On the least size of a graph with a given Degree Set

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Abstract

The *degree set* of a finite simple graph G is the set of distinct degrees of vertices of G. A theorem of Kapoor, Polimeni & Wall asserts that the least order of a graph with a given degree set $\mathcal{D}$ is $1 + \max(\mathcal{D})$. We look at the analogous problem concerning the least size of a graph with a given degree set $\mathcal{D}$. We determine the least size for the sets $\mathcal{D}$ when (i) $|\mathcal{D}| \leq 3$; (ii) $\mathcal{D} = \{1, 2, \ldots, n\}$; and (iii) every element in $\mathcal{D}$ is at least $|\mathcal{D}|$. In addition, we give sharp upper and lower bounds in all cases.

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A sequence $d_1, d_2, \ldots, d_p$ of nonnegative integers is said to be *graphic* if there exists a simple graph G with vertices $v_1, v_2, \ldots, v_p$ such that v_k has degree d_k for each k. Any graphic sequence clearly satisfies the two conditions $d_k \leq p - 1$ for each k and $\sum_{k=1}^{p} d_k$ is *even*. However, these two conditions together do not ensure that a sequence will be graphic. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphic are well known [1, 2, 3].

The *degree set* of a simple graph G is the set $\mathcal{D}(G)$ consisting of the distinct degrees of vertices

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in G. It is a simple observation that any set of positive integers forms the degree set of a graph. A natural question then is to investigate the order and size of such graphs. An initial step in that direction is to determine the least order and the least size among graphs with a specified degree set. The following result answers that question for the order of a graph:

Theorem KPW ([4]). For each nonempty finite set $\mathcal{D}$ of of positive integers, there exists a simple graph G for which $\mathcal{D}(G) = \mathcal{D}$. Moreover, there is always such a graph of order $\Delta + 1$, where $\Delta = max(\mathcal{D})$, and there is no such graph of smaller order.

Theorem KPW therefore asserts that there is always a graph with least possible order corresponding to each set of positive integers since any graph with maximum degree Δ requires $\Delta + 1$ vertices. The aim of this paper is to attempt to answer the analogous question for the size of a graph. We say that a graph G is a $(q, \mathcal{D})$-graph if it has q edges and has degree set $\mathcal{D}$. Thus, for a given finite set of positive integers $\mathcal{D}$, we determine the least q for which there exists a $(q, \mathcal{D})$-graph. We denote this least number by $\ell_q(\mathcal{D})$. In this paper, we determine $\ell_q(\mathcal{D})$ when

(i) $|\mathcal{D}| \leq 3$;
(ii) $\mathcal{D} = \{1, 2, \ldots, n\}$;
(iii) $\min(\mathcal{D}) \geq |\mathcal{D}|$.

We also give upper and lower bounds for $\ell_q(\mathcal{D})$ in all cases, and exhibit cases where each bound is achieved.

Throughout this paper we shall employ the notation $(a)_m$ to denote m occurrences of the integer a. Thus, we may denote a typical degree sequence by

$$s := (d_1)_{m_1}, (d_2)_{m_2}, \ldots, (d_n)_{m_n}, \tag{1}$$

where $d_1 > d_2 > \cdots > d_n$, and each $m_k \geq 1$ with $m_1 + m_2 + \cdots + m_n = p$.

We shall write

$$\sigma_k := \sum_{i=1}^{k} m_i, \quad \text{with} \quad \sigma_0 := 0, \quad \text{and} \quad S_{r,t} := \sum_{i=r}^{t} m_i d_i.$$

The characterization of graphic sequence due to *Erdös & Gallai* [1] requires verification of as many inequalities as is the order of the graph. We use a refined form of their result [5] that requires verification of only as many inequalities as the number of distinct terms in the sequence. For the sake of completeness, we recall this result:

Theorem A ([5]). A sequence (1) is graphic if and only if $S_{1,n}$ is even and if the inequalities

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{p} \min(k, d_i)$$

hold for $k = \sigma_1, \sigma_2, \ldots, \sigma_n$. Moreover, the inequality need only be checked for $1 \leq k \leq s$, where s is the *largest* positive integer for which $d_s \geq s - 1$.

Henceforth, we shall take $\mathcal{D} = \{d_1, d_2, \ldots, d_n\}$, with $d_1 > d_2 > \cdots > d_n$. The value $\ell_q(\mathcal{D})$ is determined by *minimizing* the sum of the terms of all graphic sequences with distinct terms given by the set $\mathcal{D}$. In case s in (1) is a degree sequence, $q = \frac{1}{2} \sum_{k=1}^{n} m_k d_k$, so that the problem of determining sizes of graphs is the same as determining the set of values taken by

$$m_1 d_1 + m_2 d_2 + \cdots + m_n d_n, \quad m_k \geq 1,$$

such that $s := (d_1)_{m_1}, (d_2)_{m_2}, \ldots, (d_n)_{m_n}$ is a graphic sequence.

We begin by solving the problem in the simple case when $|\mathcal{D}| = 1$. We give a proof of the following lemma, which is well known and appears as an exercise in several standard textbooks in Graph Theory, for the sake of completeness:

Lemma 1. *Let p be a positive integer and r be such that $0 \leq r \leq p-1$. Then there exists an r-regular graph of order p if and only if pr is even.*

Proof. The necessity of the condition follows from the fact that the sum of vertex degrees of a finite graph is always even. For the sufficiency, it is enough to prove the case when $0 \leq r \leq \frac{p-1}{2}$ since the complement of an r-regular graph of order p is a $(p-1-r)$-regular graph of the same order. Consider p vertices placed in a circle. If r is even, join each vertex to exactly $\frac{r}{2}$ vertices immediately to its left and immediately to its right. If r is odd, p must be even so that there is a vertex diagonally opposite each vertex. Join each vertex to exactly $\frac{r-1}{2}$ vertices immediately to its left and immediately to its right, and also to the vertex diagonally opposite it. This constructs an r-regular graph of order p in all cases when pr is even. □

Lemma 1 also immediately follows from Theorem A. Indeed, from Theorem A, there is a r-regular graph of order p if and only pr is even and the inequality $pr \leq p(p-1)$ is satisfied. Lemma 1 immediately implies

Theorem 1. *If $\mathcal{D} = \{a\}$, there exists a $(q, \mathcal{D})$-graph if and only if*

$$q \in \begin{cases} \left\{ ma : m \geq \dfrac{a+1}{2} \right\} & \text{if } a \text{ is odd;} \\ \left\{ m\dfrac{a}{2} : m \geq a+1 \right\} & \text{if } a \text{ is even.} \end{cases}$$

In particular, $\ell_q(\{a\}) = \frac{1}{2}a(a+1)$.

Theorem A can be easily applied to determine the analogue of Theorem 1 to the case $|\mathcal{D}| = 2$. We prove this result next.

Theorem 2. *If $\mathcal{D} = \{a, b\}$, with $a > b$, there exists a $(q, \mathcal{D})$-graph if and only if q is of the form $\frac{1}{2}(ma + nb)$, where $m, n \geq 1$, $m + n \geq a + 1$, and*

$$1 \leq m \leq b \ \text{ or } \ m \geq a + 1 \ \text{ or } \ m(a + 1 - m) \leq nb \ \text{ with } \ b + 1 \leq m \leq a.$$

In particular,

$$\ell_q(\{a, b\}) = \begin{cases} \frac{1}{2}a(b + 1) & \text{if } a \text{ is even or } b \text{ odd;} \\ \frac{1}{2}\big(a(b + 1) + (a - b)\big) & \text{if } a \text{ is odd and } b \text{ is even.} \end{cases}$$

Proof. We apply Theorem A to the sequence $s := (a)_m, (b)_n$, where $a > b$ and $a, b, m, n \geq 1$. Such a sequence is graphic if and only if $ma + nb$ is even and

$$ma \leq m(m - 1) + n. \min(m, b) \quad \text{and} \quad ma + nb \leq (m + n)(m + n - 1). \tag{2}$$

Since the number of terms in any graphic sequence must be more than the largest term, $ma + nb < (m + n)a \leq (m + n)(m + n - 1)$, so that the second inequality in (2) is always satisfied. If $m \leq b$, the first inequality reduces to $a \leq m + n - 1$, which is necessary anyway. Thus s is graphic whenever $m \leq b$, irrespective of n. If $m > b$, the first inequality reduces to $ma \leq m(m - 1) + nb$, or to $m(a + 1 - m) \leq nb$, which is satisfied whenever $m \geq a + 1$. This proves the main result.

The assertion about the least size in the first case is due to the sequence $s_1 := a, (b)_a$ and in the second case to the sequence $s_2 := (a)_2, (b)_{a-1}$. It is clear that there can be no smaller sized graph in the second case since there has to be an even number of odd vertices. This completes the proof. □

Theorem 2 can be extended to cases where $|\mathcal{D}| \geq 3$ along similar lines but is more tedious. We treat the special case of $\mathcal{D} = \{1, 2, \ldots, n\}$ next. To determine $\ell_q(\mathcal{D})$ in this case, we need the following lemma:

Lemma 2. *For any $p > 1$, the sequence $1, 2, \ldots, p$, with only the number $\lceil p/2 \rceil$ appearing twice is graphic.*

Proof. We prove only the case p is even, say $p = 2k$; the case p is odd is analogous. The sequence $1, 1, 2$ is the degree sequence of the path $\mathcal{P}_3$. If the sequence $1, 2, \ldots, k - 1, k, k, k + 1, \ldots, 2k$ corresponds to a graph G, add vertices u and v and edges connecting v to every vertex of G and to u. This gives a graph with degree sequence $1, 2, \ldots, k, k+1, k+1, k+2, \ldots, 2k+2$, and the proof is complete by induction. □

Theorem 3. *Let $S_n = \{1, 2, \ldots, n\}$. Then there exists a (q, S_n)-graph if and only if $q \geq \lceil \frac{n}{2} \rceil \left(\lfloor \frac{n}{2} \rfloor + 1\right)$. In particular, $\ell_q(S_n) = \lceil \frac{n}{2} \rceil \left(\lfloor \frac{n}{2} \rfloor + 1\right)$.*

Proof. The graphic sequence of Lemma 2 with $p = n$ has size q_0, where $2q_0 = \frac{1}{2}n(n+1) + \lceil n/2 \rceil$. It is easy to verify that this reduces to $q_0 = \lceil \frac{n}{2} \rceil \left(\lfloor \frac{n}{2} \rfloor + 1 \right)$. Let G be (q_0, S_n)-graph. Since $1 \in S_n$, a graph of any size greater than or equal to q_0 can be obtained by adding copies of $\mathcal{K}_2$.

Let H be a (q, S_n)-graph, where $q < q_0$. Thus the degree sequence of H must contain the numbers greater than or equal to $\lceil n/2 \rceil$ exactly once. Applying the inequality of Theorem A, we must then have

$$n + (n-1) + \cdots + \lceil n/2 \rceil \leq \lfloor n/2 \rfloor \left(\lfloor n/2 \rfloor + 1 \right) + \sum_{k=\lfloor n/2 \rfloor + 2}^{n+1} \min \left(1 + \lfloor n/2 \rfloor, d_k \right),$$

or,

$$\frac{1}{2} \left(n + \lceil n/2 \rceil \right) \left(\lfloor n/2 \rfloor + 1 \right) \leq \lfloor n/2 \rfloor \left(\lfloor n/2 \rfloor + 1 \right) + \sum_{k=\lfloor n/2 \rfloor + 2}^{n+1} d_k.$$

This implies

$$\frac{1}{2} \left(n - 2\lfloor n/2 \rfloor + \lceil n/2 \rceil \right) \left(\lfloor n/2 \rfloor + 1 \right) \leq \sum_{k=\lfloor n/2 \rfloor + 2}^{n+1} d_k,$$

which is impossible since the sum on the right is *at most*

$$1 + 2 + \cdots + \left(\lceil n/2 \rceil - 1 \right) + \left(\lceil n/2 \rceil - 2 \right) < \frac{1}{2} \left(\lceil n/2 \rceil \right) \left(\lceil n/2 \rceil + 1 \right),$$

which equals the expression on the left. This proves $\ell_q(S_n) = \lceil \frac{n}{2} \rceil \left(\lfloor \frac{n}{2} \rfloor + 1 \right)$, and completes the proof of the theorem. □

We now address the problem of giving lower and upper bounds for $\ell_q(\mathcal{D})$ in the general case. Any simple graph with maximum degree Δ must have at least $\Delta + 1$ vertices, and Theorem KPW asserts that there is always a simple graph of order $\Delta + 1$ whose degree set is $\mathcal{D}$, where $\Delta = \max(\mathcal{D})$. Suppose $\mathcal{D} = \{d_1, d_2, \ldots, d_n\}$, with $d_1 > d_2 > \ldots > d_n$. In order to minimize the sum of degrees of a graph G with degree set $\mathcal{D}$, the minimum possible degree sequence would consist of the $d_1 + 1$ terms consisting of $d_1, d_2, \ldots, d_{n-1}$ occurring once each and d_n occurring $(d_1 + 1) - (n - 1)$ times. In case the sum of these terms is odd (so that the sequence cannot be graphic), we either replace d_n by the smallest member of the sequence of opposite parity or else add another d_n to the sequence. In case d_n is even, only the first option is available, but when d_n is odd, both options are possible. Thus we set

$$\ell(\mathcal{D}) = \begin{cases} \frac{1}{2} p_0(\mathcal{D}) & \text{if } p_0(\mathcal{D}) \text{ is even;} \\ \frac{1}{2} \Big(p_0(\mathcal{D}) + d_r - d_n \Big) & \text{if } p_0(\mathcal{D}) \text{ is odd and } d_n \text{ is even;} \\ \min \left(\frac{1}{2} \left(p_0(\mathcal{D}) + d_r - d_n \right), \frac{1}{2} \left(p_0(\mathcal{D}) + d_n \right) \right) & \text{if both } p_0(\mathcal{D}) \text{ and } d_n \text{ are odd,} \end{cases}$$

where $p_0(\mathcal{D}) = \sum_{k=1}^{n} d_k + d_n(d_1 - n + 1)$ and $r := \max_{1 \le k \le n} \{d_k \not\equiv d_n \,(\text{mod } 2)\}$ if such a k exists, and $r = n$ otherwise. It is now clear that

$$\ell_q(\mathcal{D}) \ge \ell(\mathcal{D})$$

for any finite set $\mathcal{D}$. The following result shows that this lower bound is achieved for an infinite class of sets.

Theorem 4. *Let* $\mathcal{D}$ *be a finite set of positive integers such that* $\min(\mathcal{D}) \ge |\mathcal{D}|$. *Then* $\ell_q(\mathcal{D}) = \ell(\mathcal{D})$.

Proof. Consider the sequence s_0: $d_1, d_2, \ldots, d_{n-1}, (d_n)_{d_1-n+2}$. Observe that *if* s_0 is a graphic sequence, the corresponding graph is a $(\ell_q(\mathcal{D}), \mathcal{D})$-graph, where $\ell_q(\mathcal{D}) = \frac{1}{2}p_0(\mathcal{D})$ and $\mathcal{D} = \{d_1, d_2, \ldots, d_n\}$.

Suppose first that $p_0(\mathcal{D})$ is *even*. By Theorem A, the sequence s_0 is graphic if and only if

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n-1} \min(k, d_i) + (d_1 - n + 2)\min(k, d_n)$$

holds for $1 \le k \le n-1$. However, since $d_i \ge d_n \ge n > k$ for each i, the right side equals $k(k-1) + k(d_1 - k + 1) = kd_1$. Thus the sequence s_0 is graphic in this case. Therefore $\ell_q(\mathcal{D}) = \frac{1}{2}p_0(\mathcal{D})$ if $p_0(\mathcal{D})$ is even.

Suppose now that $p_0(\mathcal{D})$ is *odd*. If all d_i are of the same parity, then $p_0(\mathcal{D})$ is even; so $r < n$. Two cases arise:

Case I: (d_n is *even*) Consider the sequence s_1: $d_1, d_2, \ldots, d_{r-1}, (d_r)_2, d_{r+1}, \ldots, d_{n-1}, (d_n)_{d_1-n+1}$. It is graphic because the sum of the numbers in this sequence equals $p_0(\mathcal{D}) + (d_r - d_n)$, which is *even*, and because the inequalities given in Theorem A hold as above. Moreover, since s_1 is a sequence of $d_1 + 1$ numbers, the least change that can be made to the sequence s_0 so as to ensure an even sum is to replace a d_n by d_r. This gives $\ell_q(\mathcal{D}) = \frac{1}{2}\Big(p_0(\mathcal{D}) + d_r - d_n\Big)$ in this case.

Case II: (d_n is *odd*) Observe that the sequence s_1 of Case I is again graphic. Consider the sequence s_2: $d_1, d_2, \ldots, d_{n-1}, (d_n)_{d_1-n+3}$. This is graphic as the sum of the numbers in it equal $p_0(\mathcal{D}) + d_n$, and because the inequalities in Theorem A holds for the same reasons as before. We claim that in this case, one of the sequences s_1, s_2 give the minimum degree sum. Any graphic sequence must contain *at least* $d_1 + 1$ terms; if it contains exactly $d_1 + 1$ terms, as in Case I, s_1 is the sequence with minimum sum. On the other hand, among all sequences with $d_1 + 2$ terms or more, s_2 has the minimum sum. Thus $\ell_q(\mathcal{D}) = \frac{1}{2}\min\Big(p_0(\mathcal{D}) + d_r - d_n, p_0(\mathcal{D}) + d_n\Big)$ in this case. This completes the proof. □

We now turn our attention to obtaining an upper bound for $\ell_q(\mathcal{D})$. By Theorem KPW, there is necessarily a graphic sequence with $d_1 + 1$ terms. The following result gives an example of one such, thereby providing an alternate direct proof of Theorem KPW.

Theorem 5. *For each $n > 1$, the sequence*

$$(d_1)_{m_1}, (d_2)_{m_2}, \dots, (d_n)_{m_n},$$

with $m_1 = d_n$, $m_k = d_{n+1-k} - d_{n+2-k}$ for $2 \le k \le n$, $k \ne r$, and $m_r = d_{\lceil (n+1)/2 \rceil} - d_{\lceil (n+1)/2 \rceil + 1} + 1$ is a graphic sequence of order $d_1 + 1$, where $r = \lfloor (n+1)/2 \rfloor$.

Proof. We prove the result only in the case of odd n by using Theorem A, the case of even n being similar. Observe that there are

$$d_n + \sum_{k=2}^{n} (d_{n+1-k} - d_{n+2-k}) + 1 = d_1 + 1$$

terms in the sequence, and that their sum

$$\begin{aligned} d_1 d_n + \sum_{k=2}^{n} (d_{n+1-k} - d_{n+2-k}) d_k + d_r &= \sum_{k=1}^{n} d_k d_{n+1-k} - \sum_{k=2}^{n} d_k d_{n+2-k} - d_r(d_r - 1) \\ &= 2\sum_{k=1}^{r-1} d_k d_{n+1-k} - 2\sum_{k=2}^{r} d_k d_{n+2-k} - d_r(d_r - 1) \end{aligned}$$

is even.

The inequality of Theorem A amounts to proving the inequality

$$m_1 d_1 + \cdots + m_k d_k \le \sigma_k(\sigma_k - 1) + m_{k+1} \min(\sigma_k, d_{k+1}) + \cdots + m_n \min(\sigma_k, d_n), \tag{3}$$

where $\sigma_k = m_1 + \cdots + m_k$ equals d_{n+1-k} for $1 \le k \le r-1$ and $d_{n+1-k} + 1$ for $r \le k \le n$. Suppose first that $k \le r - 1$. Then the right-side of (3)

$$\begin{aligned} &= d_{n+1-k}(d_{n+1-k} - 1) + d_{n+1-k}(\sigma_{n+1-k} - \sigma_k) + d_{n+2-k}(d_{k-1} - d_k) + \cdots + d_n(d_1 - d_2) \\ &= d_{n+1-k}(d_{n+1-k} - 1) + d_{n+1-k}(d_k - d_{n+1-k} + 1) + d_{n+2-k}(d_{k-1} - d_k) + \cdots + d_n(d_1 - d_2) \\ &= d_1 d_n + d_2(d_{n-1} - d_n) + \cdots + d_k(d_{n+1-k} - d_{n+2-k}), \end{aligned}$$

which is the left-side of (3). If $k \ge r$, the first term on the right-side of (3) is $d_{n+1-k}(d_{n+1-k}+1)$ instead of $d_{n+1-k}(d_{n+1-k} - 1)$, so that the inequality still holds. This completes the proof in the case when n is odd. □

Lemma 2 shows that the upper bound for $\ell_q(\mathcal{D})$ as given by Theorem 5 is achieved. We denote this upper bound by $u(\mathcal{D})$. From the proof of Theorem 5,

$$u(\mathcal{D}) = \sum_{k=1}^{r-1} d_k d_{n+1-k} - \sum_{k=2}^{r} d_k d_{n+2-k} - \frac{1}{2} d_r(d_r - 1),$$

where $r = \lfloor (n+1)/2 \rfloor$. We summarize the results of Theorems 4 and 5 in

Theorem 6. *With the notation of* (1)*, we have*

$$\ell(\mathcal{D}) \leq \ell_q(\mathcal{D}) \leq u(\mathcal{D}),$$

where $\ell(\mathcal{D})$ *and* $u(\mathcal{D})$ *are as previously defined. Moreover, each of the bounds are sharp.*

It is sometimes possible to determine $\ell_q(\mathcal{D})$ exactly. Theorem 4 implies that the cases in which $\ell_q(\mathcal{D})$ remains to be determined are those for which $\min(\mathcal{D}) < |\mathcal{D}|$. These cases are easily taken care of when $|\mathcal{D}|$ is small, specially when $|\mathcal{D}| \leq 3$. A casewise breakup for larger sets is more tedious.

Theorem 7. *If* $b < c$*, then*

$$\begin{aligned}
\ell_q(\{1,b\}) &= b; \\
\ell_q(\{1,b,c\}) &= b+c-1; \\
\ell_q(\{2,b,c\}) &= \begin{cases} \frac{1}{2}(b+3c-2) & \text{if } b \equiv c \pmod 2; \\ b+2c-3 & \text{if } b \text{ is even and } c \text{ is odd}; \\ c+2b-3 & \text{if } b \text{ is odd and } c \text{ is even.} \end{cases}
\end{aligned}$$

Proof. The first case is a special case of Theorem 2.

Let $\mathcal{D} = \{1, b, c\}$. The sequence $c, b, (1)_{b+c-2}$ is graphic; it corresponds to the graph $\mathcal{K}_{1,b} \bigcup \mathcal{K}_{1,c}$ with the non-pendant vertices in each joined, and has $b + c - 1$ edges. Any degree sequence with *at least* three occurrences of numbers greater than 1 has sum greater than or equal to $2b+c+(c-2) = 2(b+c-1)$. If $c, b, (1)_k$ is graphic, by Theorem A, we must have $b+c \leq k+2$, so that $k \geq b + c - 2 \geq 3$. Thus, $\ell_q(\mathcal{D}) = b + c - 1$ in this case.

Finally, let us consider $\mathcal{D} = \{2, b, c\}$. If b, c are of the same parity, the sequence $c, b, (2)_{c-1}$ has even degree sum, and satisfies the inequalities of Theorem A. Since this sequence clearly has minimum possible degree sum, $\ell_q(\mathcal{D}) = \frac{1}{2}(b + 3c - 2)$ in this case.

We now consider the case when b, c are of opposite parity. Suppose b is even and c odd; then there must be *at least* two vertices of degree c. Sequences of the form $(c)_2, b, (2)_k$ are graphic if and only if

$$2c \leq b + 2k + 2 \ \text{ and } \ b + 2c \leq 2k + 6.$$

These are simultaneously satisfied precisely when $k \geq \frac{1}{2}(b + 2c - 6)$, and the sequence with minimum size of this form has $b + 2c - 3$ edges. Any other graphic sequence with the same degree set must have a minimum of two vertices of degrees b and c each, and $c + 1$ vertices in all; thus its degree sum must be *at least* $2b+2c+2(c-3) = 2(b+2c-3)$. Thus $\ell_q(\mathcal{D}) = b+2c-3$ in this case, and the case when b is odd and c even is analogous. This completes our theorem. □

We end this article with a general remark. For each k, $1 \leq k \leq p$, we set

$$f(k) := k(k-1) + \sum_{i=k+1}^{p} \min(k, d_i) - \sum_{i=1}^{k} d_i \tag{4}$$

According to Theorem A, any sequence given by (1) is graphic if and only if $S_{1,n} = \sum_{i=1}^{n} m_i d_i$ is even and $f(k) \geq 0$ for those $k = \sigma_1, \sigma_2, \ldots, \sigma_n$ which are at most s, where s is the *largest* positive integer for which $d_s \geq s - 1$. Observe that $f(1) = p - 1 - d_1 \geq 0$ as long as the basic necessary condition that the number of terms in the sequence be at least one more than the first term is met. This automatically forces $f(p) = p(p-1) - \sum_{i=1}^{n} m_i d_i \geq 0$. Thus, if each successive difference

$$f(k+1) - f(k) = 2k + \sum_{i=k+2}^{p} [\min(k+1, d_i) - \min(k, d_i)] - \min(k, d_{k+1}) - d_{k+1} \qquad (5)$$

is nonnegative, $f(k) \geq 0$ is ensured. This gives a sufficient condition to test whether a given sequence is graphic. It is sometimes easier to work with (5) instead of (4), as in the case of the sequence used in Theorem 4.

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