directed graph is the number of directed edges going out of v. Show that if all vertices in a tournament T have positive outdegree, T has at least two different Hamiltonian paths. [10 points]

8. a) Show that a graph on 16 vertices that does not contain any 7-cliques has at most 106 edges. [5 points]

b) Let G be the graph whose vertices are the 16 subsets of $\{1, 2, 3, 4\}$ with an edge between two vertices if and only if neither of the corresponding subsets is contained in the other. Show that G does not contain any 7-cliques. [5 points]

9. Consider the following game: Two players alternately choose a number in $\{1, 2, ..., 2n\}$. No number can be chosen more than once. The numbers chosen by the first player are coloured red, and the numbers chosen by the second player are coloured blue. The winner is the first player to get a 4-term arithmetic progression of his/her own colour.

a) Show that the second player does not have a winning strategy. [5 points]

b) Show that for sufficiently large n, the first player has a winning strategy. [5 points]